Techniques in Cryptography and Cryptanalysis

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Meta information

“Techniques in Cryptography and Cryptanalysis”: will cover (a choice of) important areas of cryptography.

- Lattices 📍 still here

- Zero-Knowledge Proofs ...*here soon*

~~**- Oblivious algorithms**~~

Teachers:

Phong Nguyễn

8 x 1.5h, 1st period

Brice Minaud

8 x 1.5h, 2nd period

More meta information

Exams: 1.5h, same time slot as lectures. A few exercises.

See also website for additional material.

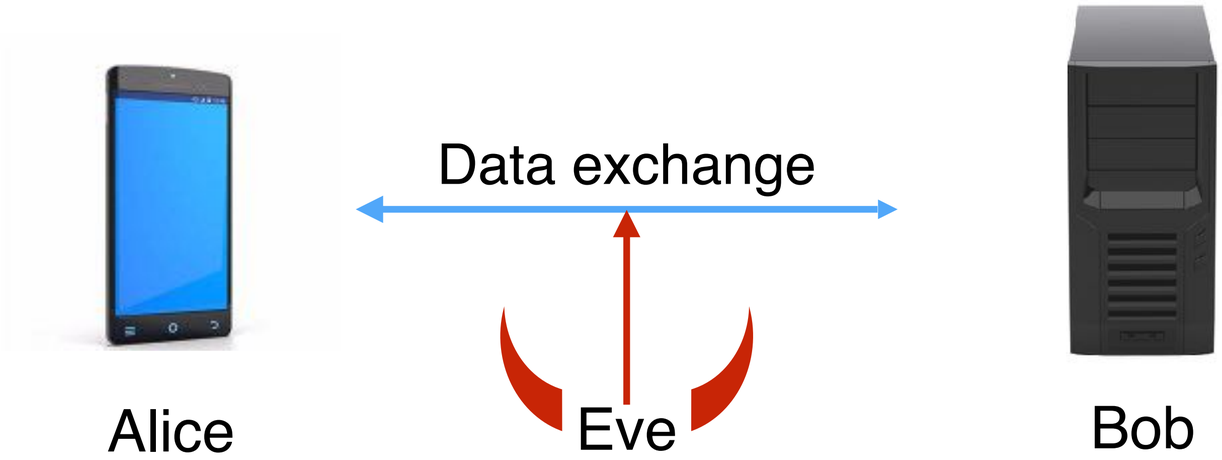
Interested in crypto? Have questions? Looking for an internship?

Don't hesitate to write to Phong and/or me.

Contact: `brice.minaud@ens.fr, phong.nguyen@ens.fr`

What is security?

Historically, most basic goal = protecting the confidentiality of data exchanges.

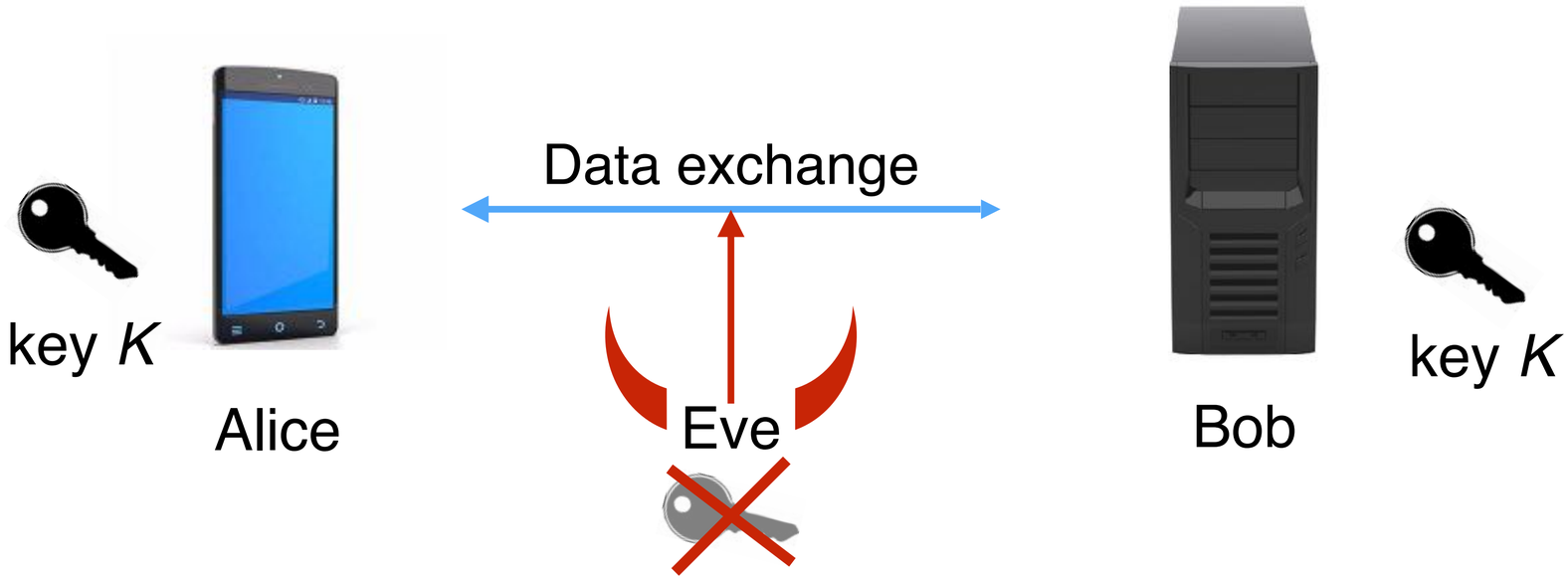


Kerckhoff's (first three) principles:

1. The system must be practically, if not mathematically, indecipherable.
2. It should not require secrecy, and it should not be a problem should it fall into enemy hands.
3. It must be possible to […] change or modify [the key] at will.

One-Time Pad

Modern version: the algorithms are public. They are parametrized by a (secret) key.



One-Time Pad.

Message space: $M \leftarrow \{0,1\}^n$ Key space: $K \leftarrow \{0,1\}^n$

Encryption(M): $C = M \oplus K$

Decryption(C): $M = C \oplus K$

Security of One-Time Pad

One-Time Pad.

Message space: $M \leftarrow \{0,1\}^n$ Key space: $K \leftarrow \{0,1\}^n$

Encryption(M): $C = M \oplus K$

Decryption(C): $M = C \oplus K$

Naive security: impossible for Eve to find M from C.

Not great. Encryption could leak last bit of M and still be secure by that definition.

We want to express that Eve learns *nothing* about M.

Perfect secrecy

Perfect secrecy, historical version, Shannon, 1949.

Prior distribution: distribution of M known a priori to Eve.

Posterior distribution: distribution of M known to Eve after seeing the encryption $\mathrm{Enc}_K(M)$ of M (for uniform K).

Perfect secrecy: posterior distribution = prior distribution.

Perfect secrecy, equivalent modern version, folklore, 20th century.

Let M_0 and M_1 be two arbitrary messages.

Perfect secrecy: $\mathrm{Enc}_K(M_0) = \mathrm{Enc}_K(M_1)$.

The equality is an equality of distributions. The randomness is over the uniform choice of K.

OTP and perfect secrecy

Proposition. The One-Time Pad achieves perfect secrecy.

Proof. $\mathrm{Enc}(M_0) = C$ iff $K = C \oplus M_0$.

So there is exactly one K that yields each possible C. Since K is uniform, so is C. Thus:

$$\mathrm{Enc}(M_0) = \mathrm{Unif}(\{0,1\}^n) = \mathrm{Enc}(M_1).$$

(Note: this would hold in any group.)

Theorem (Shannon '49). If perfect secrecy holds, it must be the case that the two parties share some prior information (a key) with:

$$\text{length(key)} \geq \text{length(message)}$$

where length denotes the bit length.

So OTP is essentially the only perfectly secure scheme.

Measuring Security

Advantage

- Previous solution is infeasible in most cases.

  → we must be content with *imperfect* security.

- The relevant notion to formally express that Eve cannot learn anything is often about the *indistinguishability* of two distributions.

Roadmap of a security definition: the **adversary** is an algorithm attempting to infer secret information.

Often, this will be expressed as the adversary trying to distinguish two distributions.

Advantage.

Let D_0 and D_1 be two probability distributions. The advantage of an adversary A (i.e. an algorithm, here with output in {0,1}) is:

$$\mathrm{Adv}^{D_0,D_1}(A) = \left|2\Pr_{b\leftarrow\$\{0,1\}}(A(D_b) = b) - 1\right|$$

Types of security

let M_0 and M_1 be two arbitrary messages…

Perfect security:

$\mathrm{Enc}_K(M_0) = \mathrm{Enc}_K(M_1)$ (as distributions, for uniform K).

Equivalently: $\mathrm{Adv}^{\mathrm{EncK}(M_0),\ \mathrm{EncK}(M_1)}(A) = 0$, for every A.

Statistical security:

$\mathrm{Adv}^{\mathrm{Enc}_K(M_0),\ \mathrm{Enc}_K(M_1)}(A)$ is negligible, for every A.

Computational security:

$\mathrm{Adv}^{\mathrm{Enc}_K(M_0),\ \mathrm{Enc}_K(M_1)}(A)$ is negligible, for every efficient adversary A.

Quantifying negligibility, efficiency

Security parameter, often denoted λ: used to quantify security.

- "Asymptotic" security: used in more theoretical results. λ remains a variable.
- "Concrete" security: used in more practical results. Typically λ = 80, 128, or 256. (e.g. "128-bit" security.)

	"Asymptotic" security	**"Concrete" security**
Negligible (probability)	$O(\lambda^{-c})$ for all c	usually $\leq 2^{-\lambda/2}$ or $2^{-\lambda}$
Efficient (adversary)	Poly(λ)	significantly less than 2^{λ} operations

Caveats: computation model (TM, RAM, circuits), "basic" operation, memory, etc.

Concreteness of security

Bits of security	Practical significance
32	Your phone can do it, instantly.
66	Bitcoin hashes *per second* worldwide.
80	Bitcoin hashes per year worldwide. *(Some state actors could do it?)*
128	Considered secure. Standard choice. *(Watch out for trade-offs, like time/data or multi-target)*
256	Arguments for impossibility based on physics. *(Relevant for very long-term or quantum security.)*

Bitcoin data from https://www.blockchain.com/en/charts/hash-rate in 2019.

Types of security, again

let M_0 and M_1 be two arbitrary messages…

Perfect security:

$\mathrm{Enc}_K(M_0) = \mathrm{Enc}_K(M_1)$ (as distributions, for uniform K).

Equivalently: $\mathrm{Adv}^{\mathrm{EncK}(M_0),\ \mathrm{EncK}(M_1)}(A) = 0$, for every A.

Statistical security:

$\mathrm{Adv}^{\mathrm{Enc}_K(M_0),\ \mathrm{Enc}_K(M_1)}(A)$ is negligible, for every A.

Computational security:

$\mathrm{Adv}^{\mathrm{Enc}_K(M_0),\ \mathrm{Enc}_K(M_1)}(A)$ is negligible, for every efficient adversary A.

Statistical distance

Good tool to bound or analyze advantage.

Statistical distance.

Let D_0 and D_1 be two probability distributions over some set X.

$$\mathrm{Dist}(D_0, D_1) = \frac{1}{2} \sum_{x \in X} |D_0(x) - D_1(x)|$$

Proposition 1. This is, in fact, a distance.

Proof. x, y ↦ |y - x| is a distance. So Dist(·,·) is a sum of distances. (Can also write it out.)

Statistical distance, cont'd

Proposition 2. The statistical distance Dist(D_0,D_1) is equal to the advantage of the best adversary trying to distinguish D_0 from D_1.

Statistical distance, cont'd

Proposition 2. The statistical distance Dist(D_0,D_1) is equal to the advantage of the best adversary trying to distinguish D_0 from D_1.

Proof. Let A be the adversary such that, given $x \leftarrow D_b$, A outputs 0 iff $D_0(x) \geq D_1(x)$. A is clearly best possible.

$$
\begin{aligned}
\mathrm{Adv}^{D_0,D_1}(A) &= 2\mathrm{Pr}_{x\leftarrow D_b(x),b\leftarrow_\$\{0,1\}}(A(x)=b)-1 \\
&= 2\sum_{x'}\sum_{b'}\Pr(A(x)=b|x=x',b=b') \\
&\qquad\qquad \cdot \Pr_{x\leftarrow D_b}(x=x'|b=b')\Pr_{b\leftarrow_\$\{0,1\}}(b=b')-1 \\
&= \sum_{x'}\sum_{b'}\mathbb{1}_{A(x')=b'}D_b(x')-1 \\
&= \sum_{x'}\max(D_0(x'),D_1(x'))-1 \\
&= \mathrm{Dist}(D_0,D_1) \qquad \text{using: } \max(a,b)=\frac{1}{2}(a+b+|b-a|).
\end{aligned}
$$

Statistical distance, cont'd

Corollary. Let A be any algorithm. Then:

$$\text{Dist}(A(D_0),A(D_1)) \leq \text{Dist}(D_0,D_1)$$

Statistical distance, cont'd

Proposition 3. For all n, $\mathrm{Dist}(D_0^n, D_1^n) \leq n\mathrm{Dist}(D_0, D_1)$.

Statistical distance, cont'd

Corollary. Let A be any algorithm. Then:

$$\mathrm{Dist}(A(D_0),A(D_1)) \leq \mathrm{Dist}(D_0,D_1)$$

Proof. Let B be the best adversary distinguishing D_0 from D_1, and C be the best adversary distinguishing $A(D_0)$ from $A(D_1)$.

$$\mathrm{Dist}(A(D_0),A(D_1)) = \mathrm{Adv}^{A(D_0),A(D_1)}(C) = \mathrm{Adv}^{D_0,D_1}(C \circ A)$$

$$\leq \mathrm{Adv}^{D_0,D_1}(B) = \mathrm{Dist}(D_0,D_1).$$

Proposition 3. For all n, $\mathrm{Dist}(D_0^n,D_1^n) \leq n\mathrm{Dist}(D_0,D_1)$.

Proof.

$\mathrm{Dist}(A^n,B^n) \leq \mathrm{Dist}(A^n,A^{n-1}B) + \mathrm{Dist}(A^{n-1}B,A^{n-2}B^2) + \ldots + \mathrm{Dist}(AB^{n-1},B^n)$.

Sometimes called the “hybrid” argument, although the same term is also used in more general settings.

Computational version

Advantage of the best adversary = statistical distance.

By extension:

Advantage of a class of adversaries.

Let D_0 and D_1 be two probability distributions, and **A** a set of adversaries. Define:

$$\mathrm{Adv}^{D_0,D_1}(\mathbf{A}) = \sup\{\mathrm{Adv}^{D_0,D_1}(A) : A \in \mathbf{A}\}$$

Define $\mathbf{A}(t)$ the set of adversaries that terminate in time t. Let:

$$\mathrm{Adv}^{D_0,D_1}(t) = \mathrm{Adv}^{D_0,D_1}(\mathbf{A}(t))$$

NB For asymptotic security, what matters usually is to distinguish two *families* of distributions. We want (abuse of notation):

$$\mathrm{Adv}^{D_0,D_1}(\mathrm{Poly}(\lambda)) = \mathrm{Negl}(\lambda)$$

with D_0, D_1 (implicitly) parametrized by λ.

Computational version, cont'd

Types of security, revisited

let M_0 and M_1 be two arbitrary messages…

Perfect security:

$\text{Enc}_K(M_0) = \text{Enc}_K(M_1)$ (as distributions, for uniform K).

Equivalently: $\text{Dist}(\text{Enc}_K(M_1), \text{Enc}_K(M_2)) = 0$.

Equivalently: $\text{Adv}^{\text{EncK}(M_0),\ \text{EncK}(M_1)}(\{\text{all } A\}) = 0$.

Statistical security:

$\text{Dist}(\text{Enc}_K(M_1), \text{Enc}_K(M_2))$ is negligible.

Equivalently: $\text{Adv}^{\text{Enc}_K(M_0),\ \text{Enc}_K(M_1)}(\{\text{all } A\})$ is negligible.

Computational security:

$\text{Adv}^{\text{Enc}_K(M_0),\ \text{Enc}_K(M_1)}(\{\text{efficient } A\})$ is negligible.

Exercise

Consider a Bernoulli (coin flip) distribution B with $B(0) = 1/2 - \varepsilon$ and $B(1) = 1/2 + \varepsilon$. Let U be the uniform distribution on $\{0,1\}$. Observe:

$$\text{Dist}(B,U) = \varepsilon.$$

Assume we are doing a one-time pad with an imperfect randomness source, where the key bits are drawn according to B:

$$K \leftarrow B^n \text{ (instead of } U^n)$$

Say ε is negligible (asymptotic sense).

Is this still secure?

Perfect security? Statistical? Computational?

Solution

Let's encrypt a message $M \in \{0,1\}^n$.

$$\begin{aligned}\mathrm{Dist}(\mathrm{Enc}_K(M),U^n) &= \mathrm{Dist}(K\oplus M,U^n)\\ &\le \textstyle\sum_{i<n} \mathrm{Dist}((K\oplus M)_i ,U)\\ &= n\varepsilon\end{aligned}$$

$(K\oplus M)_i$: i-th bit of $K\oplus M$

For $M_0, M_1 \in \{0,1\}^n$.

$$\begin{aligned}\mathrm{Dist}(\mathrm{Enc}_K(M_0),\mathrm{Enc}_K(M_1)) &\le \mathrm{Dist}(\mathrm{Enc}_K(M_0),U^n) + \mathrm{Dist}(\mathrm{Enc}_K(M_1),U^n)\\ &\le 2n\varepsilon\end{aligned}$$

Note that $n \cdot \mathrm{Negl}(n) = \mathrm{Negl}(n)$ so this is (statistically) secure!

(A more refined analysis shows this grows in $\sqrt{n\epsilon}$. The hybrid argument is a little crude here.)

Shannon's impossibility, revisited

Theorem (Shannon ’49). If perfect secrecy holds, it must be the case that the two parties share some prior information (a key) with:

$$\text{length(key)} \geq \text{length(message)}$$

where length denotes the bit length.

Saying perfect security is impossible.

What about *statistical* security?

Modern definition

Statistical secrecy.

Let M_0 and M_1 be two arbitrary messages.

Statistical secrecy: $\text{Dist}(\text{Enc}_K(M_0), \text{Enc}_K(M_1))$ is negligible.

More formally:

Statistical secrecy.

$\forall\, p \in \text{Poly}(\lambda)$, $\forall\, M_0, M_1$ of size p, $\text{Dist}(\text{Enc}_K(M_0), \text{Enc}_K(M_1)) = O(\lambda^{-c})$,
(where the distributions are induced by $K \leftarrow_{\$} \{0,1\}^{\lambda}$).

Is this possible?

No. Hint: for $p > 3\lambda$, $\exists\, M_0, M_1$ such that the set of possible encryptions are disjoint.

Conclusion

For many goals of cryptography (even simply symmetric encryption), can't have statistical security.

