Reflection and the Singular Cardinal Hypothesis

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Stationary set reflection is a simple combinatorial principle that comes as a consequence of certain strong forcing axioms, such as Martin's Maximum (MM) or PFA$^+$ [8]. In a slightly weaker form (which will be considered in Corollary 2), it also holds above a strongly a compact cardinal.

In recent years, several studies have tackled the problem of whether stationary set reflection implies another common principle, the singular cardinal hypothesis, beginning with B. Veličković [8], then M. Foreman and S. Todorčević [3], all of whom considered somewhat stronger versions of stationary set reflection, until the problem was closed, positively, by S. Shelah [7] in 2004. In this article, we propose a simpler variant of Shelah's proof.

The proof presented here is self-contained, aside from one reference to Shelah's PCF theory. As such, we briefly recall some of the basic definitions. Let X be a set and λ a cardinal. The following definitions are proper extensions of the usual definitions of club and stationary sets to structures of the form $[X]^\lambda = \{x \in X : Card(x) = \lambda\}$. For $X \subseteq ORD$, $otp(X)$ denotes the order type of X.

Definition 1. $C \subseteq [X]^\lambda$ is said to be *closed* iff, for any continuous increasing sequence $\langle c_\xi : \xi < \alpha \rangle$ of elements of C of length $\alpha \leq \lambda$, $\bigcup_{\xi<\alpha} c_\xi \in C$.

C is said to be *unbounded* iff, for any $x \in [X]^\lambda$, there exists $c \supseteq x$ in C.

C is said to be *club* iff it is both closed and unbounded.

Definition 2. $S \subseteq [X]^\lambda$ is said to be *stationary* iff for any club set $C \subseteq [X]^\lambda$, $C \cap S \neq \emptyset$.

Definition 3. *Stationary set reflection* states that, for any set X and stationary subset S of $[X]^\omega$, we can *reflect* S in some $A \in [X]^{\aleph_1}$; that is, $S \cap [A]^\omega$ is stationary in $[A]^\omega$.

Definition 4. The singular cardinal hypothesis SCH is the following assertion: for any singular cardinal κ, if $2^{\mathrm{cof}(\kappa)} < \kappa$, then $\kappa^{\mathrm{cof}(\kappa)} = \kappa^+$.

The layout of this article is as such. First, we shall need Lemma 2 to set off the induction in the main proof (Lemma 1 is used to prove Lemma 2). Then we state and prove the main result in Theorem 1. The fact that stationary set reflection implies the SCH comes in Corollary 1 as a consequence of the main theorem. Finally, in Corollary 2, we show that the previous results still hold without any constraint on the size of the reflecting sets.

Lemma 1. *If stationary subsets of $[\omega_2]^\omega$ reflect, then they reflect in some $\alpha \in \omega_2$.*

Proof. Let S be a stationary subset of $[\omega_2]^\omega$, and suppose that S does not reflect in any $\alpha \in \omega_2$. Then for each $\alpha \in \omega_2$ there is a club set $C(f_\alpha)$ in $[\alpha]^\omega$ containing the closure points in $[\alpha]^\omega$ of some function $f_\alpha : [\alpha]^{<\omega} \to \alpha$, and such that $S \cap [\alpha]^\omega \cap C(f_\alpha) = \emptyset$. Letting $f_\alpha(e) = \min(e)$ for $e \in [\omega_2]^\omega - [\alpha]^\omega$, we can look at f_α as a function from $[\omega_2]^{<\omega}$ into ω_2.

By making simple definitions by cases, we can build two functions f and g from $[\omega_2]^{<\omega}$ into ω_2 such that for all $X \subseteq \omega_2$:

1. if X is closed by f, then for all $\alpha \in X$, X is closed by f_α;
2. if X is closed by g and $Card(X) = \aleph_1$, then either $X \in \omega_2$ or $otp(X) = \omega_1$.

The construction of g, for instance, may go as follows.

1. For $e \in [\omega_2]^n$ with $n > 2$, $g(e) = n - 3$. Thus, if $X \subseteq \omega_2$ is closed by g, then $\omega \subseteq X$.
2. For $n \in \omega$ and $\xi \in \omega_1 - \omega$, $g(\{n, \xi\}) = h_\xi(n)$, where $h_\xi : \omega \to \xi$ is a fixed bijection. Thus, if $X \subseteq \omega_2$ is closed by g, then $X \cap \omega_1 \in \omega_1 + 1$.
3. For $\xi \in \omega_1$ and $\alpha \in \omega_2 - \omega_1$, $g(\{\xi, \alpha\}) = i_\alpha(\xi)$, where $i_\alpha : \omega_1 \to \alpha$ is a fixed bijection. Thus, if $X \in [\omega_2]^{\aleph_1}$ is closed by g and $\omega_1 \subseteq X$, then $X \in \omega_2$.
4. For $\alpha < \beta$ in $\omega_2 - \omega_1$, $g(\{\alpha, \beta\}) = i_\beta^{-1}(\alpha)$. Thus, if $X \in [\omega_2]^{\aleph_1}$ is closed by g and $otp(X) > \omega_1$, then $X \cap \omega_1$ is unbounded, hence $\omega_1 \subseteq X$ by point 2, hence $X \in \omega_2$ by point 3.
5. In all other cases $g(e)$ equals 0.

Let $C(f)$ and $C(g)$ be the respective club sets of closure points of f and g in $[\omega_2]^\omega$.

Finally, let $A \in [\omega_2]^{\aleph_1}$ such that $S \cap C(f) \cap C(g)$ reflects in A. A is closed by g, but $A \notin \omega_2$ by hypothesis, so $otp(A) = \omega_1$. Let then h be the unique isomorphism from ω_1 into A; $\{h(\xi) : \xi < \omega_1\}$ is club in A, so there exists $\alpha \in A$ such that $A \cap \alpha \in S$. Since A is closed by f and $\alpha \in A$, by choice of f we know that A is closed by f_α, hence, due to the definition of f_α, so is $A \cap \alpha$. On the other hand, since $S \cap [\alpha]^\omega \cap C(f_\alpha) = \emptyset$ and $A \cap \alpha \in S \cap [\alpha]^\omega$, $A \cap \alpha$ cannot closed by f_α; so there is a contradiction. □

Lemma 2. *If stationary subsets of $[\omega_2]^\omega$ reflect, then $\aleph_2^{\aleph_0} = \aleph_2$.*

Proof. For each $\alpha \in \omega_2$, let us pick $\langle X^\alpha_\xi : \xi < \omega_1\rangle$ a continuous increasing sequence of countable subsets of α cofinal in α. Let $C = \bigcup_{\alpha<\omega_2}\{X^\alpha_\xi : \xi < \omega_1\}$. Notice that $[\omega_2]^\omega - C$ cannot reflect in any $\alpha \in \omega_2$ by choice of C, so by Lemma 1 it does not reflect at all; hence it is not stationary, hence C contains a club set. Since club sets in $[\omega_2]^\omega$ are of size $\aleph_2^{\aleph_0}$ (see [1], Theorem 3.2) and $Card(C) = \aleph_2$, we get $\aleph_2^{\aleph_0} = \aleph_2$. □

Theorem 1 (Shelah [7]). *If stationary set reflection holds, then for any regular cardinal $\lambda \geq \aleph_2$, $\lambda^{\aleph_0} = \lambda$.*

Proof. Assume that the theorem does not hold, and let λ be the least counterexample. Basic cardinal arithmetic (along with Lemma 2) shows that λ is the successor of some κ of cofinality $\aleph_0$, and $\kappa^{\aleph_0} > \lambda$. Furthermore, Lemma 2 implies that $2^{\aleph_0} < \kappa$. Our goal is to show that stationary set reflection does not hold in $[\lambda]^\omega$.

To that end we need to borrow the following notion from PCF theory [6]. We borrow the terminology from [2].

Definition 5. Given a sequence $\langle \mu_\alpha : \alpha < \beta\rangle$ of regular ordinals and an ideal I on β, a *scale* on $\langle \mu_\alpha : \alpha < \beta\rangle$ is a I-strictly increasing and cofinal sequence $\langle f_\xi : \xi < \gamma\rangle$ in $\prod_{\alpha<\beta} \mu_\alpha$.

The scale $\langle f_\xi : \xi < \gamma\rangle$ is said to be *better* iff, for every cardinal $\alpha < \gamma$ with $\mathrm{cof}(\alpha) > \beta$, there exists a club set $C \subseteq \alpha$, and, letting $\langle c_i : i < \delta\rangle$ be an enumeration of C in increasing order, we can define for each $i < \delta$ a set $Z_i \in I$ such that for all $i < j$ in δ, $f_i \restriction (\beta - (Z_i \cup Z_j)) < f_j \restriction (\beta - (Z_i \cup Z_j))$.