→ **Strategy:**

1) Reduce security of crypto protocols to the security of their basic bricks (known as primitives: encryption, signature etc).

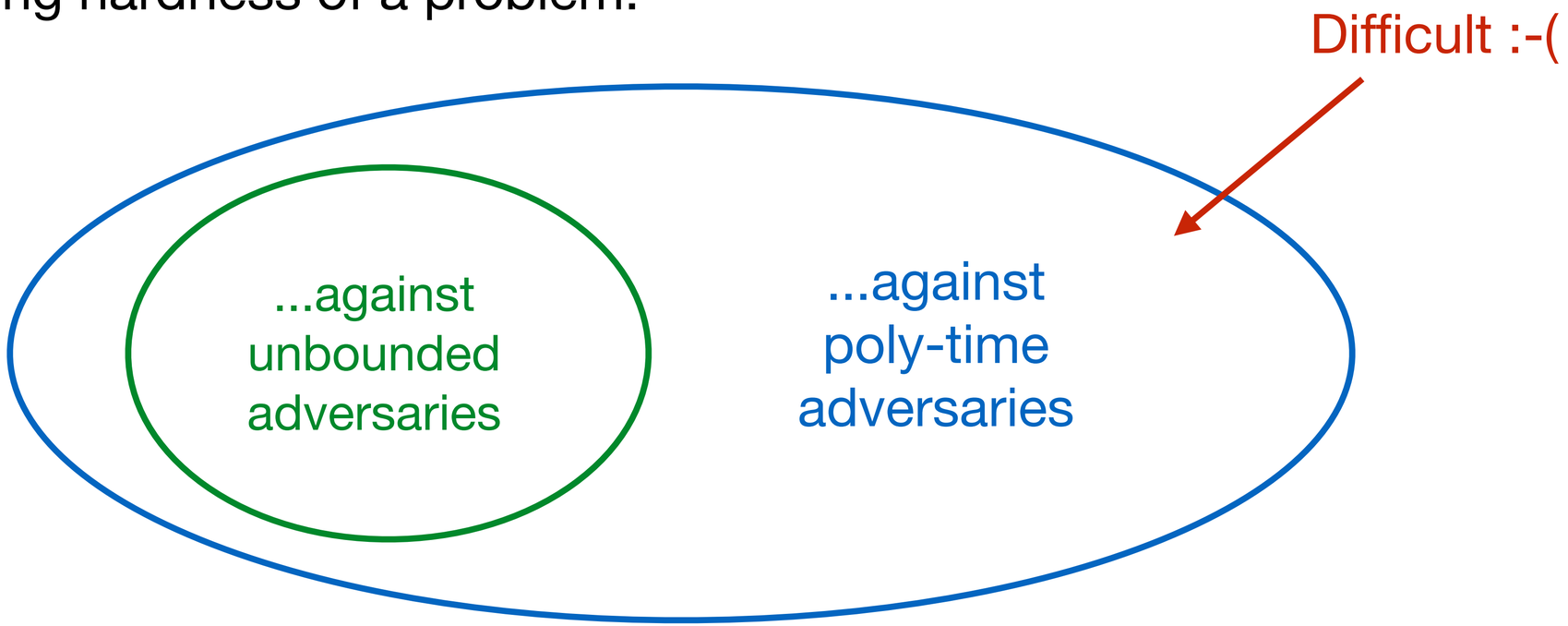
2) (When possible) reduce security of primitives to *known* mathematical hard problem, e.g. discrete logarithm, LWE etc.

How? use arguments based on advantage, statistical distance, etc: see MPRI course 2.30 “Proofs of security protocols”!

Conclusion

Corollary: crypto requires problems that are **computationally** hard, but not information theoretically hard (= against unbounded adversaries).

Proving hardness of a problem:



Remark. Information-theoretical arguments don't even really care what an algorithm *is*.
(Turing machine? RAM ? Quantum? Family of circuits? ...)

Conclusion

Crypto requires problems that are **computationally hard**, but not information theoretically hard. *Which we don't know how to do.*

Corollary 1. Cryptography requires hardness assumptions.

Corollary 2. Entire modern world relies on statements we don't know how to prove (or if they are provable).

...and it would be catastrophic if they were wrong. (Payment ecosystem, secure Internet, private messaging, etc)

Remark. Crypto is rather unique within Computer Science in requiring that some problems should be *hard*.

Hard problem zoo

Hard problems relate to cryptographic **primitives**. Higher-level constructions can be proved secure assuming secure underlying primitives.

Hard problems for **asymmetric** primitives.

- The RSA problem (≠factorization).
- Discrete Logarithm over certain groups.
- Hard problems in lattices.
- Syndrome decoding for random codes.
- Etc...

See MPRI 2.12.2 “Arithmetic algorithms for cryptology”

See MPRI 2.12.1 = us :)

See MPRI 2.13.2 right before :)

Hard problems for **symmetric** primitives.

- Ad-hoc assumptions: the primitive (AES, DES etc) is secure.

Wait, no proofs for symmetric primitives?

Functionality: no special structure (roughly, only need to “obfuscate”).

Speed: handles massive data: needs to be extremely fast.

Strategy: take extremely efficient operations (XOR, add, bit shift etc), combine them in carefully designed, but algebraically “incoherent” ways (on purpose).

⇒ No proof (for *primitives*) but well-studied, and works very well.

vs assymetric crypto:

Functionality: special structure (trapdoor, morphism etc).

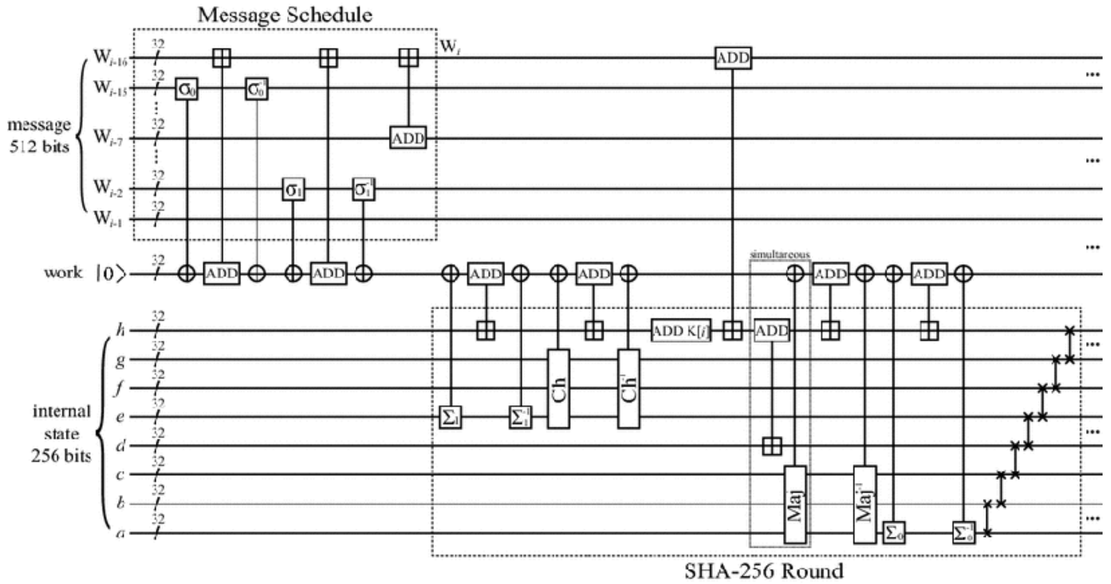
Speed: critical but punctual operations.

Strategy: take mathematical objects with special properties, use those properties.

⇒ Proofs reducing to some hard problem in underlying math object.

In practice

SHA-2 hash:



vs

RSA encryption:

Public parameter $N = pq$ with prime p, q.

Encryption of $m = m^3 \bmod N$.

What is *this* course about then?

In combination, existing MPRI crypto courses cover all major types of primitives (cf. earlier slide), both in symmetric and asymmetric crypto + proofs built on top (MPRI 2.30).

This course:

- first half: lattices.
- *this* half: advanced primitives (especially zero-knowledge proofs).

Key point: cryptography can do *much more* than simple encryptions, signatures, key exchanges, etc.

> *Think: electronic voting, ~~cryptocurrencies~~, delegated computation, homomorphic encryption...*

→ We will see some important constructions beyond basic primitives.

Brief interlude: crypto magic

Challenge:

Define an injective mapping $F: \{0,1\}^* \rightarrow \{0,1\}^\lambda$.

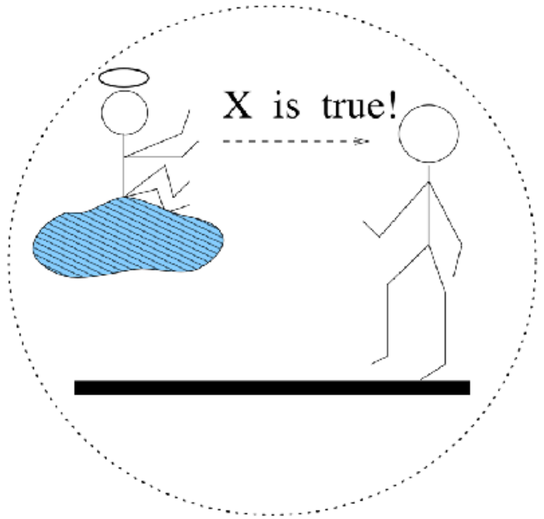
How about if injectivity is only computational?

i.e. computationally hard to find $x \neq y$ s.t. $F(x) = F(y)$.

Then it's fine! It's a (cryptographic) hash function.

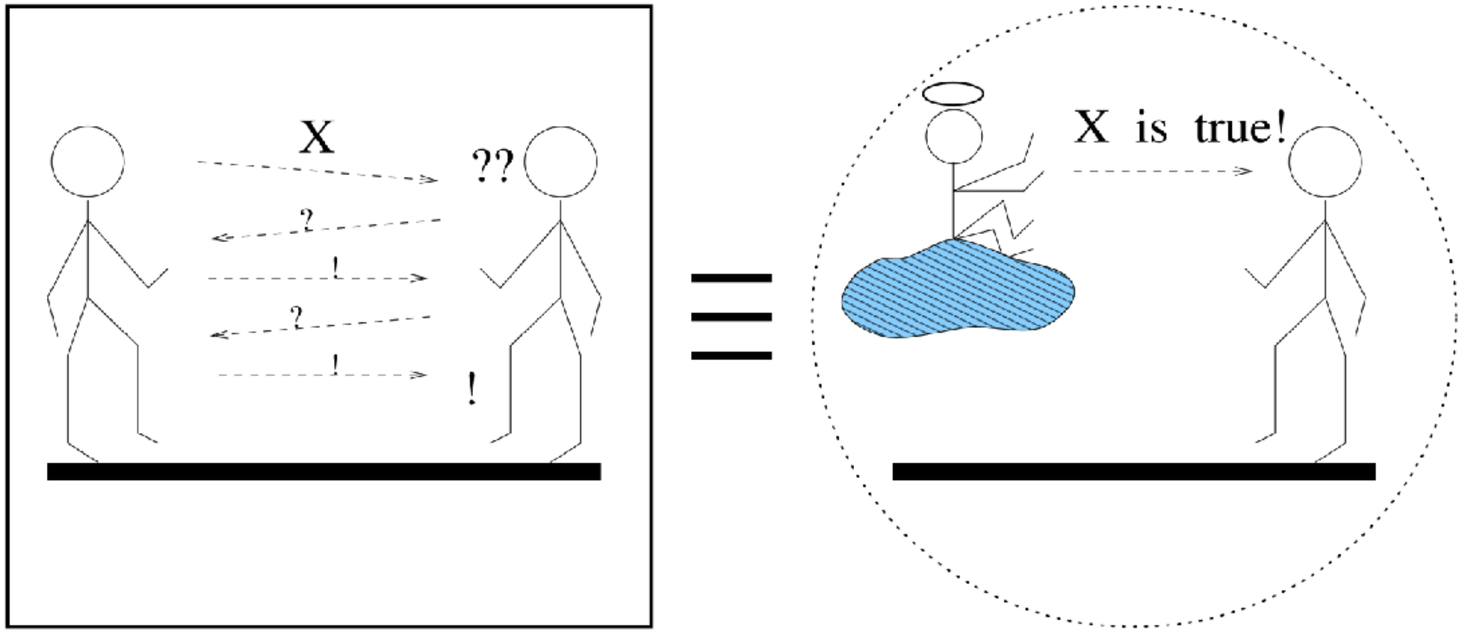
(Story for another time: hardness as sketched above is ill-defined.)

Zero knowledge proofs



Zero Knowledge

Goldwasser, Micali, Rackoff '85.



A zero-knowledge course would be a very bad course.

Image credit Oded Goldreich www.wisdom.weizmann.ac.il/~oded/PS/zk-tut10.ps

Expressivity

Zero-knowledge (ZK) proofs are very powerful and versatile.

On an intuitive level (for now), statements you may want to prove:

- “I followed the protocol honestly.” (but want to hide the secret values involved.) *E.g. prove election result is correct, without revealing votes.*
- “I know this secret information.” (but don't want to reveal it.) *For identification purposes.*
- “The amount of money going into this transaction is equal to the amount of money coming out.” (but want to hide the amount, and how it was divided.)

What do we want to prove?

Want to prove a statement on some x: P(x) is true.

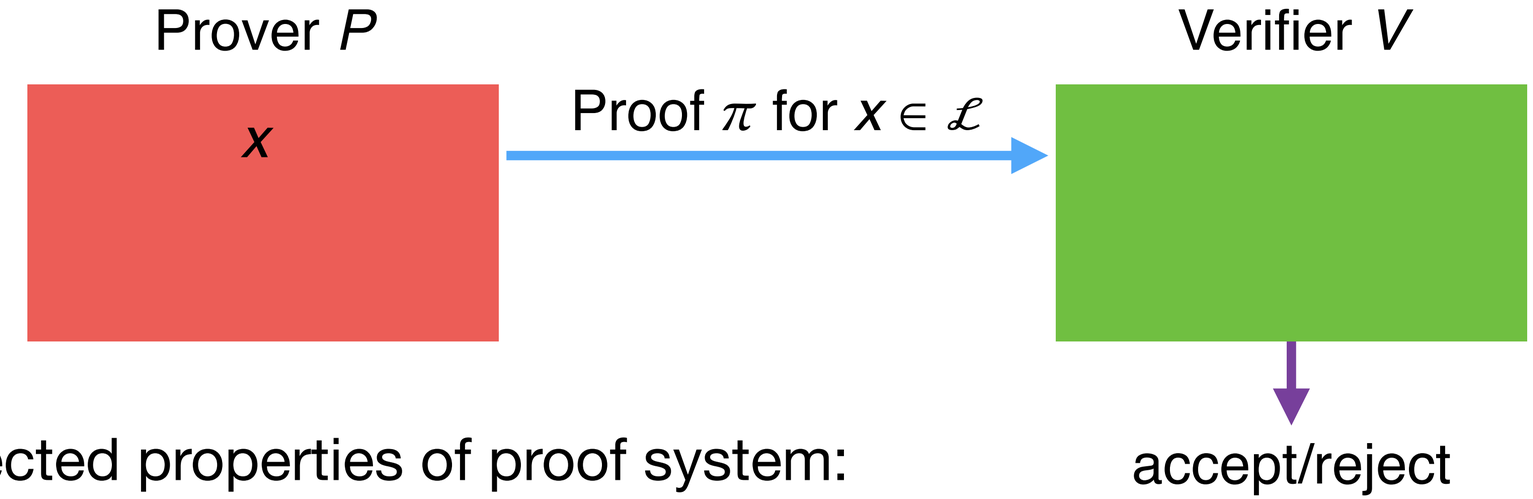
Exemple: x = list V of encryptions of all votes + election result R
P(V,R) = result R is the majority vote among encrypted votes V.

In general, can regard x as a bit string.

Equivalently: want to prove $x \in \mathcal{L}$. (set $\mathcal{L} = \{y : P(y)\}$.)

What is a proof?

For a language $\mathcal{L}$:



Expected properties of proof system:

- Completeness. If $x \in \mathcal{L}$, then $\exists$ proof π, $V(\pi)$ = accept.
- Soundness. If $x \notin \mathcal{L}$, then $\forall$ proof π, $V(\pi)$ = reject.
- Efficiency. V is PPT (Probabilistic Polynomial Time).

Without the last condition, definition is vacuous (prover is useless).

Zero knowledge

Intuitively: Verifier learns *nothing* from π other than $x \in \mathcal{L}$.

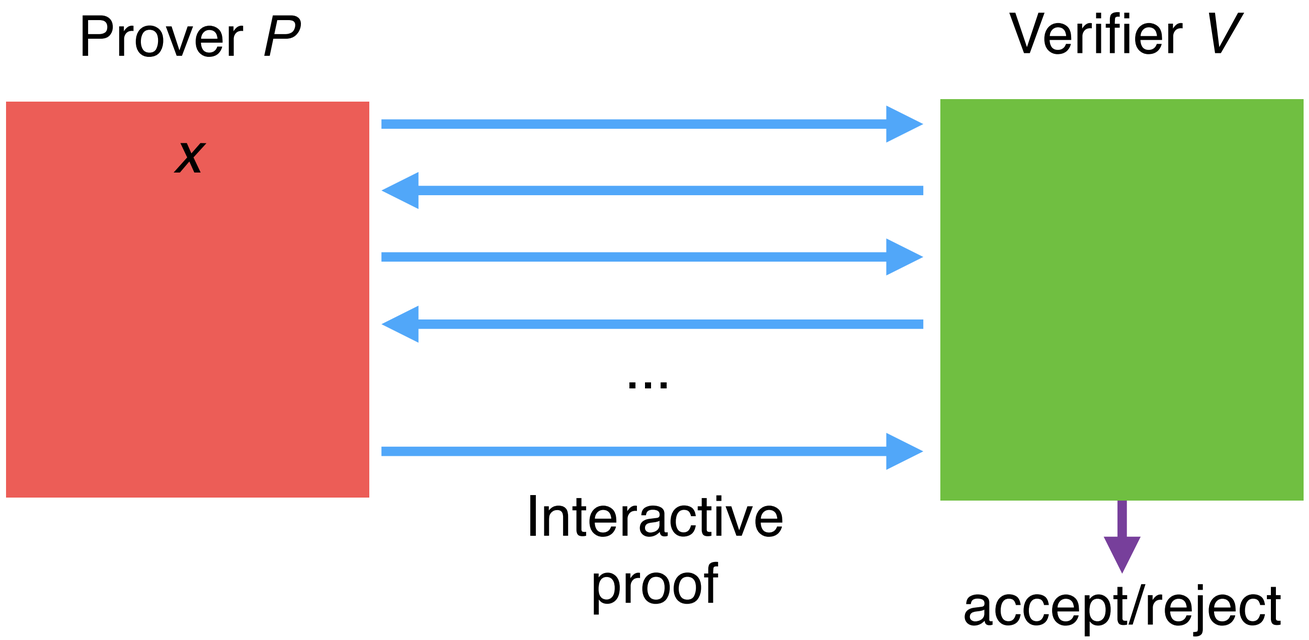
...this is impossible for previous notion of proof.