PCF theory shows that we can choose an increasing sequence $\langle \kappa_n : n < \omega\rangle$ of regular cardinals in κ with limit κ so as to have a *better scale* $\langle f_\xi : \xi < \lambda\rangle$ on $\langle \kappa_n : n < \omega\rangle$, with respect to the ideal FIN of finite subsets of ω (see [6], II, Claim 1.5A). In fact, in the scope of this proof, we shall only need the *better* scale property to hold for α of cofinality $\aleph_1$.

For $X \subseteq ORD$, let $\delta(X) = \sup(X \cap \lambda)$, and $\chi(X)(n) = \sup(X \cap \kappa_n)$. Most of the proof will hinge on the comparison, for $X \in [\lambda]^\omega$, between $\chi(X)$ and $f_{\delta(X)}$. Let us then define:

$$E_X = \{n < \omega : \chi(X)(n) \leq f_{\delta(X)}(n)\}$$

Let $\{A_\xi : \xi < \omega_1\}$ be a set of almost-disjoint subsets of ω; that is, for all $\xi \neq \zeta$ in ω_1, $A_\xi \cap A_\zeta$ is finite; and let ϕ be a partial function:

$$\begin{aligned}\phi : \mathcal{P}(\omega) &\to \omega_1\\ E &\mapsto min\{\xi < \omega_1 : Card(A_\xi \cap E) = \aleph_0\}\end{aligned}$$

Finally, let us consider the set:

$$\begin{aligned}\mathcal{S} = \{X \in [\lambda]^\omega : \ &\phi(E_X) \text{ is defined and } \phi(E_X) \geq otp(X),\\ &X \text{ is closed by } x \mapsto f_x(n), \text{ for all } n\}\end{aligned}$$

We are going to show that $\mathcal{S}$ is stationary, yet does not reflect in any $A \in [\lambda]^{\aleph_1}$.

Claim 1. *$\mathcal{S}$ does not reflect in any $A \in [\lambda]^{\aleph_1}$.*

Proof. Let us assume to the contrary that $\mathcal{S}$ reflects in some $A \in [\lambda]^{\aleph_1}$. Let $\langle X_i : i < \omega_1 \rangle$ be a continuous cofinal sequence of increasing countable subsets of A, and let $R = \{i < \omega_1 : X_i \in \mathcal{S}\}$. Since $\{X_i : i < \omega_1\}$ is club in $[A]^\omega$, saying that $\mathcal{S}$ reflects in A is the same as saying that $\{X_i : i \in R\}$ is stationary in $[A]^\omega$, or that R is stationary in ω_1. First, we show that $\mathrm{cof}(\sup(A)) = \aleph_1$.

Let us assume towards contradiction that $\mathrm{cof}(\sup(A)) < \aleph_1$. Then there exists $\alpha \in \omega_1$ such that $\sup(X_\alpha) = \sup(A)$. Now for any $\beta \geq \alpha$ with $\beta \in R$, $f_{\delta(X_\beta)} = f_{\delta(X_\alpha)}$, while $\chi(X_\beta) \geq \chi(X_\alpha)$; hence $E_{X_\alpha} \subseteq E_{X_\beta}$, and in particular $\phi(E_{X_\beta}) \leq \phi(E_{X_\alpha})$. However, since $X_\beta \in \mathcal{S}$, we also have $\phi(E_{X_\beta}) \geq otp(X_\beta)$, and $otp(X_\beta)$ grows towards ω_1, so there is a contradiction.

Since $\mathrm{cof}(\sup(A)) = \aleph_1$, we are free to assume that $\delta(X_i) = \sup(X_i)$ is stricly increasing, trimming $\langle X_i : i < \omega_1 \rangle$ if necessary. Let $\delta_i = \delta(X_i)$, and let $\beta_i = min(A - \delta_i)$. Trimming $\langle X_i : i < \omega_1 \rangle$ two more times, we can ensure that:

$$\forall i < j \in R,\ (\beta_i < \delta_j) \wedge (\beta_i \in X_j) \tag{1}$$

Now let us apply the *better* scale property of $\langle f_\xi : \xi < \gamma \rangle$ to $\delta(A)$: there exists a club set $C \subseteq \delta(A)$, with $\langle c_i : i < \omega_1 \rangle$ an increasing enumeration of C, along with a sequence $\langle n_i : i < \omega_1 \rangle$ of elements of ω such that for $i < j \in R$ and $n \geq n_i, n_j$, we have $f_{\delta_i}(n) < f_{\delta_j}(n)$. As $i \mapsto n_i$ infers a division of $C \cap R$ into $\aleph_0$ subsets, one of them is stationary: let us rename it R. Thus, there exists $k \in \omega$ such that for all $i < j$ in R, $f_{\delta_i} \restriction [k, \omega) < f_{\delta_j} \restriction [k, \omega)$.

Because of (1), we know that for i in R and $j = min(R - (i+1))$, $f_{\delta_i} \leq_{FIN} f_{\beta_i} <_{FIN} f_{\delta_j}$, so there exists $m_i \in \omega$ such that for all $n \geq m_i$, $f_{\delta_i}(n) \leq f_{\beta_i}(n) < f_{\delta_j}(n)$. Using the same reasoning as before, we can thin R so as to have $m_i = m$ a constant, and increase k so that $k \geq m$. As a result:

$$\forall i < j \in R,\ f_{\delta_i} \restriction [k, \omega) \leq f_{\beta_i} \restriction [k, \omega) < f_{\delta_j} \restriction [k, \omega) \tag{2}$$

Now let $f \in \prod_{n<\omega} \kappa_n$ with $f(n) = \bigcup_{i \in R} f_{\beta_i}(n)$ if $n \geq k$ and 0 otherwise. Because of (1) and the closure properties of $\mathcal{S}$, for $i < j \in R$ and $n \in \omega$ we have $f_{\beta_i}(n) \in X_j$, so $f(n) \leq \bigcup_{i \in R}(X_i \cap \kappa_n) = \chi_A(n)$. Let $B = \{n \in [k, \omega) : f(n) = \chi_A(n)\}$ and $\overline{B} = \{n \in [k, \omega) : f(n) < \chi_A(n)\} = [k, \omega) - B$. We are going to prove that, for all i in some stationary subset of R, $f_{\delta_i} \restriction B \geq \chi(X_i) \restriction B$ and $f_{\delta_i} \restriction \overline{B} < \chi(X_i) \restriction \overline{B}$.

Let $n \in B$. Since $f(n) = \chi(A)(n)$ and $\mathrm{cof}(f(n)) = \aleph_1$, we can define a club set C_n in $\chi(A)(n)$ such that for all $i < j$ in C_n, $\chi(X_i)(n) < f_{\delta_j}(n)$, and also $f_{\delta_i}(n) \in X_j$. As a result, for l a limit point of C_n, we get $f_{\delta_l}(n) \geq \bigcup_{i \in C_n \cap l} f_{\delta_i}(n) = \bigcup_{i \in C_n \cap l} \chi(X_i)(n) = \chi(X_l)(n)$. Let D_n be the club set of limit points of C_n: as $R \cap (\bigcap_{n<\omega} D_n)$ is stationary, we rename it R; and for all $i \in R$ we have $f_{\delta_i} \restriction B \geq \chi(X_i) \restriction B$.

Let $n \in \overline{B}$. Since $f(n) < \chi(A)(n)$, there exists $i(n) \in \omega_1$ such that $f(n) < \chi(X_{i(n)}(n))$. As $\sup_{n \in B}(i(n)) < \omega_1$, $R - \sup_{n \in B}(i(n))$ is stationary and we can rename it (again) R. Thus, for all $i \in R$ we have $f_{\delta_i} \restriction \overline{B} < \chi(X_i) \restriction \overline{B}$.

We have shown that for $i \in R$, $\{n \in [k, \omega) : \chi(X_i)(n) \leq f_{\delta_i}(n)\} = B$, so $E_{X_i} =_{FIN} B$. In particular, $\phi(E_{X_i})$ remains constant on R. That is contradictory, since $\phi(E_{X_i}) \geq otp(X_i)$ and $otp(X_i)$ tends to ω_1. □

Claim 2. *$\mathcal{S}$ is stationary.*

Proof. Let C be a club set in $[\lambda]^\omega$. By Kueker's theorem, C contains the set of closure points of some function $f_C : [\lambda]^{<\omega} \to \lambda$. We are going to look for $X \in \mathcal{S}$ such that $f_C''[X] \subseteq X$.