(only possible languages are those in BPP, i.e. when the proof is useless...)

→ going to generalize/relax notion of proofs in a few ways:

- Interactive proof, probabilistic prover, imperfect (statistical) soundness...

Interactive proof



An Interactive Proof (P,V) for $\mathcal{L}$ must satisfy:

- (Perfect) Completeness. If $x \in \mathcal{L}$, then $P \leftrightarrow V$ accepts.
- (Statistical) Soundness. If $x \notin \mathcal{L}$, then $\forall$ prover P^*, $\Pr[P^* \leftrightarrow V \text{ rejects}] = \text{non-negl}(|x|)$. (i.e. $\geq 1/p(|x|)$ for some fixed polynomial p.)
- Efficiency. V is PPT.

Caveat: prover is unbounded.

Interactive Proofs

Concept discovered independently in **cryptography** (Goldwasser, Micali, Rackoff), and **complexity theory** (Babai).

Complexity theory view:

NP = languages that have a (non-interactive) proof with (deterministic) verifier in **P**.

IP = languages that have an interactive proof with probabilistic verifier.

Remark:

- "interactive proofs with deterministic verifier" =
- "non-interactive proofs with probabilistic verifier" = AM.

Connection with **Probabilistically Checkable Proofs** (rich theory).

IP

IP: complexity class of languages that admit an interactive proof.

Public-coin proof: verifier gives its randomness to prover (AM).
Private-coin proof: no such restriction (IP). No more expressive.

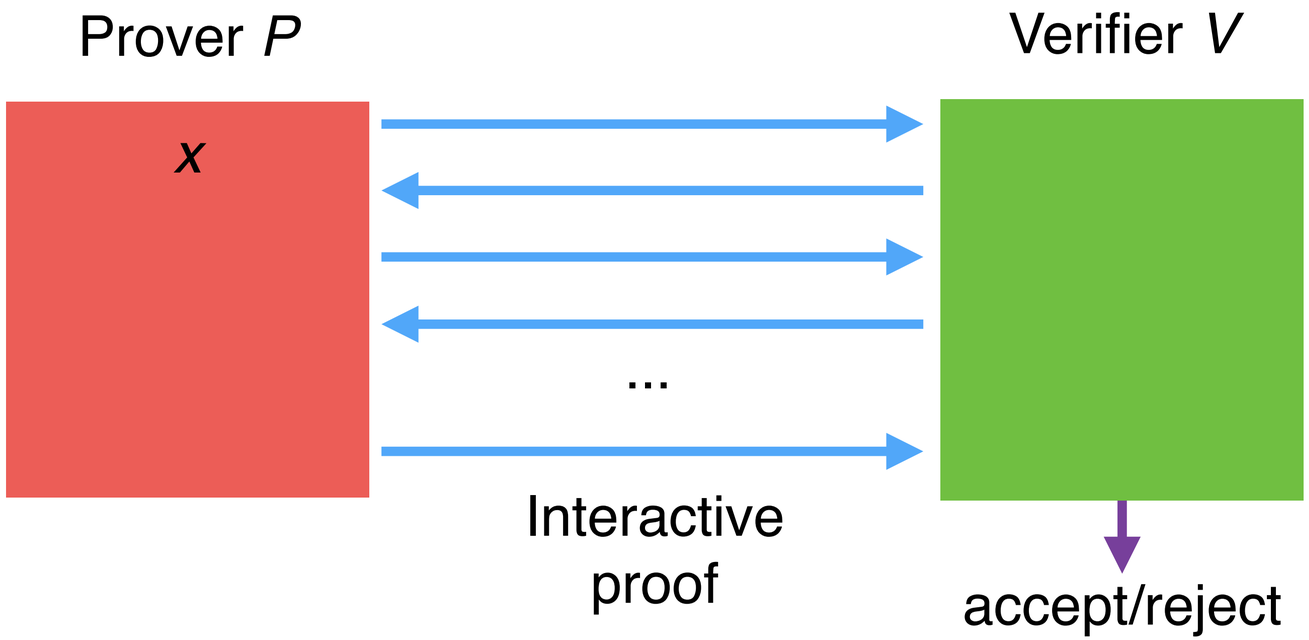
Theorem. Shamir, LKFN at FOCS '90.

IP = PSPACE.

Very powerful but in crypto, for usability, we want efficient (PPT) prover.

→ **argument** of knowledge.

Interactive **argument** of knowledge



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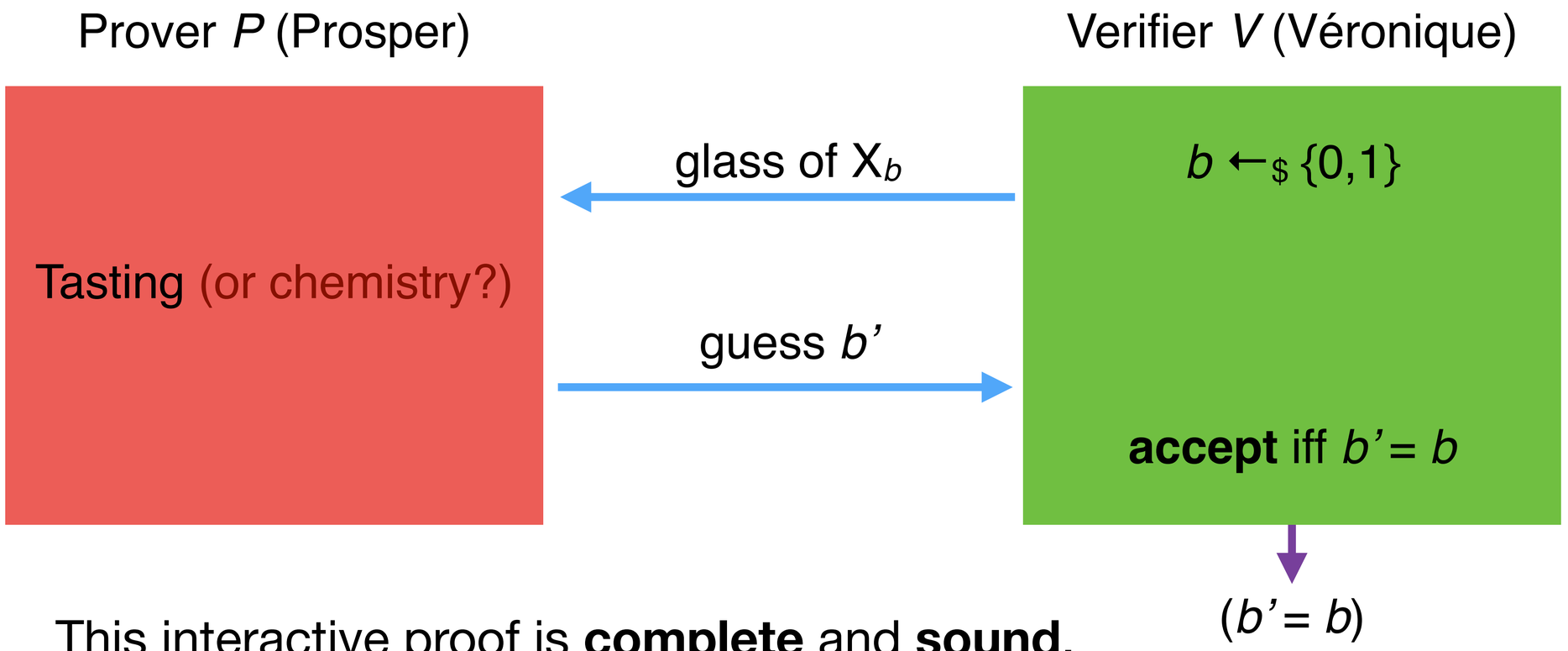
- (Perfect) Completeness. If $x \in \mathcal{L}$, then $P \leftrightarrow V$ accepts.
- (Statistical) Soundness. If $x \notin \mathcal{L}$, then $\forall$ PPT prover P^*, $\Pr[P^* \leftrightarrow V \text{ rejects}]$ = non-negl($|x|$). (i.e. $\geq 1/p(|x|)$ for some fixed polynomial p.)
- Efficiency. V is PPT.

(only difference)

Preliminary examples

Pepsi vs Coke is in IP

Prosper (*P*) wants to prove to Véronique (*V*) that she can distinguish Pepsi from Coke. Let (X_0,X_1) = (Pepsi,Coke).



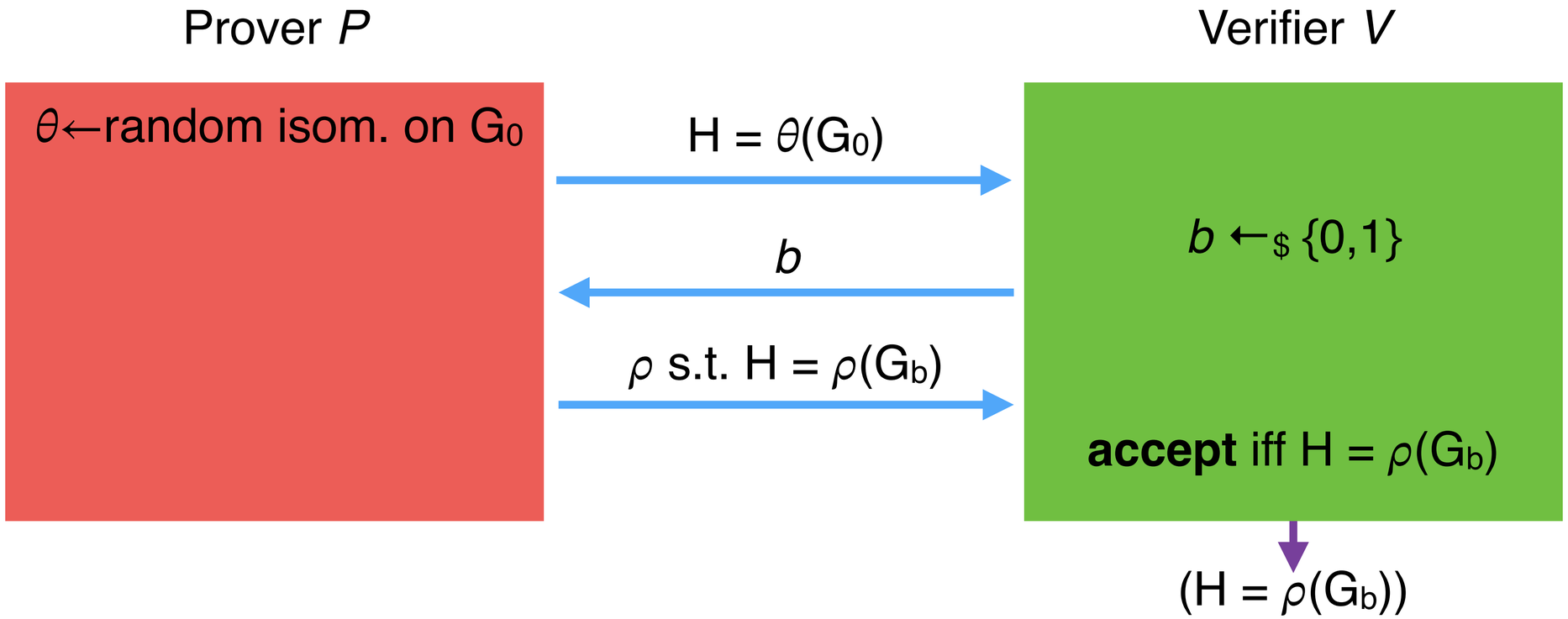
This interactive proof is **complete** and **sound**.

Soundess error = 1/2. Reduce to $2^{-\lambda}$: iterate the protocol λ times.

Graph isomorphism (unbounded prover)

- Suppose two graphs G_0, G_1 are isomorphic: $\exists\sigma$, $\sigma(G_0) = G_1$.
- Prover wants to prove $G_0 \sim G_1$ without revealing anything about the isomorphism.

Formally: $\mathcal{L} = \{(G,G') : G \sim G'\}$, want to prove $(G_0,G_1) \in \mathcal{L}$.



Analysis

- (Perfect) Completeness.
  "If $x \in \mathcal{L}$, then $P \leftrightarrow V$ accepts".

  Clearly true.

- (Statistical) Soundness.
  "If $x \notin \mathcal{L}$, then $\forall$ prover P^*, Pr[$P^* \leftrightarrow V$ rejects] = non-negl($|x|$)".

  True: V will reject with probability $\geq 1/2$.

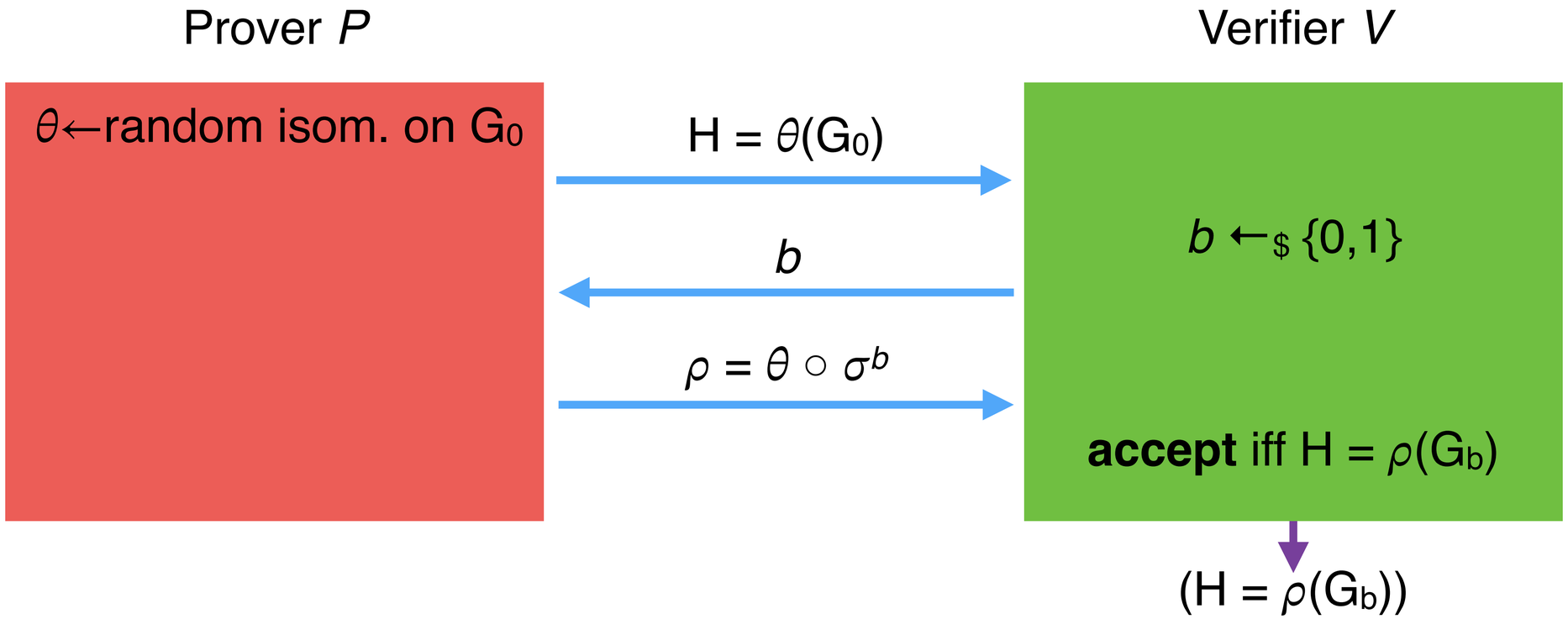
- Efficiency. V is PPT.

We want to actually use this → want a bounded prover (PPT).

Graph isomorphism (bounded prover)

- Prover knows an isomorphism σ between G_0, G_1: $\sigma(G_0) = G_1$.
- Prover wants to prove $G_0 \sim G_1$ without revealing anything about the isomorphism.

Formally: $\mathcal{L} = \{(G,G') : G \sim G'\}$, want to prove $(G_0,G_1) \in \mathcal{L}$.



Bounded prover who knows a *witness*. Public coin. Perfect ZK.

Proofs of knowledge

?

Motivation

We want a bounded prover (PPT).
Corollary: secret prover knowledge is necessary.

In Graph isomorphism, intuitively, proof “shows” that not only statement is true, but prover knows a witness: the permutation σ.

This difference is meaningful!

Example: cyclic group G = <g>. For this language...

$$\mathcal{L} = \{h \mid \exists s,\ h = g^s\}$$

Interactive Proof: trivial ($\mathcal{L}$ = G !)

Knowing a witness: hard (Discrete Log problem)

Defining knowledge

To generalize, we want witnesses.

NP languages are great: $\mathcal{L} = \{x \mid \exists w, \mathrm{R}(x,w)\}$ for efficient R.

→ Want to formalize: the prover **knows** a witness.

*What does it mean for an algorithm to **know** something?*

Defining knowledge

Attempt 1.

Algorithm $\mathcal{A}$ knows a secret s if it outputs s.

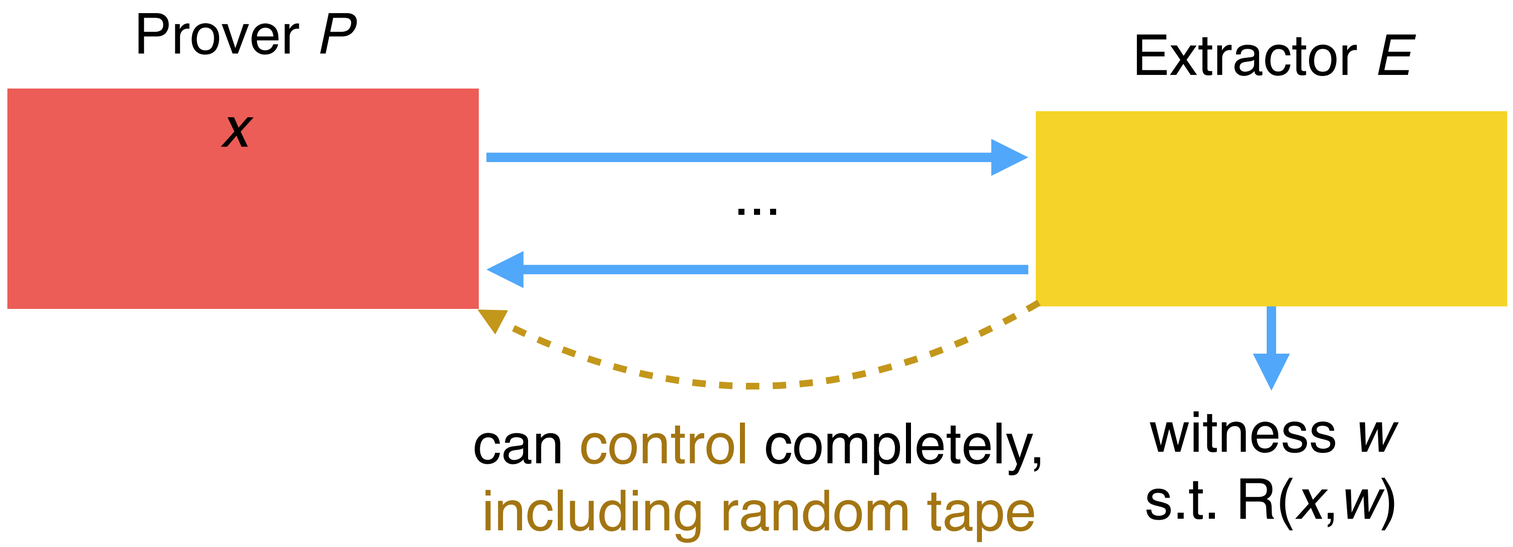
Attempt 2.