In order to build a set $X \in \mathcal{S}$, the main issue is to control both E_X and $otp(X)$, so that $\phi(E_X) \geq otp(X)$. For that purpose, we consider a closed two-player game G_ε for each choice of $\varepsilon \in \omega_1$. Player 1 sets up constraints that will, later on, allow us to control E_X and ensure that $\phi(E_X) \geq \varepsilon$; meanwhile, player 2 tries to meet these constraints, build the set X, bound $\chi(X)$, as well as prove that $otp(X) \leq \varepsilon$.

In the first part of the proof, we show that player 2 has a winning strategy for some $\varepsilon \in \omega_1$. In the second part, we show how player 1 should play against that strategy in order to obtain X as required. We begin by describing G_ε.

Let θ be a sufficiently large regular cardinal, say $\theta = (2^\lambda)^+$, and let $\mathcal{H}(\theta) = \{X : tc(X) < \theta\}$, where $tc(X)$ is the transitive closure of the set X. Let $\lhd$ be a well-order on $\mathcal{H}(\theta)$. For $X \subseteq ORD$, we define $sk(X)$ as the Skolem hull of X in $\langle \mathcal{H}(\theta), \in, \lhd \rangle$, $sk_\lambda(X) = sk(X) \cap \lambda$, and $cl(X) = sk_\lambda(X \cup \{\langle f_\xi : \xi < \lambda \rangle, f_C\})$. Moreover, let $\langle t_n : n < \omega \rangle$ be an enumeration of each Skolem term in $\langle \mathcal{H}(\theta), \in, \lhd \rangle$ applied to every possible combination of functions $x \mapsto f_x(n)$ for $n < \omega$, f_C, with variables v_i, $i \in \mathbb{N}$. The idea is that, if we interpret the v_i's as the elements of some countable set X, the t_i's enumerate all possible elements of $cl(X)$.

The game G_ε proceeds as follows.

1. (a) At step $2n$: player 1 picks an ordinal $\xi_{2n} \in \kappa_n$. Player 2 then picks α_{2n} and γ_{2n} in κ_n such that $\xi_{2n} \leq \alpha_{2n} \leq \gamma_{2n}$.
   (b) At step $2n+1$: player 1 picks an ordinal $\xi_{2n+1} \in \lambda$. Player 2 then picks α_{2n+1} in λ such that $\xi_{2n+1} \leq \alpha_{2n+1}$.
2. Player 2 chooses an ordinal ζ_n in ε.

Once the game is over, let $X = cl(\{\alpha_n : n \in \omega\})$. Interpreting the variables v_i in t_n as α_i, one can compute the value of each t_n. Let τ_n be the value of t_n whenever it is an element of λ. The τ_n's thus constitute an enumeration of X.

Player 2 is said to win the game iff:

1. For all $n \in \omega$, $X \cap \kappa_n \subseteq \gamma_{2n}$.
2. The mapping $g : X \to \varepsilon$ with $g(\tau_n) = \zeta_n$ is well-defined and strictly increasing. As such g witnesses $otp(X) \leq \varepsilon$.

Fact 1. *There exists $\varepsilon \in \omega_1$ such that player 2 has a winning strategy for G_ε.*

Proof. Let $\varepsilon \in \omega_1$. The first point is that the game G_ε is closed, because if player 2 loses, that loss is appearent in a finite number of moves. Indeed, if at the end of the game, for some $n \in \omega$, $X \cap \kappa_n \not\subseteq \gamma_n$, then some element τ_n of X witnesses it; but the value τ_n can be computed as soon as all α_i (recursively) appearing in t_n have been determined, and there is only of finite number of them. The same goes for the second winning condition.

Since G_ε is closed, the Gale-Stewart theorem [4] guarantees that one of the two players has a winning strategy. Let us assume towards contradiction that player 1 has a winning strategy σ_ε for all $\varepsilon \in \omega_1$. The crux of the matter here is that player 1's best interest is always to play ξ_n as high possible. In particular, if we modify σ_ε to increase player 1's answer ξ_n to some sequence of moves by player 2, we still get a winning strategy.