Algorithm $\mathcal{A}$ knows a secret s if s appears somewhere in the code.

Attempt 3.

Algorithm $\mathcal{A}$ knows a secret s if there exists an efficient algorithm $\mathscr{E}$ that can extract s from $\mathcal{A}$.

Soundness of a **knowledge** proof



Knowledge soundness.

∃ efficient extractor *E* that, given access to *P* and *x*, can compute *w* such that R(*x*,*w*).

More formally

NP Language $\mathcal{L} = \{x \mid \exists w, R(x,w)\}$.

Knowledge soundness (attempt 1, right idea, not yet perfect).

A proof system is **knowledge-sound** if and only if:

∃ efficient extractor E^P with oracle access to a prover P such that ∀ x, ∀ P that convinces V (with probability 1), E^P outputs w such that $R(x,w)$ (with probability ≥ 1/2).

Remark: this is a property of the verifier.

Modern definition

NP Language $\mathcal{L} = \{x \mid \exists w, R(x,w)\}$.

Knowledge soundness (simple).

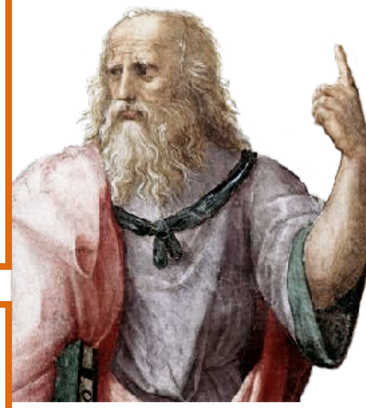
A proof system is **knowledge-sound** if and only if:

$\exists$ efficient extractor E^P with oracle access to a prover P such that $\forall$ x, $\forall$ P that convinces V with non-negligible probability, E^P outputs w such that $R(x,w)$ with non-negligible probability.

Knowledge soundness (better).

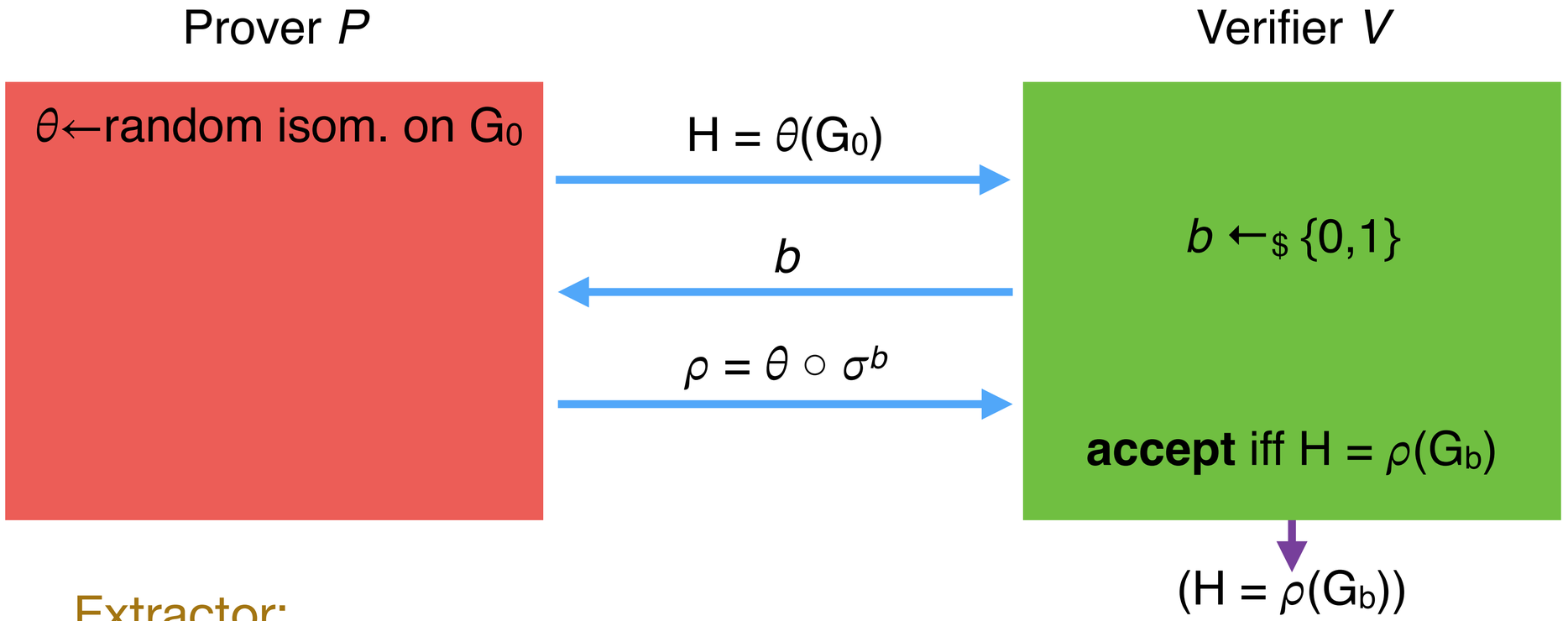
A proof system is **knowledge-sound** with soundess κ iff:

$\exists$ efficient extractor E^P such that if $\epsilon = \Pr[P \leftrightarrow V \text{ accepts}] > \kappa$, then E^P succeeds with probability at least $\epsilon - \kappa$.



More information: “On Defining Proofs of Knowledge”, Bellare and Goldwasser
https://www.wisdom.weizmann.ac.il/~oded/pok.html

Knowledge soundness for Graph Isomorphism

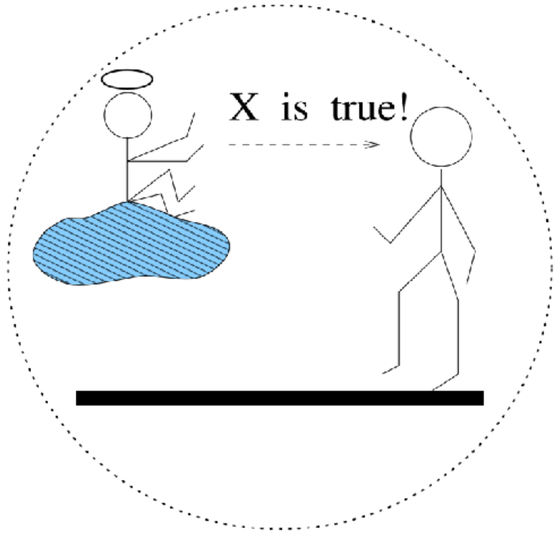


Extractor:

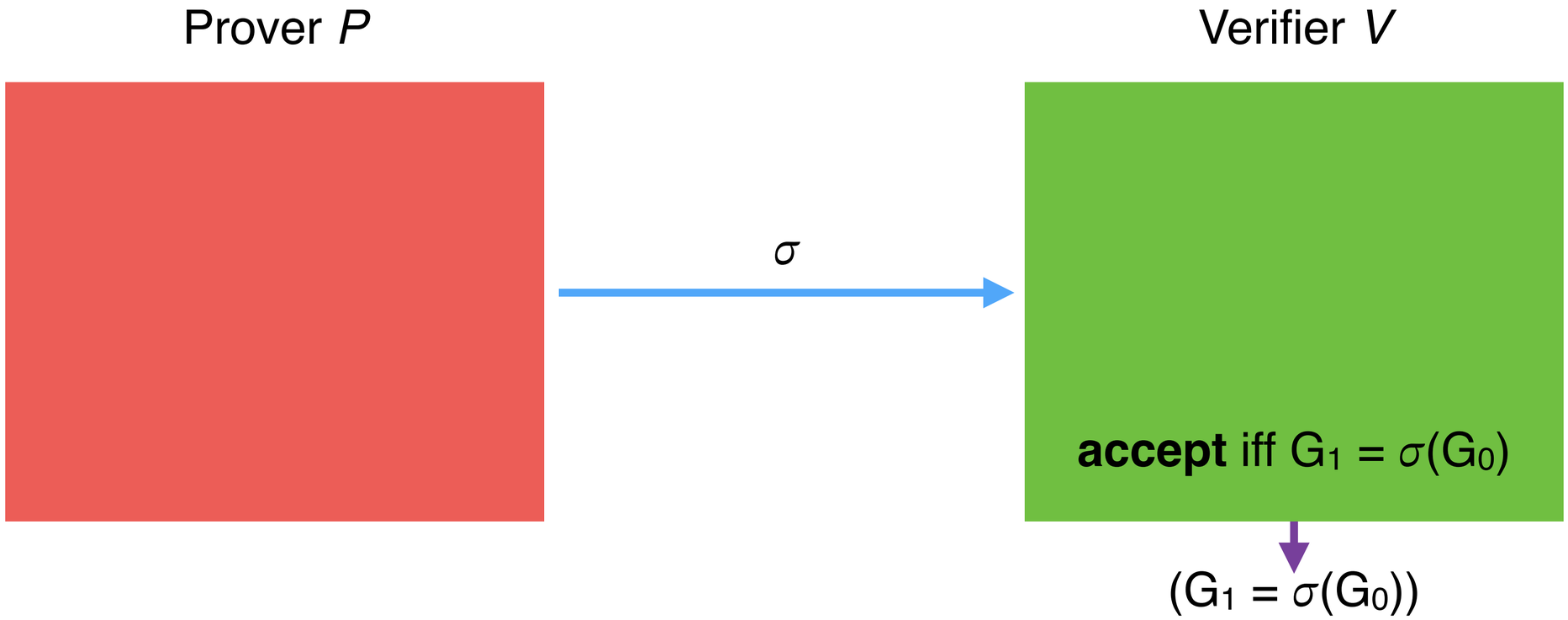
- calls P, gets $H = \theta(G_0)$.
- asks $b = 0$, **and** $b = 1$. This is legitimate due to randomness control! Gets back ρ_0, ρ_1 with $H = \rho_0(G_0) = \rho_1(G_1)$.
- $G_1 = \rho_1^{-1} \circ \rho_0(G_0) \rightarrow$ witness $\sigma = \rho_1^{-1} \circ \rho_0$.

Special soundness: two challenges reveals witness, cf. next section.

Zero knowledge



Towards zero knowledge

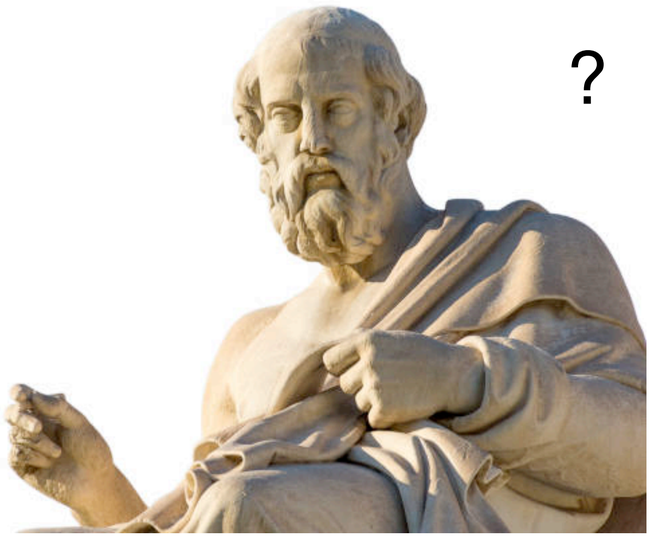


For language in NP, witness itself *is* a proof of knowledge...

- Zero-knowledge: prove membership or knowledge while revealing *nothing else*.

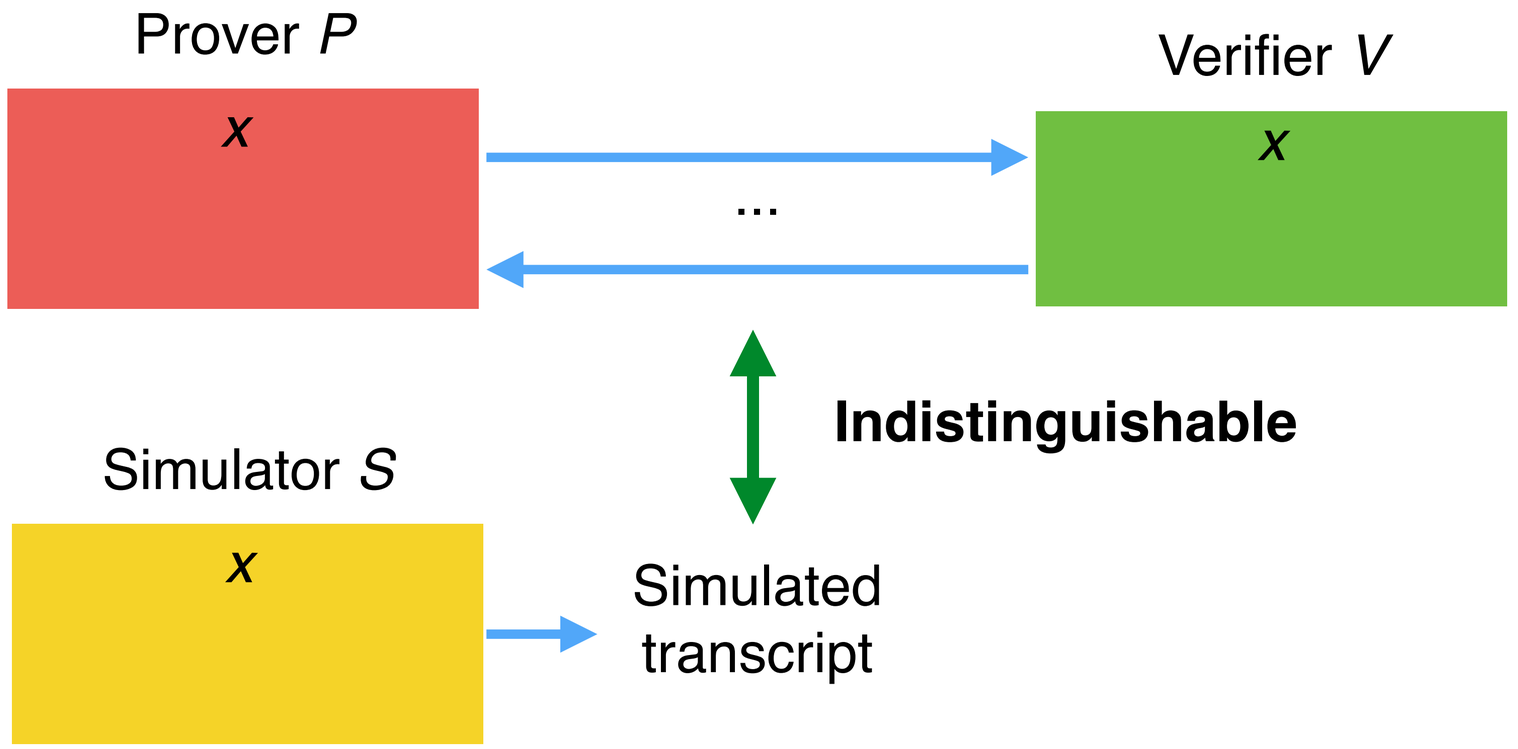
Towards zero knowledge

Need to formalize: *the verifier learns nothing*.



?

Honest-verifier zero-knowledge



Honest-verifier zero-knowledge.

The (interactive) proof system (P,V) is **zero-knowledge** iff:

$\exists$ efficient (PPT) simulator S s.t. $\forall\, x \in \mathcal{L}$, transcript of P interacting with V on input x is indistinguishable from the output of S(x).

Analysis

Point of definition:

- anything *V* could learn from interacting (honestly) with *P*, could also learn by just running *S*.
- *S* is efficient and knows no secret information.

⇒ Anything *V* can compute with access to *P*, can compute without *P*.

That expresses formally: "*V* learns nothing from *P*".

- Is the Graph Isomorphism proof ZK?

Yes. Simulator: choose b in {0,1}, and random permutation π of G_b. Publish as simulated transcript: ($\pi(G_b)$,b, π). This is identically distributed to a real transcript → perfect zero-knowledge.

Key argument: $\pi(G_b)$ for uniform π does not depend on *b*.

Types of zero knowledge

Let ρ be the distribution of real transcrpits, σ simulated transcript.

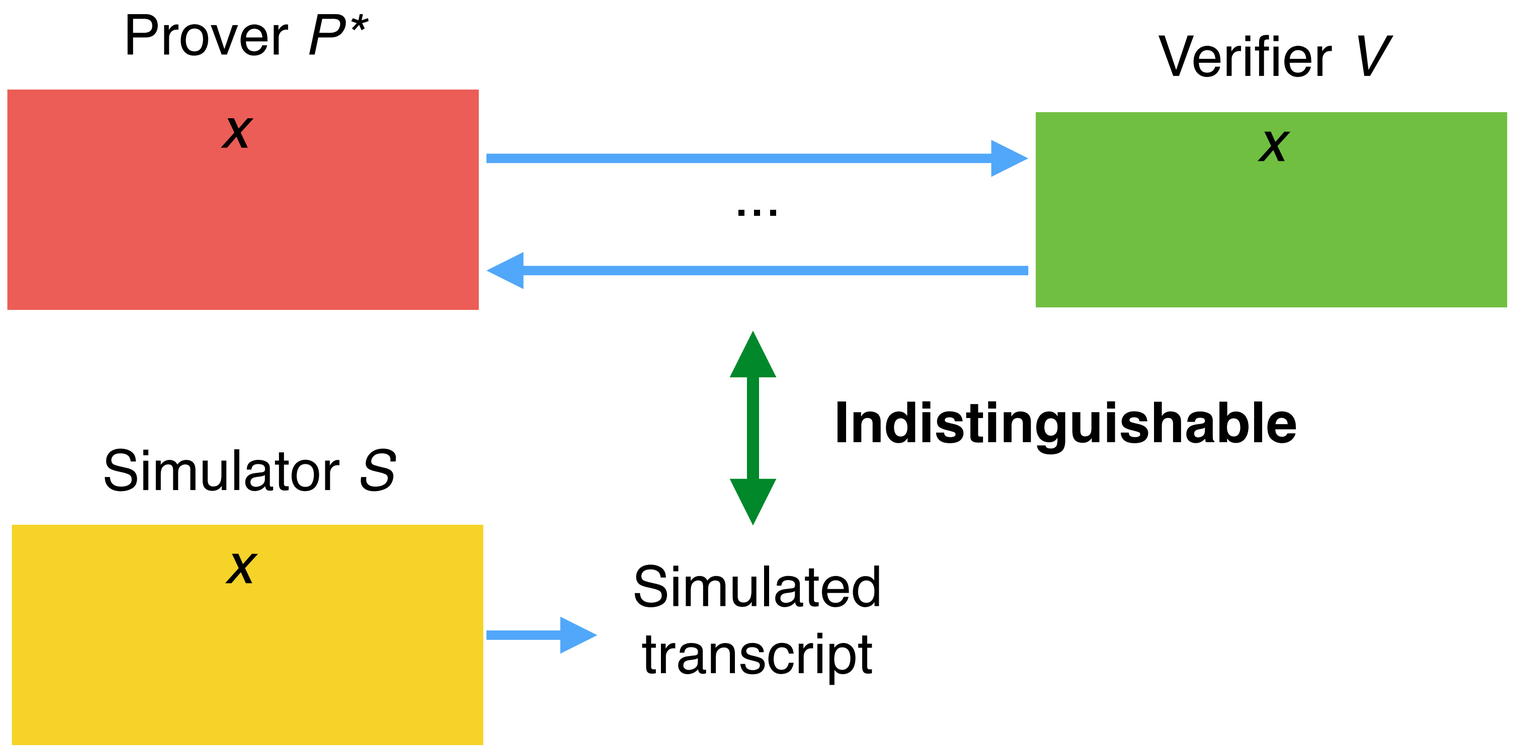
- Perfect ZK: $\rho = \sigma$.
- Statistical ZK: dist(ρ,σ) is negligible. (dist = statistical distance)
- Computational ZK: advantage of efficient adversary trying to distinguish ρ from σ is negligible.

implies

Likewise: completeness, soundness can be perfect/statistical/computational.

What if the prover is **malicious** (does not follow the protocol?)

~~Honest-verifier~~ Zero-knowledge



Zero-knowledge.

The (interactive) proof system (P,V) is **zero-knowledge** iff:

$\forall$ prover P^*, $\exists$ PPT simulator S s.t. $\forall\, x \in \mathcal{L}$, transcript of P^* interacting with V on input x is indistinguishable from output of $S(x)$.

Summary

A ZK proof is (perfectly/statistically/computationally):

1. Complete
2. Sound
3. Zero-knowledge.

"Proof of membership"

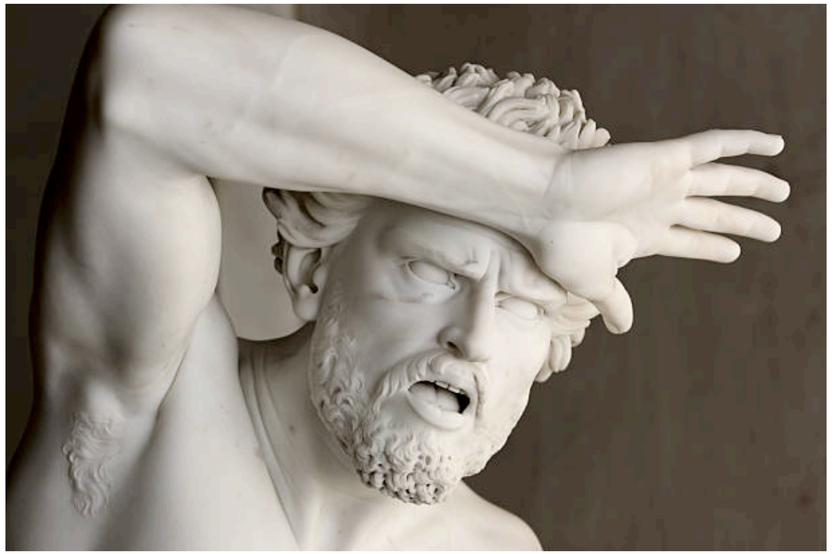
A ZK proof *of knowledge* is (perfectly/statistically/computationally):

1. Complete
2. *Knowledge*-Sound
3. Zero-knowledge.

"Proof of knowledge"

Question

Is a zero-knowledge proof of knowledge possible?



Yes. Graph Isomorphism is an example.

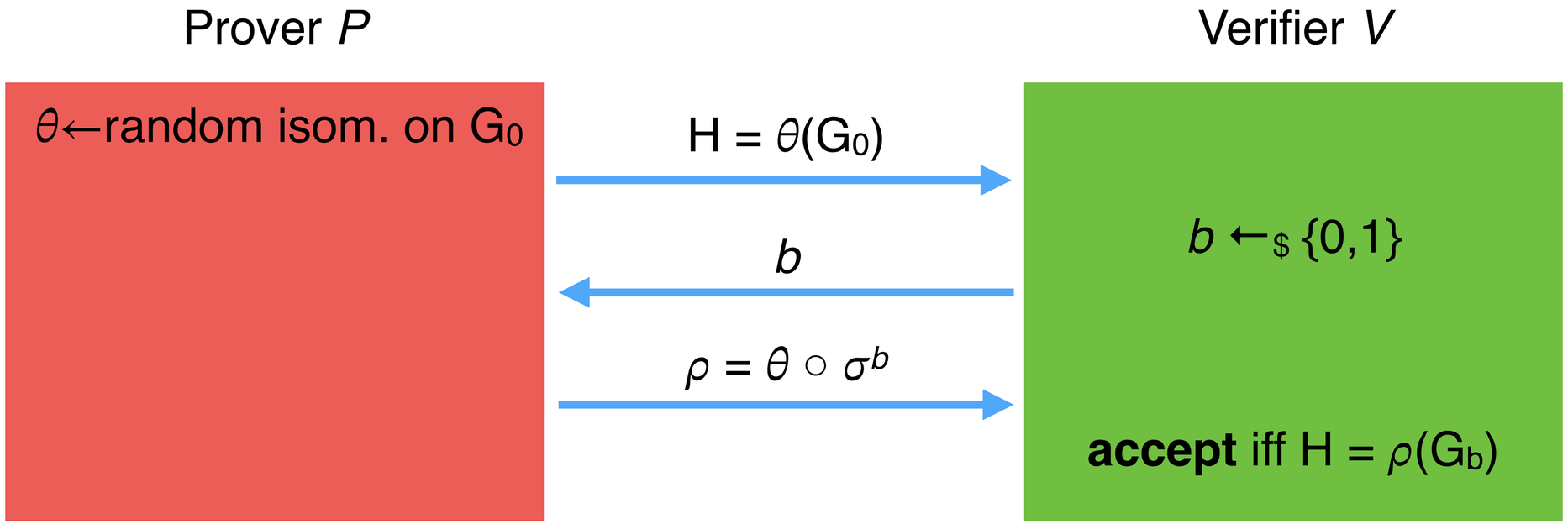
Subtlety: Knowledge Extractor can control the prover's random tape, Verifier cannot.

Examples

Graph isomorphism

- I know an isomorphism σ between two graphs G_0, G_1: $\sigma(G_0) = G_1$.
- I want to prove $G_0 \sim G_1$ without revealing anything about the isomorphism.

Formally: $\mathcal{L} = \{(G,G') : G \sim G'\}$, want to prove $(G_0,G_1) \in \mathcal{L}$.

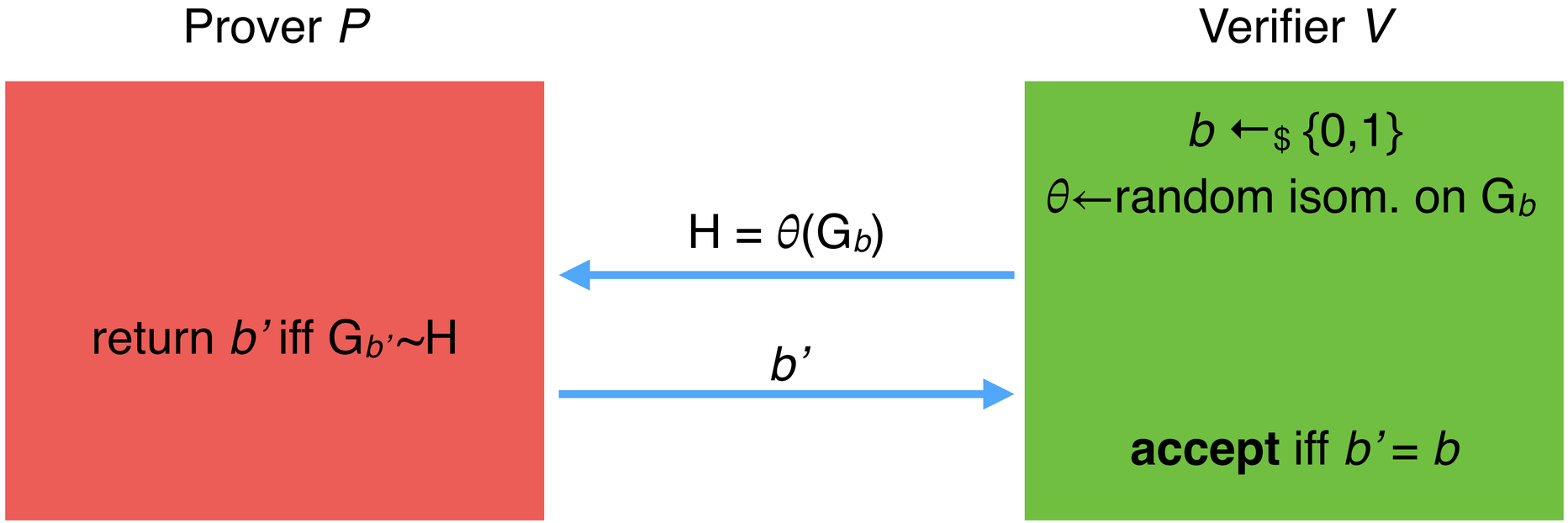


Bounded prover who knows a *witness*. Public coin. Perfect ZK.

Graph **non** isomorphism

- I am an unbounded prover who knows $G_0 \not\sim G_1$.
- I want to prove $G_0 \not\sim G_1$ without revealing anything else.

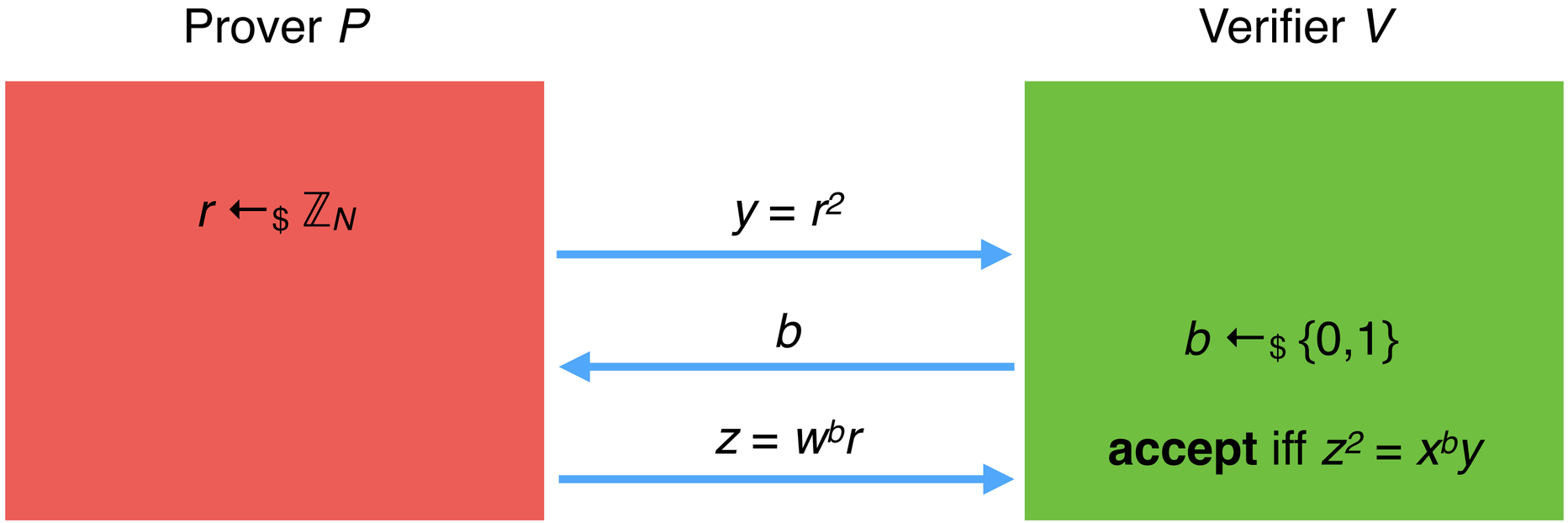
Formally: $\mathcal{L} = \{(G,G') : G \not\sim G'\}$, want to prove $(G_0,G_1) \in \mathcal{L}$.



Unbounded prover. Private coin. Not ZK for malicious *V*.

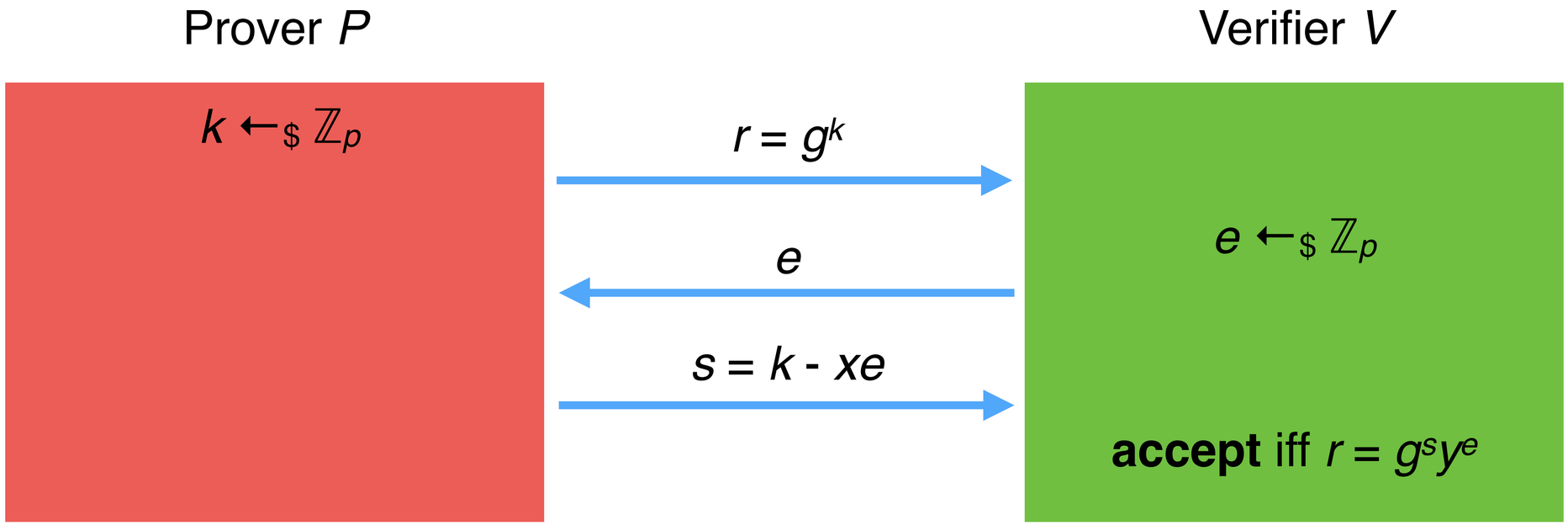
Knowledge of a square root

- Public $N = pq$ for large primes p, q, public x in $\mathbb{Z}_N$.
- I am a bounded prover who knows w such that $x = w^2 \bmod N$.
- I want to prove that knowledge without revealing anything else.



Knowledge of a discrete log

- Let $\mathbb{G} = \langle g \rangle \sim \mathbb{Z}_p$ and $y \in \mathbb{G}$. I know $x \in \mathbb{Z}_p$ such that $y = g^x$.
- Corresponding language is trivial! $\forall y\ \exists x,\ y = g^x$. But proof of knowledge still makes sense.



Known as Schnorr protocol.

Analysis of Schnorr protocol

- (Perfect) Completeness.

  Clear.

- (Special) Knowledge soundness.

  Extractor: gets $r = g^k$, asks two challenges $e \neq e'$, gets back s, s' with $r = g^s y^e = g^{s'} y^{e'}$. Yields $y = g^{(s-s')/(e'-e)}$.

- (Perfect) Honest-verifier zero knowledge.

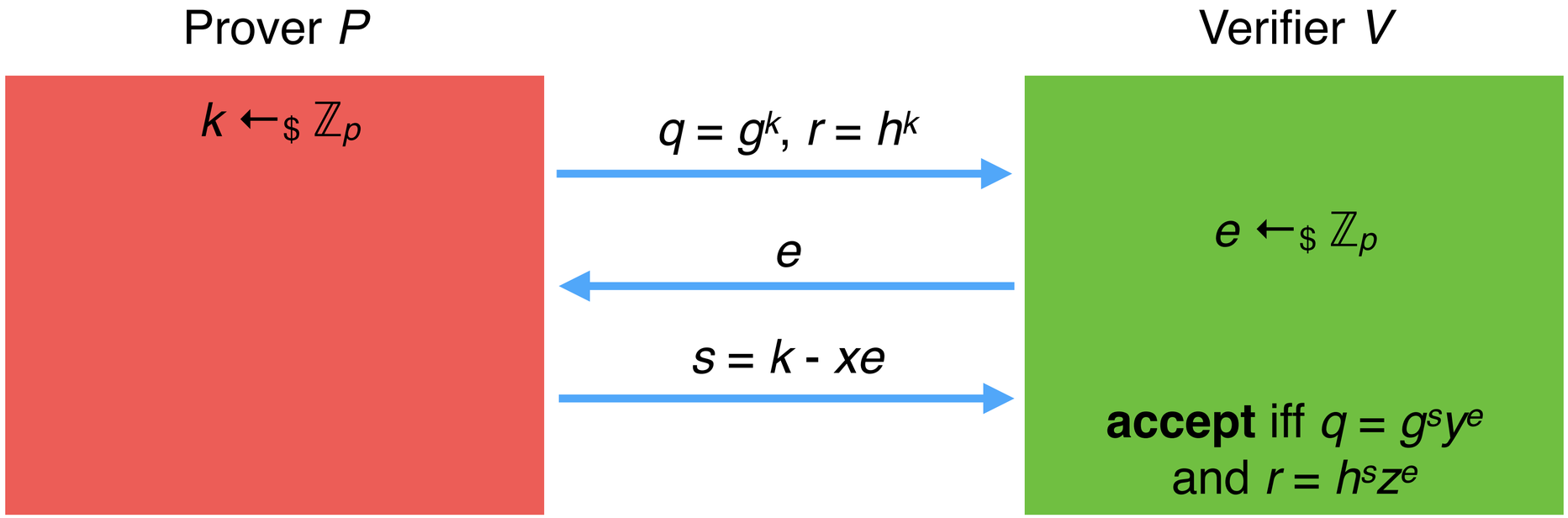
  Simulator: draw $e \leftarrow_\$ \mathbb{Z}_p$, $s \leftarrow_\$ \mathbb{Z}_p$, **then** $r = g^s y^e$. Return transcript (r,e,s). Note r, e still uniform and independent → distribution is identical to real transcript.