Assuming that player 1 follows the strategy σ_ε, for any given sequence s of moves by player 2 up to step n of the game, let $\sigma_\varepsilon(s)$ be the answer ξ_n dictated to player 1 by his strategy σ_ε (letting $\sigma_\varepsilon(s) = 0$ if s is not a possible sequence of moves for player 2 when player 1 applies σ_ε). We can define a new strategy σ for player 1 by $\sigma(s) = \sup_{\varepsilon\in\omega_1} \sigma_\varepsilon(s)$. Due to the remark above, σ is a winning strategy for player 1 for all games G_ε.

From here on we assume that player 2 always plays $\alpha_n = \xi_n$, and player 1 answers with σ. Thus, up to step $2n$, this subgame is determined by player 2's choices of $\gamma_{2i} \in \kappa_i$, for $i \leq n$, and $\zeta_n \in \varepsilon < \omega_1$. As a result, there are only κ_n possible sequence of moves up to step $2n + 1$ (we are free to assume $\omega_1 < \kappa_0$); so the set of all possible plays ξ_{2n+2} by player 1 is bounded in κ_{n+1}. Thus, improving σ if necessary, we can assume that player 1's moves are independent of all previous moves. Let $\langle \xi_n : n < \omega \rangle$ be the sequence of player 1's moves.

Let us now turn back to the regular games G_ϵ, and play as player 2 against strategy σ using the following strategy of our own. We are going to play $\alpha_n = \xi_n$ every turn, so we know from the start the set $X = cl(\{\alpha_n : n < \omega\}) = cl(\{\xi_n : n < \omega\})$. Let $\varepsilon = otp(X)$. We play in the game G_ε.

Since we know X and have a mapping $g : X \to \varepsilon$, we can compute in advance the value of each $g(\tau_n)$. Each turn, we play $\alpha_n = \xi_n$, $\gamma_n = \sup(X \cap \kappa_n)$ when n is even, and $\zeta_n = g(\tau_n)$. This is clearly a winning strategy for player 2, so σ is not a winning strategy, which is a contradiction. □

Let then $\varepsilon \in \omega_1$ be such that player 2 has a winning strategy τ for G_ε. If we play against τ, we know that we will get a set X such that $otp(X) \leq \varepsilon$, $X \in C$, and X is closed by all relevant functions; thus the last remaining point is to ensure $\phi(E_X) \geq \varepsilon$. Letting $A = A_\varepsilon$, we are going to achieve this by having $E_X = A$.

First, we need $M \prec \mathcal{H}(\theta)$ such that $\chi(M) \leq_{FIN} f_{\delta_M}$, and M contains all relevant objects: $\langle f_\xi : \xi < \lambda \rangle$, $\langle \kappa_i : i < \omega \rangle$, τ, f_C. To obtain such an M we build an increasing continuous sequence $\langle M_\zeta : \zeta < \omega_1 \rangle$ of elementary submodels of $\mathcal{H}(\theta)$ with $\langle M_\alpha : \alpha < \zeta \rangle \in M_\zeta$ for all ζ, and put the aforementioned relevant objects in M_0. Since $M_\zeta \in M_{\zeta+1}$ and $M_{\zeta+1} \prec \mathcal{H}(\theta)$, there exists a continuous increasing sequence $\langle \alpha(\zeta) : \zeta < \omega_1 \rangle$ with $\alpha_\zeta \in M_\zeta$ such that for each $\zeta < \omega_1$, $\chi(M_\zeta) <_{FIN} f_{\alpha(\zeta+1)}$. Using the *better* scale property of $\langle f_\xi : \xi < \lambda \rangle$ applied to the point $\sup_{\zeta<\omega_1}(\alpha(\zeta))$, we can extract a subsequence $\langle \alpha_{\eta(i)} : i < \omega \rangle$ of $\langle \alpha_\zeta : \zeta < \omega_1 \rangle$ such that, letting $\eta = \sup_{i<\omega}(\eta(i))$, there exists $k < \omega$ such that $f_{\alpha_{\eta(i)}} \restriction [k, \omega) < f_{\alpha_{\eta(i+1)}} \restriction [k, \omega) < f_{\alpha_\eta} \restriction [k, \omega)$. Then $M = M_\eta$ satisfies the condition, since $\chi(M) = \bigcup_{i<\omega} \chi(M_{\eta(i)}) =_{FIN} \bigcup_{i<\omega} f_{\alpha_{\eta(i)}} \leq_{FIN} f_{\alpha_\eta} \leq_{FIN} f_{\delta_M}$. Let $\langle m_i : i < \omega \rangle$ be an enumeration of $M \cap \lambda$.