We will use this for a signature!

Equality of exponents = DH language

- Let $\mathbb{G} \sim \mathbb{Z}_p$, $g, h \in \mathbb{G}$. I know $x \in \mathbb{Z}_p$ such that $(y, z) = (g^x, h^x)$.
- Corresponding language is Diffie-Hellman language (for fixed g, h)!

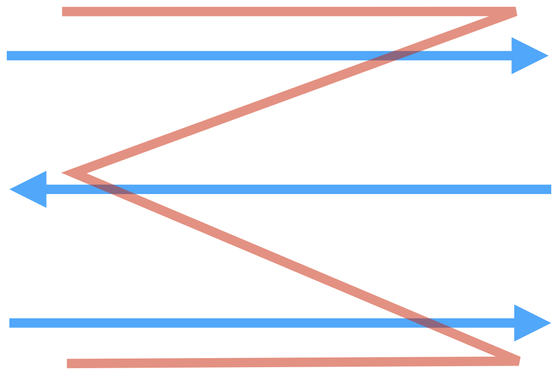
$$\mathcal{L} = \{(g, g^a, g^b, g^{ab}) : a, b \in \mathbb{Z}_p\} \leftrightarrow \mathcal{L}' = \{(g^a, h^a) : a \in \mathbb{Z}_p\} \text{ for } h = g^b$$



This is two ‘simultaneous’ executions of Schnorr protocol, with same (k,e). Soundness and ZK proofs are the same.

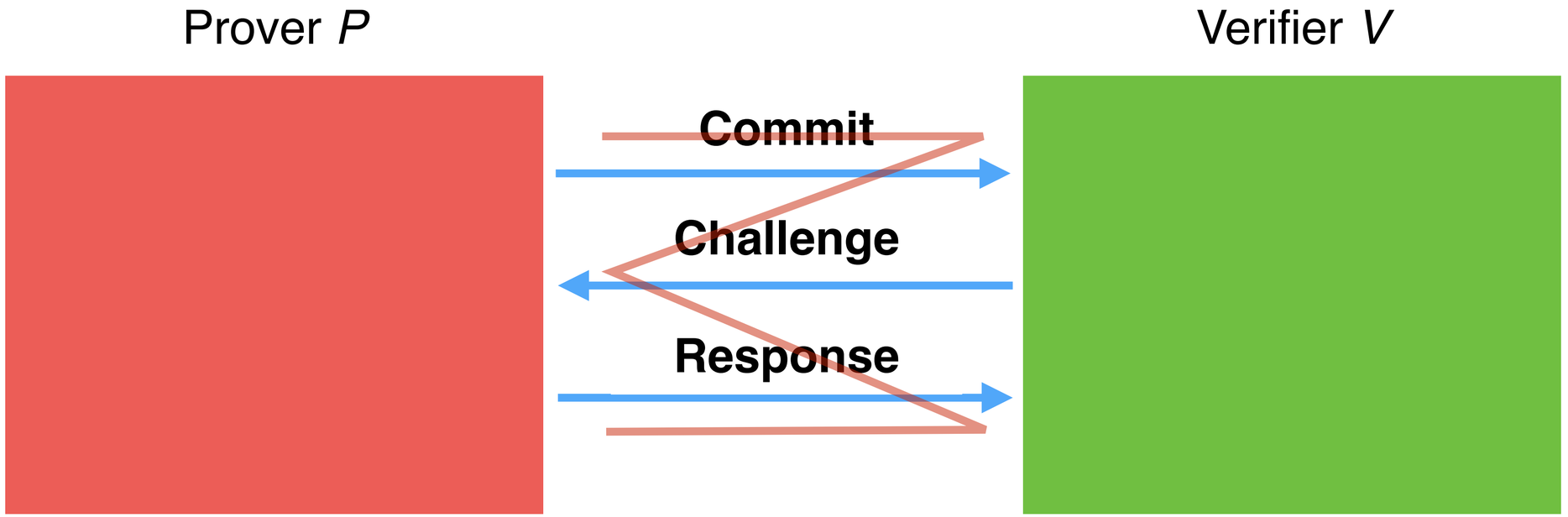
We will use this in a voting protocol!

Sigma protocols and NIZK



Sigma protocol

Schnorr protocol:

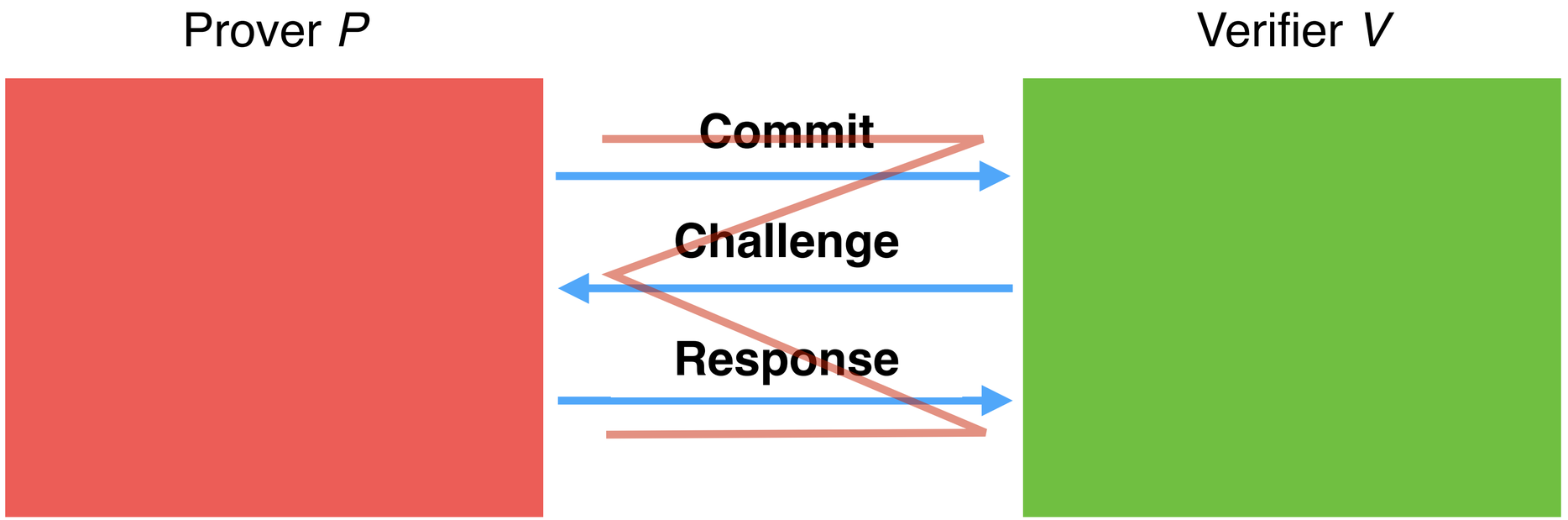


Public-coin ZK protocols following this pattern = Sigma Protocols.

Fiat-Shamir transform:

By setting **Challenge** = Hash(**Commit**), can be made non-interactive
→ Non-Interactive Zero-Knowledge (NIZK)

Special soundness



Special soundness: “answering two distinct uniform challenges for the same commit ⇒ knowing witness”

Special soundness ⇒ Knowledge soundness ⇒ Soundness

Sigma protocol → signature

NIZK knowledge proof: “I know a witness *w* for R(*x*,*w*)” and can prove it non-interactively without revealing anything about *w*.

This is an identification scheme.

Sigma protocol → can integrate message into challenge randomness.

This yields a signature scheme!

Public key: *x*

Secret key: *w*

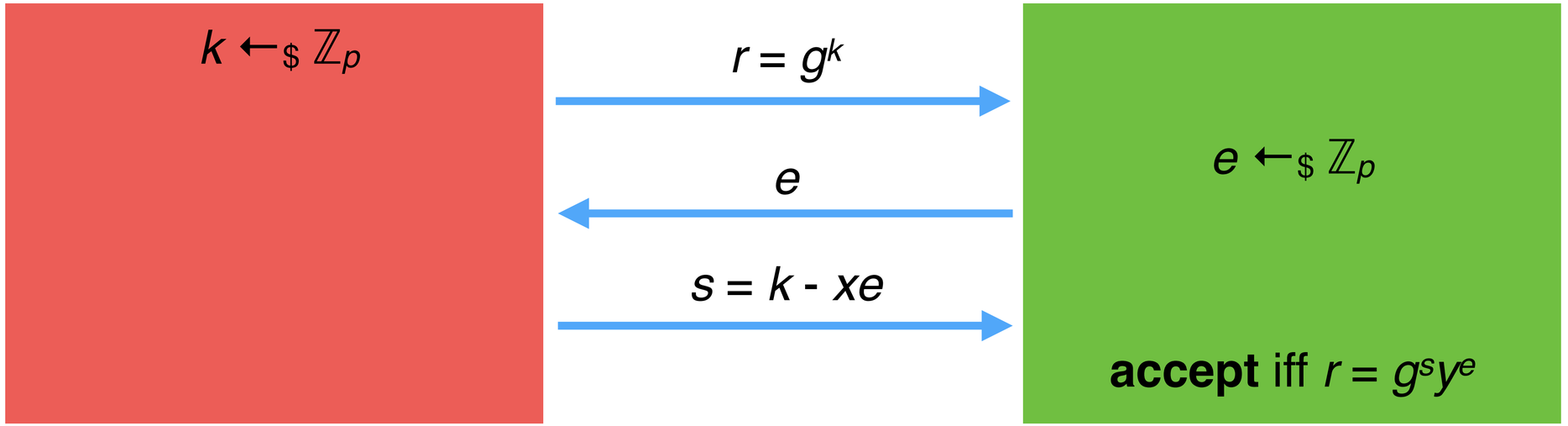
Sign(*m*): signature = NIZK proof with challenge = hash(commit,*m*)

Verify signature = verify proof.

That is the Fiat-Shamir transform.

Example: Schnorr signature

Schnorr protocol:



Schnorr signature:

Public key: $y = g^x$

Secret key: x

Sign(m): signature $\sigma = (r,s)$ with $r = g^k$ for $k \leftarrow_\$ \mathbb{Z}_p$, $s = k - xH(r,m)$.

Verify(σ,m): accept iff $r = g^s y^{H(r,m)}$.

Security reduces to Discrete Log, in the Random Oracle Model.

ZK proofs for arbitrary circuits

Reductions

Suppose there exists an efficient (polynomial) reduction from $\mathcal{L}'$ to $\mathcal{L}$:
$\exists$ efficient f such that $x \in \mathcal{L}'$ iff $f(x) \in \mathcal{L}$. (Karp reduction.)

If I can do ZK proofs for $\mathcal{L}$, I can do ZK proofs for $\mathcal{L}'$!

To prove $x \in \mathcal{L}'$, do a ZK proof of $f(x) \in \mathcal{L}$.

Also works for knowledge proofs (via everything being constructive).

$\Rightarrow$ **The dream:** if we can do ZK proof for an NP-complete language, we can prove everything we ever want!

Notably circuit-SAT.

Commitment scheme

A **commitment scheme** is a (family of) functions $C: X \times A \to V$ s.t.:

- Binding: it is hard to find $x \neq x'$ and a, a' s.t. $C(x,a) = C(x',a')$.
- Hiding: for all x, x', the distributions $C(x,a)$ for $a \leftarrow_\$ A$ and $C(x',a)$ for $a \leftarrow_\$ A$ are indistinguishable.

Usage:

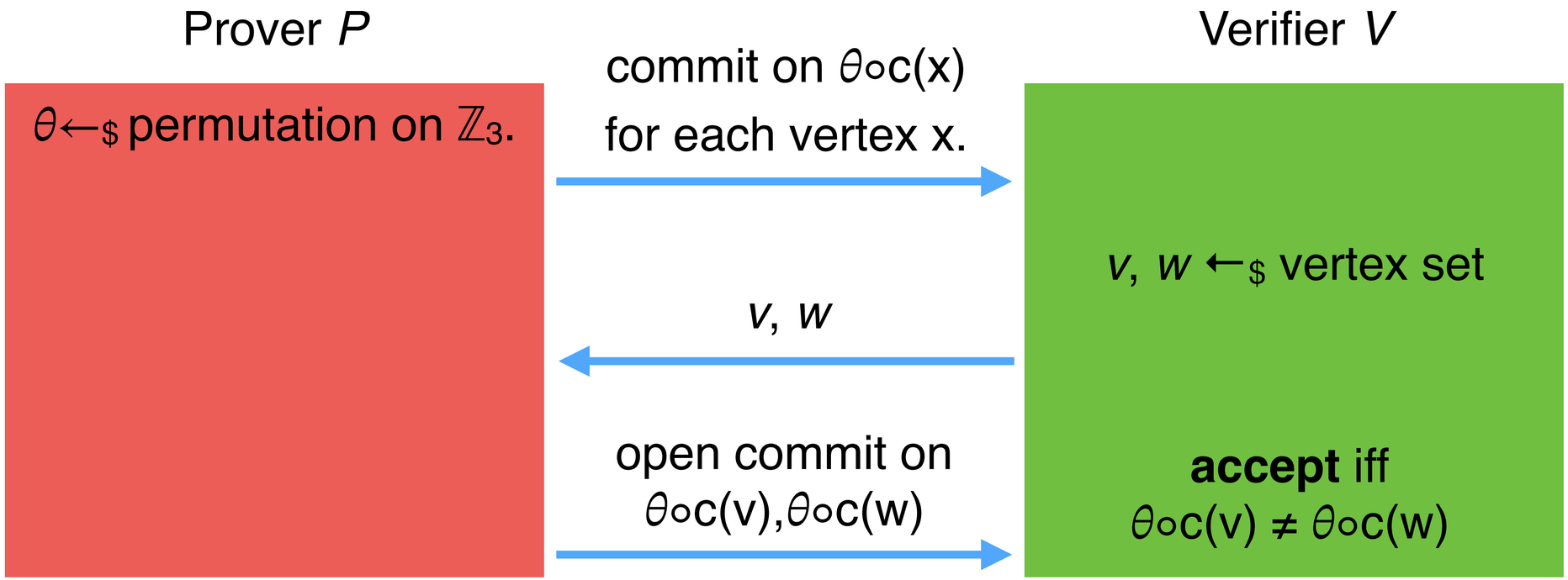
- Alice **commits** to a value x by drawing $a \leftarrow_\$ A$ and sending $C(x,a)$.
- Later, Alice **opens** the commitment by revealing the inputs x,a.

Instantiation: pick a hash function.

The dream: ZK proof for 3-coloring

- I know an 3-coloring *c* of a graph G (into $\mathbb{Z}_3$).
- I want to prove that such a coloring exists, without revealing anything about the coloring.

Formally: $\mathcal{L}$ = {(G): G admits a 3-coloring}



Bounded prover with a *witness*. Public coin. Computational ZK.

The wake-up

...this is incredibly inefficient.

- transform circuit-SAT instance into 3-coloring instance.

- run previous protocol *many* times (roughly #circuit size × security parameter) → gigantic proofs, verification times...

SNARKs

SNARK(?) tile by William Morris.

Finite Fields

Most of what follows is going to happen in a finite field.

For a short presentation of finite fields, see:

https://www.di.ens.fr/brice.minaud/cours/ff.pdf

A **key idea** we will use:

If $P \neq Q$ are two degree-d polynomials over $\mathbb{F}_q$, then for $\alpha \leftarrow \mathbb{F}_q$ drawn uniformly at random, $\Pr[\, P(\alpha) \neq Q(\alpha) \,] \geq 1 - d/q$.

→ to check if two bounded-degree polynomials are equal, it is enough to check at a random point!

Proof: P-Q is a non-zero polynomial of degree at most d, so it can be zero on at most d points.

A toy example

Véronique wants to compute the 1000th Fibonacci number in $\mathbb{Z}_p$.

She doesn't have time, so she asks Prosper to to it. But she wants a *proof* that the computation was correct.

"Solution": agree on whole computation circuit → encode as SAT problem → transform into 3-coloring problem → include NIZK proof of that 3-coloring problem with the result.

Remark: size of proof is linear in the size of the circuit Véronique doesn't want to compute.

(P & V hate closed formulas and fast exponentiation.)

SNARK

We would like to achieve zero-knowledge proofs that are **succint** and non-interactive.

Succint **N**on-interactive **Ar**gument of **K**nowledge: **SNARK**.

Also a fantastical beast by Lewis Caroll:

A new approach

Prosper computes the Fibonacci sequence f_1, ..., f_{1000} in $\mathbb{Z}_p$.
He sends f_1, f_2, and f_{1000} to Véronique.

Now V. wants to check $f_{i+2} = f_i + f_{i+1}$ for all i's.

Magic claim: she will be able to check that this computation was correct, for all i, with 99% certainty, by asking Prosper for only 4 values in $\mathbb{Z}_p$.

Disclaimers:
- we assume Prosper answers queries honestly (for now).
- from now on, assume $|\mathbb{Z}_p|$ is "large enough", say $|\mathbb{Z}_p| > 100000$.
  (Otherwise, just go to a field extension.)

This line of presentation is loosely borrowed from Eli Ben-Sasson:
https://www.youtube.com/watch?v=9VuZvdxFZQo

A new approach

Setup: Prosper interpolates a degree-999 polynomial P in $\mathbb{Z}_p$ such that $P(i) = f_i$ for $i = 1, ..., 1000$.

Let $D = (X\text{-}1)\cdot(X\text{-}2)\cdot\ldots\cdot(X\text{-}998)$.

$$P(i+2) - P(i+1) - P(i) = 0 \text{ for } i = 1,\ldots,998$$
$$\Leftrightarrow \quad D \text{ divides } P(X+2) - P(X+1) - P(X)$$
$$\Leftrightarrow \quad P(X+2) - P(X+1) - P(X) = D\cdot H \text{ for some } H$$

How Véronique checks that the computation was correct:

- Véronique draws $\alpha \leftarrow \mathbb{Z}_p$ uniformly, computes $D(\alpha)$.
- She asks Prosper for $P(\alpha)$, $P(\alpha+1)$, $P(\alpha+2)$, $H(\alpha)$.
- She accepts computation was correct iff:

$$P(\alpha+2) - P(\alpha+1) - P(\alpha) = D(\alpha)\cdot H(\alpha)$$

Why the approach works

Completeness: if Prosper computed the f_i's correctly, then he can compute $H(\alpha)$ as required.

Soundness: The only requirements for soundness to hold are

- The same polynomial P was used to compute $P(\alpha)$, $P(\alpha+1)$, $P(\alpha+2)$ (as well as $P(1)$, $P(2)$, $P(1000)$);
- P and H have the correct degree (resp. 1000 and 1).

If Prosper computed the $P(i) = f_i$'s incorrectly, then as long as the previous requirements hold, we have:

$$\Pr[\, P(\alpha+2) - P(\alpha+1) - P(\alpha) = D(\alpha) \cdot H(\alpha) \,] \leq 1000/p < 0.01$$

so Véronique will detect the issue with $> 99\%$ probability.

(An implicit assumption here is: H does not depend on α.)

It remains to force Prosper to answer queries honestly.

In particular, soundness argument crucially relies on P, H being bounded-degree polys.

→ need to limit Prosper to computing polys of degree < 1000.

→ A new ingredient: **pairings.**

Quick “reminder”

Fix cyclic group $\mathbb{G} = \langle g \rangle$.

Discrete Logarithm Problem: given g^a for uniform a, compute a.

(Computational) Diffie-Hellman Problem: given (g^a, g^b) for uniform a, b, compute g^{ab}.

In crypto, it is often assumed that these problems are difficult (in the relevant group).

Example of group used in practice: prime subgroup of $\mathbb{Z}_p^*$.

Pairings

Pairings. Let $\mathbb{G} = \langle g \rangle$, $\mathbb{T} = \langle t \rangle$ be two cyclic groups of order p. A map e: $\mathbb{G} \times \mathbb{G} \rightarrow \mathbb{T}$ is a *pairing* iff for all a, b in $\mathbb{Z}_p$,

$$e(g^a, g^b) = t^{ab}.$$

Remarks:

- Definition doesn't depend on choice of generators, as long as $t = e(g,g)$.

- Assume Discrete Log is hard in $\mathbb{G}$, otherwise this is useless. On the other hand, e implies DDH cannot be hard (why?).

- First two groups need not be equal in general.

- Can be realized with $\mathbb{G}$ an elliptic curve, $\mathbb{T} = \mathbb{F}_q^*$.

Encodings

Fix $\mathbb{G} = \langle g \rangle$ of order p.

Encode a value $a \in \mathbb{Z}_p$ as g^a. We will write $[a] = g^a$.

We assume DL is hard → decoding a *random* value is hard. But encoding is deterministic → checking if $h \in \mathbb{G}$ encodes a given value is easy.

Additive homomorphism: given encodings $[a],[b]$ of a and b, can compute encoding of $a+b$: $[a+b] = [a][b]$.

→ can compute $\mathbb{Z}_p$-**linear** functions over encodings.

Idea: a pairing e: $\langle g \rangle \times \langle g \rangle \to \langle t \rangle$ allows computing **quadratic** functions over encodings (at the cost of moving to $\mathbb{T}$).

Keeping Prosper honest, using encodings

First: want to ensure P computed by Prosper is degree ≤ 1000.

Approach:

- Véronique draws evaluation point $\alpha \leftarrow \mathbb{Z}_p$ uniformly at random.

- V. publishes encodings $[\alpha]$, $[\alpha^2]$, ..., $[\alpha^{1000}]$.

→ Prosper can compute $[P(\alpha)]$, because it is a linear combination of the $[\alpha^i]$'s, $i \leq 1000$. But only for $\deg(P) \leq 1000$.
E.g. cannot compute $[\alpha^{1001}]$.

Prosper can compute in the same way $[P(\alpha)]$, $[P(\alpha+1)]$, $[P(\alpha+2)]$, $[H(\alpha)]$.

Remark: Prosper can compute $[(\alpha+1)^i]$ from the $[\alpha^j]$'s for $j \leq i$.

Remaining issues:

1) ensure value “[$P(\alpha)$]”returned by Prosper is in fact a linear combination of [α^i]'s.

2) ensure deg(H) ≤ 1, not 1000.

3) ensure [$P(\alpha)$], [$P(\alpha+1)$], [$P(\alpha+2)$] etc. are from same polynomial.

4) last issue: how does Véronique check the result? Cannot decode encodings.

Dealing with issues (1) and (2)

Goal

1) ensure [$P(\alpha)$] is in fact a linear combination of [α^i]'s.

2) ensure deg(H) ≤ 1, not 1000.

Solution:

V. publishes encodings [α], [α^2], ..., [α^{1000}]...
...and also encodings [γ], [$\gamma\alpha$], [$\gamma\alpha^2$], ..., [$\gamma\alpha^{1000}$] for a uniform γ.

→ Prosper can compute [$P(\alpha)$], and [$\gamma P(\alpha)$], and send them to V.

V. can now use the pairing e to check: $e([P(\alpha)],[\gamma]) = e([\gamma P(\alpha)],[1])$.

The point: if Prosper did not compute [$P(\alpha)$] as linear combination of [α^i]'s, he cannot compute [$\gamma P(\alpha)$]. (Note this is quadratic.)

This is an ad-hoc *knowledge assumption* (true in a generic model).

Goal

1) ensure $[P(\alpha)]$ is in fact a linear combination of $[\alpha^i]$'s.

2) ensure deg(H) ≤ 1, not 1000.

Solution:

V. publishes encodings $[\alpha]$, $[\alpha^2]$, ..., $[\alpha^{1000}]$...

...and also encodings $[\eta]$, $[\eta\alpha]$, for a uniform η.

→ Prosper can compute $[H(\alpha)]$, and $[\eta H(\alpha)]$.

V. can check: $e([H(\alpha)],[\eta]) = e([\eta H(\alpha)],[1])$.

The point: if Prosper did not compute $[H(\alpha)]$ as linear combination of $[\alpha^i]$'s, $i \leq 1$, he cannot compute $[\eta H(\alpha)]$.

Dealing with issue (3)

Goal

3) ensure $[P(\alpha)]$, $[P(\alpha+1)]$, $[P(\alpha+2)]$ etc. are from same polynomial.

Solution:

Let's deal with $[P(\alpha)]$, $[P(\alpha+1)]$.

V. publishes $[\theta]$, $[\theta((\alpha+1)^2-\alpha^2)]$, ..., $[\theta((\alpha+1)^{1000}-\alpha^{1000})]$ for a uniform θ.

→ Prosper can compute $[\theta(P(\alpha+1)-P(\alpha))]$.

V. can check: $e([\theta(P(\alpha+1)-P(\alpha))],[1]) = e([P(\alpha+1)-P(\alpha)],[\theta])$.

The point: if Prosper did not compute $[P(\alpha)]$, $[P(\alpha+1)]$ with same coefficients, he cannot compute $[\theta(P(\alpha+1)-P(\alpha))]$.

Checking divisibility

Summary of 3 previous slides: we have forced Prosper to compute $[P(\alpha)]$, $[H(\alpha)]$, ... as polys of correct degree.

Remains to check $P(\alpha+2)$-$P(\alpha+1)$-$P(\alpha) = D(\alpha)\cdot H(\alpha)$, using the encodings.

No problem! this is a **quadratic equation.** Check:

$$e([P(\alpha+2)\text{-}P(\alpha+1)\text{-}P(\alpha)],[1]) = e([D(\alpha)],[H(\alpha)])$$

Conclusion. Since $P(\alpha)$, $H(\alpha)$ etc are polys of right degree, original argument applies: checking equality at random α ensures with $\geq 1\text{-}1000/|\mathbb{Z}_p| > 99\%$ probability the equality is true on the whole polys → D divides $P(\alpha+2)$-$P(\alpha+1)$-$P(\alpha)$ → computation was correct.

Efficiency

Prosper proves correct computation by providing a **constant number** of encodings: $[P(\alpha)]$, $[\gamma P(\alpha)]$, $[H(\alpha)]$, $[\eta H(\alpha)]$ etc.

#encodings is absolute constant, independent of circuit size.

Pre-processing by Véronique was still linear in circuit size: publishes $[\alpha^i]$, $i \leq 1000$, etc. But...

- Can be amortized over many circuits.

- Exist “fully succint” SNARKs, with O(log(circ. size)) verifier pre-processing.

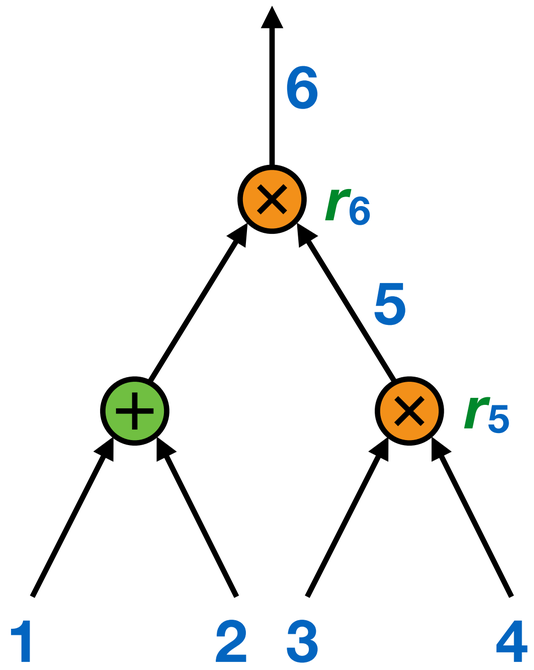
Working with circuits directly

In essence: we have seen how to do a succint proof of polynomial divisibility.

Can in principle encode valid machine state transitions as polynomial constraints → succint proofs for circuit-SAT.

Now: want to do that more concretely = get SNARKs for circuit-SAT (directly).

We are going to encode a circuit as polynomials.



For simplicity, forget about negations. Write circuit with ⊕ (XOR), ⊗ (AND) gates. Then:

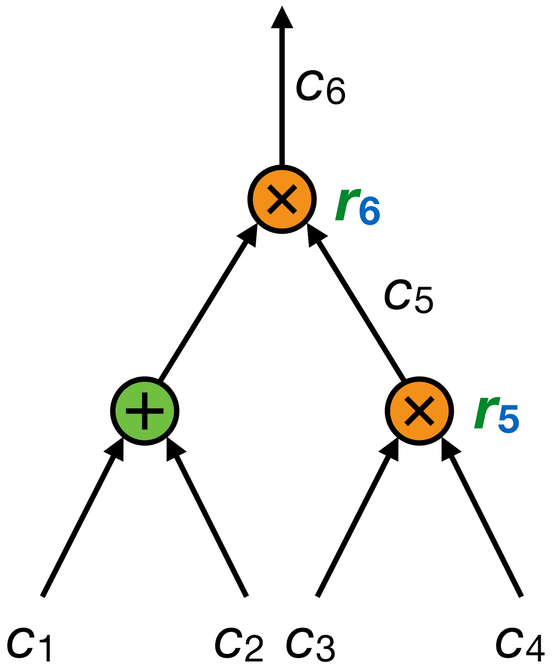
1) Associate an integer **i** to each input; and to each output of a mult gate ⊗.

2) Associate an element $r_i \in \mathbb{F}_q$ to mult gate **i**.

Now circuit can be encoded as polys. For each **i** = **1**,...,**6**, define polynomials $\mathbf{v}_i$, $\mathbf{w}_i$, $\mathbf{y}_i$:

- $\mathbf{v}_i(r_j)$=1 if value **i** is *left input* to gate **j**, 0 if not.
- $\mathbf{w}_i(r_j)$=1 if value **i** is *right input* to gate **j**, 0 if not.
- $\mathbf{y}_i(r_j)$=1 if value **i** is *output* of gate **j**, 0 if not.

Exemple.



In this case, $\mathbf{v}_i$, $\mathbf{w}_i$, $\mathbf{y}_i$ are degree 2.

Encoding mult gate **5**:

- $\mathbf{v}_3(r_5)=1$, $\mathbf{v}_i(r_5)=0$ otherwise.
- $\mathbf{w}_4(r_5)=1$, $\mathbf{w}_i(r_5)=0$ otherwise.
- $\mathbf{y}_5(r_5)=1$, $\mathbf{y}_i(r_5)=0$ otherwise.

Encoding mult gate **6**:

- $\mathbf{v}_1(r_6)=\mathbf{v}_2(r_6)=1$, $\mathbf{v}_i(r_6)=0$ otherwise.
- $\mathbf{w}_5(r_6)=1$, $\mathbf{w}_i(r_6)=0$ otherwise.
- $\mathbf{y}_6(r_6)=1$, $\mathbf{y}_i(r_6)=0$ otherwise.

The point: an assignment of variables c_1, ..., c_6 satisfies the circuit iff:

$$(\Sigma c_i \mathbf{v}_i(r_5)) \cdot (\Sigma c_i \mathbf{w}_i(r_5)) = \Sigma c_i \mathbf{y}_i(r_5) \quad \text{and} \quad (\Sigma c_i \mathbf{v}_i(r_6)) \cdot (\Sigma c_i \mathbf{w}_i(r_6)) = \Sigma c_i \mathbf{y}_i(r_6)$$

Equivalently:

$$(X-r_5)(X-r_6) \text{ divides } (\Sigma c_i \mathbf{v}_i) \cdot (\Sigma c_i \mathbf{w}_i) - \Sigma c_i \mathbf{y}_i$$

→ we have reduced:

“Prosper wants to prove he knows inputs satisfying a circuit.”

into:

“Prosper wants to prove he knows linear combinations $V = \mathbf{\Sigma} c_i \mathbf{v_i}$, $W = \mathbf{\Sigma} c_i \mathbf{w_i}$, $Y = \mathbf{\Sigma} c_i \mathbf{y_i}$, *such that* $T = (X-\mathbf{r_5})(X-\mathbf{r_6})$ *divides* $VW-Y$.”

$$\Leftrightarrow \exists\, H,\ \ T \cdot H = V \cdot W - Y$$

1. quadratic!
2. polynomial equality!

We know how to do that!

V. publishes $[\alpha^i]$, plus auxiliary $[\gamma\alpha^i]$ etc... (at setup, indep. of circuit)

P.'s proof is $[V(\alpha)]$, $[W(\alpha)]$, $[Y(\alpha)]$, $[H(\alpha)]$, plus auxiliary $[\gamma V(\alpha)]$ etc...

V. checks $e(T(\alpha),H(\alpha))=e([V(\alpha)],[W(\alpha)])e([Y(\alpha)],[1])^{-1}$ and auxiliary stuff.

Constant-size proof. Construction works for any circuit.

Succint Zero Knowledge?

Proof: ([$V(\alpha)$], [$W(\alpha)$], [$Y(\alpha)$], [$H(\alpha)$]) (+ auxiliary values)

Is it ZK?

- ZK not needed for “delegation of computation” application.
- But needed for other applications.

Zero Knowledge

Proof: ([$V(\alpha)$], [$W(\alpha)$], [$Y(\alpha)$], [$H(\alpha)$]) (+ auxiliary values)

Validity check: $T \cdot H = V \cdot W\text{-}Y$ at point α (+ auxiliary tests)

(That is, concretely we check: $e([T(\alpha)],[H(\alpha)]) = e([V(\alpha)],[W(\alpha)])e([Y(\alpha)],[1])^{-1}$)

Natural idea to build ZK simulator:

Pick proof (a,b,c,d) uniformly among values that verify the validity check.

Problem: cannot argue indistinguishability.

Solution: modify proof so that the above simulator is indistinguishable.

zk-SNARK

$[V'] = [V + d_V T]$ $\quad d_V \leftarrow_\$ \mathbb{Z}_p$

$[W'] = [V + d_W T]$ $\quad d_W \leftarrow_\$ \mathbb{Z}_p$

$[Y'] = [Y + d_Y T]$ $\quad d_Y \leftarrow_\$ \mathbb{Z}_p$

$[H'] = [(V' \cdot W' - Y')/T]$ $\quad$ well-defined: $T \mid V' \cdot W' - Y'$, because $T \mid V \cdot W - Y$.

Old proof: $([V(\alpha)], [W(\alpha)], [Y(\alpha)], [H(\alpha)])$

New proof: $([V'(\alpha)], [W'(\alpha)], [Y'(\alpha)], [H'(\alpha)])$

Validity check: unchanged!

New proof is ZK because it is a **uniform** among 4-tuples that satisfy validity*.

⚠️ Above statement assumes $[\alpha^i]$'s + auxiliary values were built honestly.

Remark: also, need to adapt auxiliary checks.

* assuming $T(\alpha) \neq 0$, which holds except with negligible probability. Building simulator left as exercise.

Subversion of preprocessing

Q: How to enforce the $[\alpha^i]$'s + auxiliary values were built honestly?

A: check $[\alpha^i][\alpha] = [\alpha^i]$ for all i.

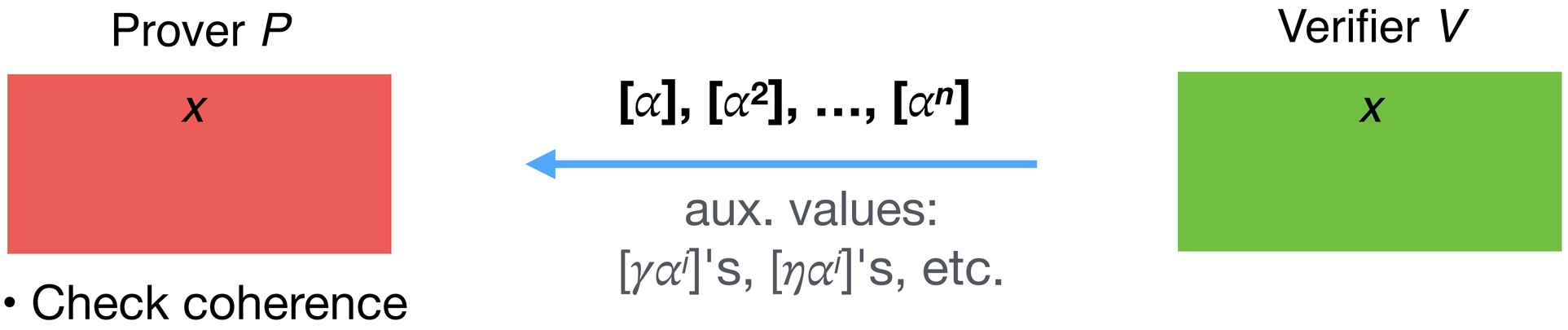
Same idea for auxiliary values: check $[\alpha^i][\gamma] = [\gamma\alpha^i]$ for all i, etc.

Q: What if the pairing is asymmetric ($\mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{T}$ instead of $\mathbb{G} \times \mathbb{G} \to \mathbb{T}$)?

A: Publish $[\alpha^i]$'s in both groups. Can then check as above.