Fact 2. *$\delta(sk_\lambda(M \cup \kappa)) = \delta(M)$. Similarly,* $\sup(sk_\lambda(M \cup \kappa_n) \cap \kappa_{n+1}) = \sup(M \cap \kappa_{n+1})$ *for all $n < \omega$.*

Proof. Let $\alpha \in sk_\lambda(M \cup \kappa)$: there exists a Skolem term z_1 with parameters $x_1, \ldots, x_m$ in M and $y_1, \ldots, y_n$ in κ such that $\alpha = z_1(\beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n)$. Now if we consider the Skolem function $z_2(\beta_1, \ldots, \beta_m)$ corresponding to the formula: $\sup(x < \lambda : \exists x_1, \ldots, x_n \in \kappa, x = z_1(\beta_1, \ldots, \beta_m, x_1, \ldots, x_n))$, it is clear that $z_2(\beta_1, \ldots, \beta_m) \geq z_1(\beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n)$ in $sk_\lambda(M \cup \kappa)$; but, since $M \prec sk_\lambda(M \cup \kappa)$ and $\kappa \in M$, $z_2^M = z_2^{sk_\lambda(M \cup \kappa)}$, so $\alpha \leq z_2(\beta_1, \ldots, \beta_m) \in M \cap \lambda$ (necessarily $z_2(\beta_1, \ldots, \beta_m) < \lambda$ for cofinality reasons, as the parameters are all in κ).

Hence $\delta(sk_\lambda(M \cup \kappa)) \leq \delta(M)$. The converse also holds since $M \subset sk_\lambda(M \cup \kappa)$, so we have an equality. We can apply the same reasoning to obtain the second equality. □

Now we play the game G_ε as player 1 against player 2's winning strategy τ as follows.

1. At step $2n$:
   (a) if $n \in A$, we play $\xi_{2n} = 0$;
   (b) if $n \notin A$, we play $\xi_{2n} = f_{\delta_M}(n)$.
2. At step $2n + 1$, we play $\xi_{2n+1} = m_n$.

This game yields a set $X = cl(\{\alpha_n : n < \omega\})$.

Fact 3. *$E_X = A$.*

Proof. Because of player 1's move $\xi_{2n+1} = m_n$ on odd n, $M \cap \lambda \subseteq X$, so $\delta_M \leq \delta_X$. Conversely, $X \subset sk_\lambda(M \cup \kappa)$, so Fact 2 yields $\delta_M = \delta_X$.

Let us look at step $2n$ of the game we have described. First, if $n \in A$, we played $\xi_{2n} = f_{\delta_M}(n)$, so player 2 was forced to respond with $\alpha_{2n} \geq f_{\delta_M}(n)$. Hence $\chi(X)(n) \geq f_{\delta_X}(n)$, so $n \in E_X$, so $A \subseteq E_X$.

Second, if $n \notin A$, we played $\xi_{2n} = 0$. Since all moves up to that point in the game belong to $M \cup \kappa_{n-1}$, and $\tau \in M$, we have $\alpha_{2n} \in sk_\lambda(M \cup \kappa_{n-1}) \cap \kappa_n$. In particular $\alpha_{2n} \leq \sup(M \cap \kappa_n)$ thanks to Fact 2; hence $\alpha_{2n} \leq f_{\delta_X}(n)$, so $n \notin E_X$, so $A \supseteq E_X$. □

Corollary 1. *Stationary set reflection implies the SCH.*

Proof. Assume to the contrary that SCH does not hold, and let us pick κ the first cardinal that contradicts SCH. Silver's theorem [5] implies that $\text{cof}(\kappa) = \aleph_0$; so we have $\kappa^{\aleph_0} > \kappa^+$. Theorem 1, on the other hand, states that $(\kappa^+)^{\aleph_0} = \kappa^+$, so there is a contradiction. □

Corollary 2. *If for all regular cardinal $\lambda \geq \aleph_2$ and stationary $S \subseteq [\lambda]^\omega$, S reflects in some $A \in [\lambda]^{<\lambda}$, then for any regular $\lambda \geq \aleph_2$, $\lambda^{\aleph_0} = \lambda$. In particular, SCH still holds.*

Proof. That is, Theorem 1 still holds if we allow reflection in any $A \in [\lambda]^{<\lambda}$. This stems from the fact that, in the proof of Theorem 1, the set $\mathcal{S}$ does not actually reflect in any $A \in [\lambda]^{<\lambda}$.

Assume to the contrary that $\mathcal{S}$ reflects in some $A \subseteq \lambda$, $Card(A) < \lambda$. First, suppose that $Card(A) < \kappa$. Then we can collapse $Card(A)$ on ω_1 with the corresponding classic forcing. Since this forcing preserves countable sequences and stationary sets, $\mathcal{S}$ is unaffected and still reflects in A, while $Card(A)$ becomes ω_1, which contradicts Theorem 1. As a side effect, however, since ordinals in λ that were formerly of cofinality greater than $\aleph_1$ may end up, after the forcing, with cofinality $\aleph_1$, and we still need to apply the *better* scale property to them, we now have to use the *better* scale property to a greater extent; namely, we need it to hold on all ordinals of cofinality greater than or equal to $\aleph_1$, and not just on those of cofinality $\aleph_1$, as was the case in Theorem 1.

Now suppose that $Card(A) = \kappa$. Let $\delta = \sup(A)$ and $\gamma = \text{cof}(Card(A))$; since κ is singular, we have $\gamma < \kappa$. Recall that in Claim 1, we have shown that $\text{cof}(\sup(A)) > \aleph_0$; the reasoning we used does not really depend on $Card(A)$ and still holds. Thus we can apply the *better* scale property of $\langle f_\xi : \xi < \lambda \rangle$ to δ: this entails that the set $\{n \in \omega : \text{cof}(f_\delta(n)) = \gamma\}$ is cofinite. Let then $m \in \omega$ such that $n \geq m$ implies $\text{cof}(f_\delta(n)) = \gamma$.

For each $n \geq m$, we are going to build a set $B_n \subseteq A \cap \kappa_n$ of size $\leq \gamma$; we consider three cases:

1. if $\chi(A)(n) = f_\delta(n)$, then we choose $B_n \subseteq A \cap \kappa_n$ such that $\sup(B_n) = \sup(A \cap \kappa_n)$;
2. if $\chi(A)(n) < f_\delta(n)$, then $B_n = \emptyset$;
3. if $\chi(A)(n) > f_\delta(n)$, then $B_n = \{\alpha_n\}$ with $\alpha_n = \min\{\chi(A)(n) - f_\delta(n)\}$.

Let $B_\lambda \subseteq A$ of size γ such that $\sup(B_\lambda) = \sup(A)$.

Furthermore, let $Y = \{n \geq m : \chi(A)(n) < f_\delta(n)\}$; since $\text{cof}(\delta) > \aleph_0$, using the *better* scale property on δ, we know that there exists $\alpha < \delta$ in A such that $f_\alpha(n) > \chi(A)(n)$ for all $n \in Y$ (removing a finite number of elements from Y if necessary). Finally, let $B = B_0 \cup \bigcup_{i<\omega}(B_{i+1} - \kappa_i) \cup (B_\lambda - \kappa) \cup \{\alpha\}$. Increasing B if necessary, we are free to assume that B is closed by $x \mapsto f_x(n)$, for all n (because A itself satisfies that condition).

The construction of B ensures that the set

$$
\begin{aligned}
D = \{X \in [A]^\omega : \delta(X) = \delta(X \cap B) \\
&\wedge \forall n \geq m, \chi(A)(n) = f_\delta(n) \implies \chi(X)(n) = \chi(X \cap B)(n) \\
&\wedge \{\alpha_n : n \geq m, \chi(A)(n) < f_\delta(n)\} \cup \{\alpha\} \subseteq X\}
\end{aligned}
$$

is club in $[A]^\omega$. The construction of D, in turn, ensures that for each $X \in \mathcal{S} \cap D$, $E_X =_{FIN} E_{X\cap B}$ (recall that $E_X = \{n < \omega : \chi(X)(n) \leq f_{\delta(X)}(n)\}$), so that we get $X \cap B \in \mathcal{S}$ by definition of $\mathcal{S}$.

As a result $\mathcal{S}$ reflects in B. Indeed, let C be a club set in $[B]^\omega$ and $C_A = \{X \in [A]^\omega : X \cap B \in C\}$, then there exists $X \in \mathcal{S} \cap D \cap C_A$, and $X \cap B \in \mathcal{S} \cap C$. □

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