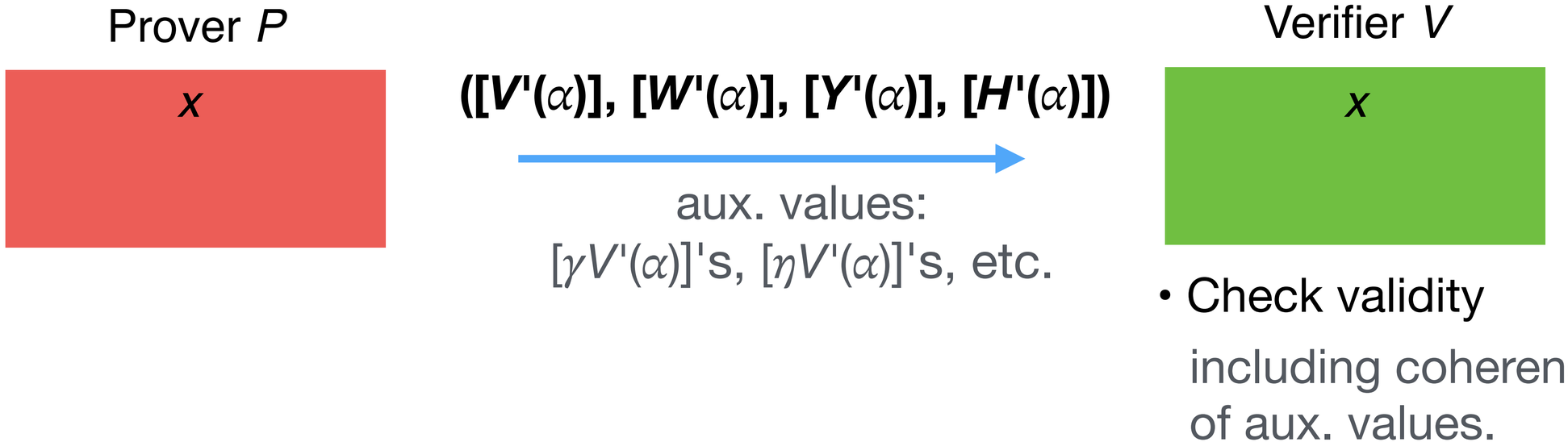
Historical note: for fully succint SNARKs subversion of preprocessing is a non-trivial issue. Initial setup required "trusted setup" → "ceremonies".

Recap: the scheme



Preprocessing

Proof(s)



Recap: the ideas

Key step: convert target computation/language into polynomials.

Why? Polynomials...

- ...enable “key lemma” about checking equality at single point (“minimal distance” property). Key in “succint proof”-style results.
- ...enable enforcing honesty efficiently, here via encodings + pairings.

Even better: arrange that checking proof = checking quadratic equation.

Enables use of pairings. More specific to this approach.

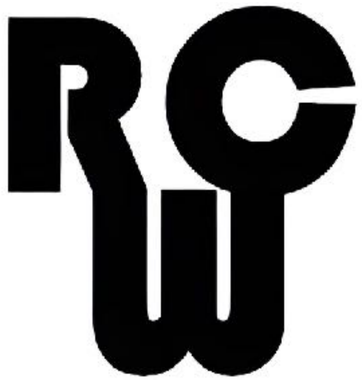
In practice

Construction was proposed in Pinocchio scheme (Parno et al. S&P 2013).

Practical: proofs ~ 300kB, verification time ~ 10 ms.

- Introduced for verifiable outsourced computation.
- Zero-knowledge variant (built as seen earlier).
- Further improvements since.

Applications



Real-World Crypto

Application #1: e-Voting

e-Voting

Are going to see (more or less) Helios voting system.

https://heliosvoting.org/

Used for many small- to medium-scale elections.
Including IACR (International Association for Cryptologic Research).

We will focus on yes/no referendum.

Nice description of Belenios variant: https://hal.inria.fr/hal-02066930/document

Goals

We want:

- Vote privacy
- Full verifiability:
  - Voter can check their vote was counted
  - Everyone can check election result is correct

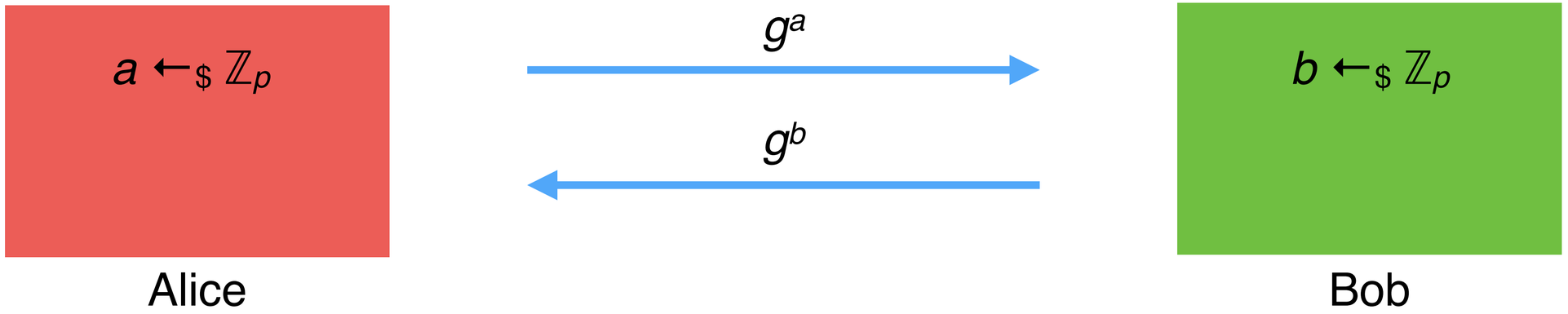
    Every voter cast ≤1 vote, result = number of yes votes

We do not try to protect against:

- Coercion/vote buying

Quick reminder: Diffie-Hellman key exchange

$\mathbb{G} = \langle g \rangle$, $|\mathbb{G}| = p$



Afterwards: Alice & Bob can both compute g^{ab}.

But computing g^{ab} is hard for external observer (DDH problem).

Quick reminder: ElGamal Encryption

$\mathbb{G} = \langle g \rangle$, $|\mathbb{G}| = p$

Secret key: $x \in \mathbb{Z}_p$

Public key: $h = g^x$

Enc$(m) = (g^r, m \cdot h^r)$, where $r \leftarrow_\$ \mathbb{Z}_p$

Dec$(k,c) = c/k^x$

Security reduces to DDH assumption over $\mathbb{G}$.

Multiplicatively homomorphic: given $(k,c) = \text{Enc}(m)$, $(k',c') = \text{Enc}(m')$, (kk',cc') is an encryption of mm'.

Basics

Election = want to add up encrypted votes...

→ just use additively homomorphic encryption!

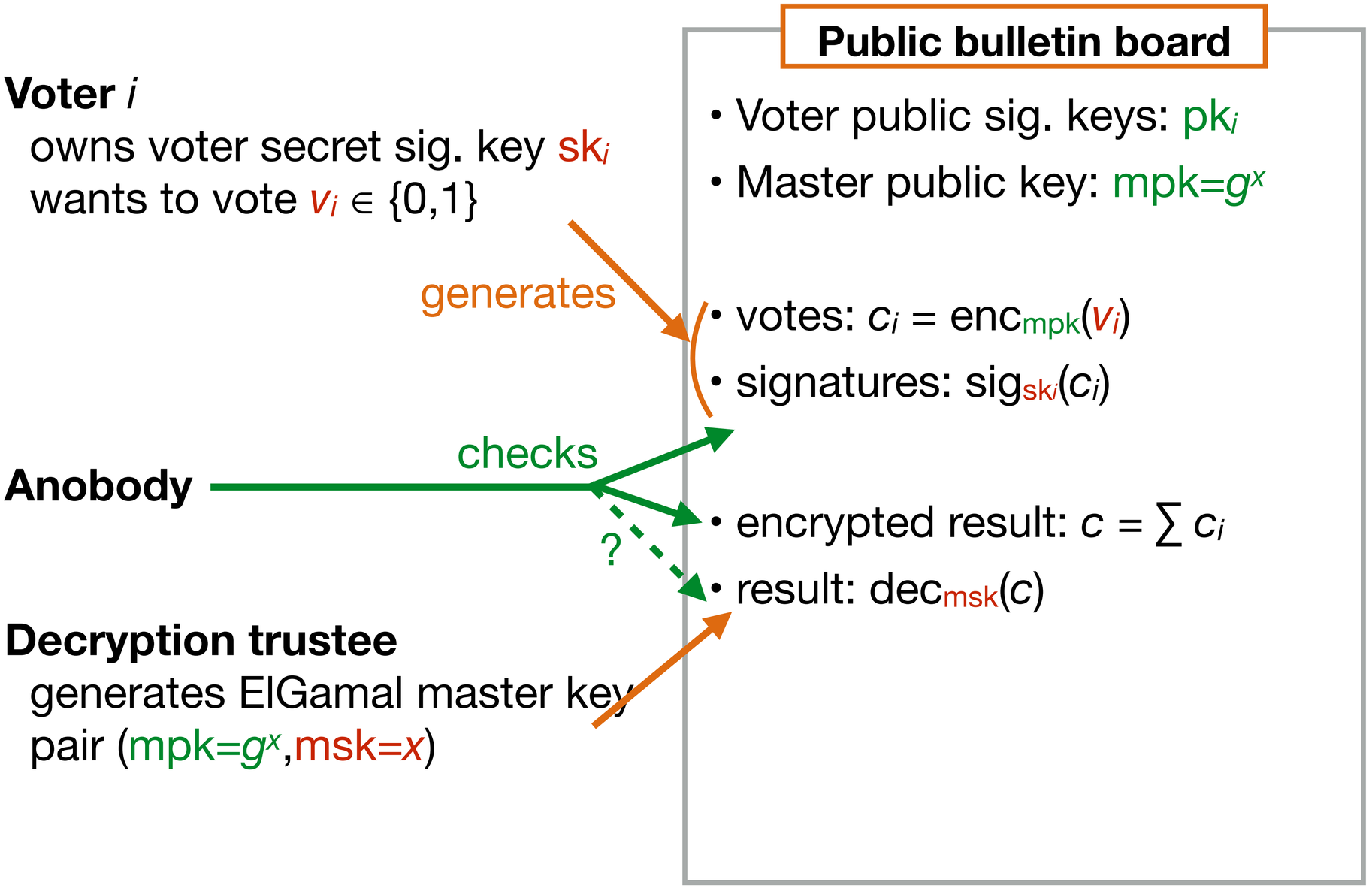
Helios: use ElGamal. Multiplicatively homomorphic.
To make it additive: vote for v is g^v.
Recovering v from g^v is discrete log, but brute force OK (v small).

In addition: voters sign their votes.

Helios: Schnorr signatures.

Who decrypts the result?

First attempt



Problem: how to verify final result.

Making election result verifiable

ElGamal encryption:

Master keys: (mpk=g^x,msk=x)

Encrypted election result $c = (c_L = g^k, c_R = m \cdot g^{xk})$

Election result = Dec(c) = $m = c_R / c_L^x$

→ giving decryption is same as giving c_L^x

→ to prove decryption is correct, prove:

discrete log of $(c_L)^x$ in base c_L = discrete log of mpk=g^x in base g

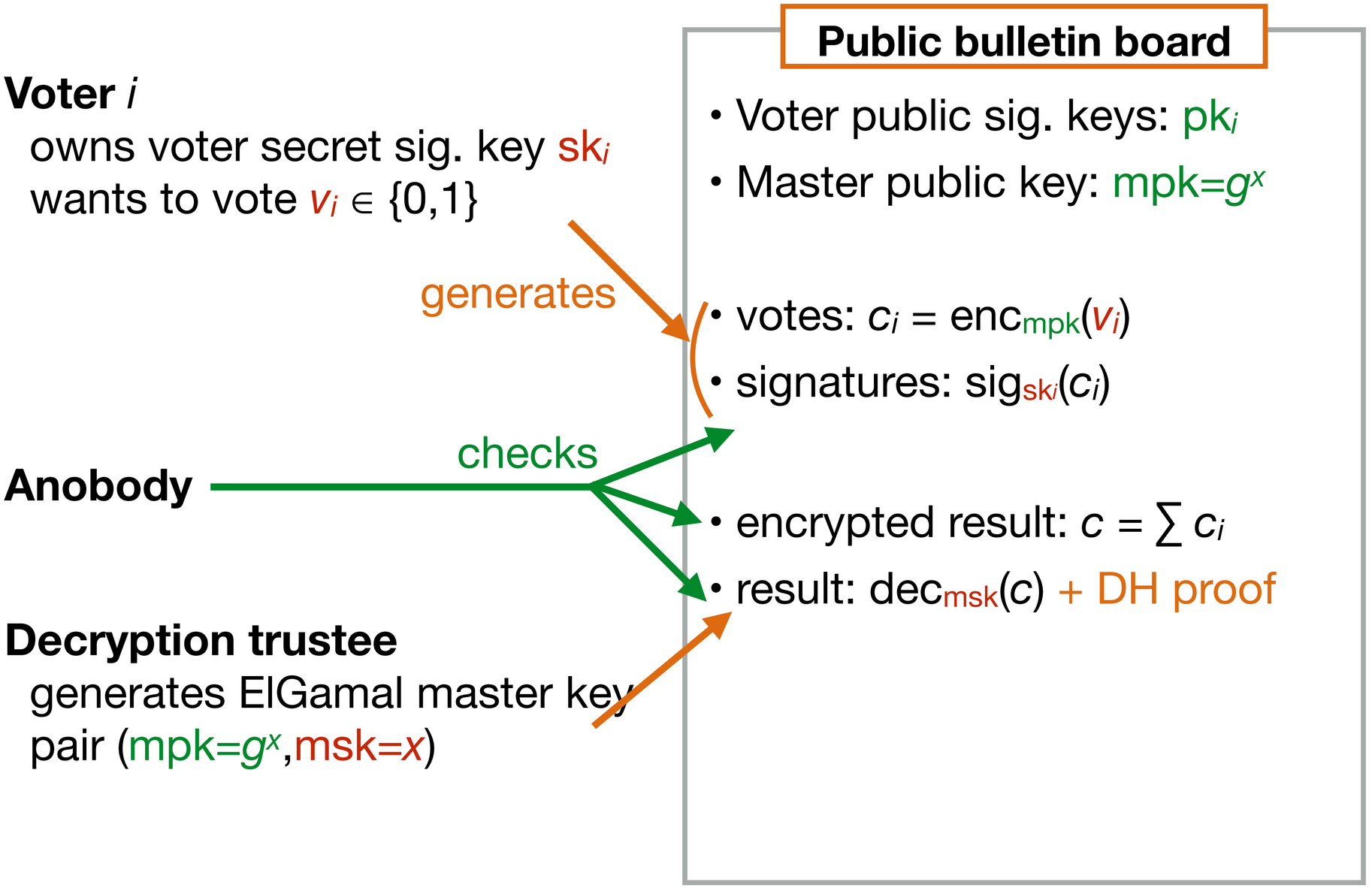
$\Leftrightarrow (g, g^x, c_L, c_L^x) \in$ Diffie-Hellman language

→ to make election result verifiable: decryption trustee just provides NIZK proof of DH language for (g, g^x, c_L, c_L^x)!

Take ZK proof of DH language from earlier + Fiat-Shamir → NIZK

Note ZK property is crucial.

Now with verifiable election result



Problem 2: how about I vote $enc_{mpk}(1000)$?

Proving individual vote correctness

In addition to vote $\text{enc}_{\text{mpk}}(v_i)$ and signature $\text{sig}_{\text{sk}_i}(c_i)$, voter provides NIZK proof that $v_i \in \{0,1\}$.

Helios doesn't use SNARK here, but more tailored proof of disjunction.

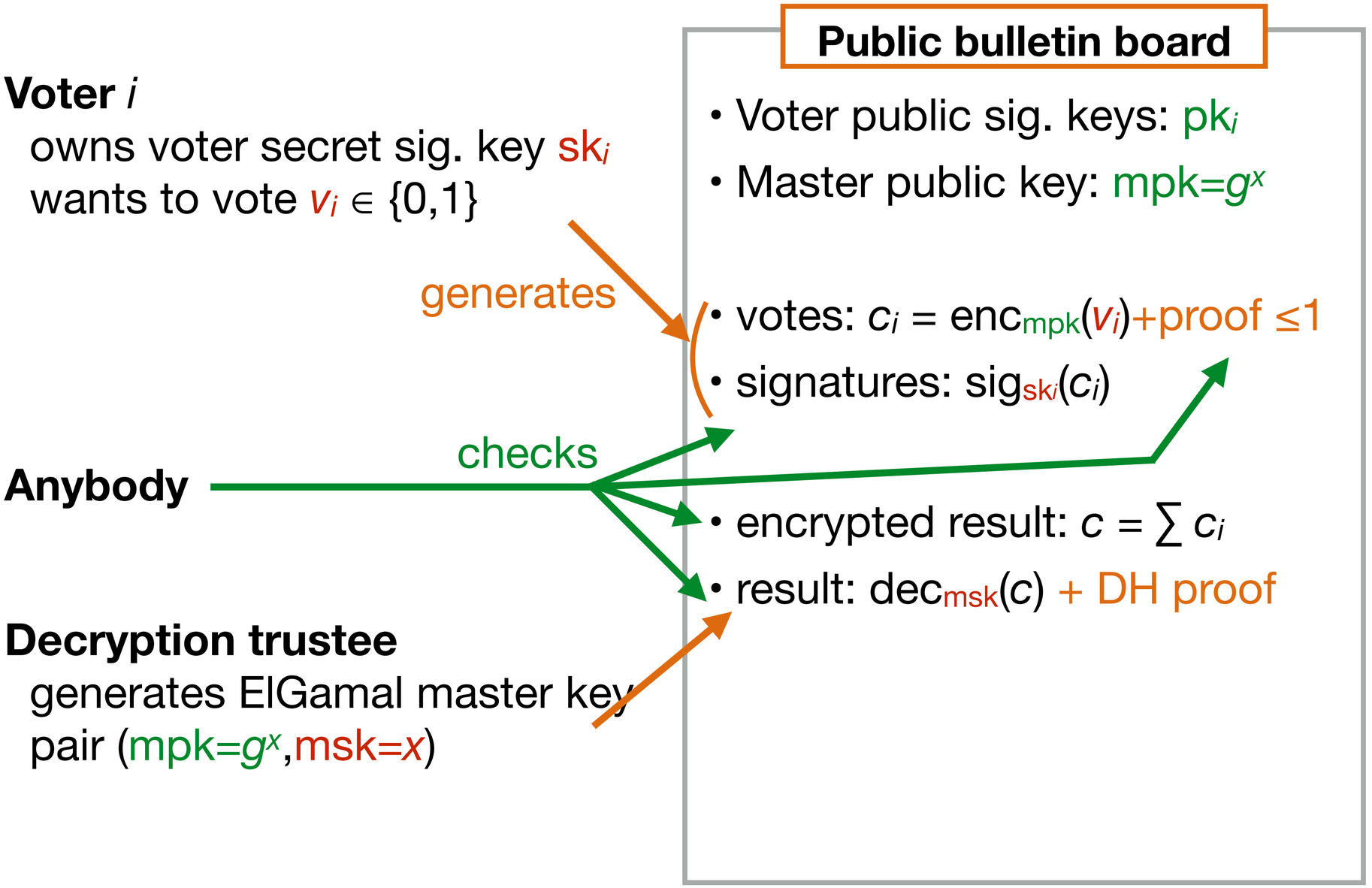
Note ZK property is crucial again.

To prevent “weeding attack” (vote replication):

NIZK proof includes g^k, pk_i in challenge randomness (hash input of sigma protocol), where g^k is the randomness used in $\text{enc}_{\text{mpk}}(v_i)$.

→ proof (hence vote) cannot be duplicated without knowing sk_i.

Now with full verifiability



Bonus problem: replace decryption trustee by threshold scheme.

Application #2: Anonymous Cryptocurrencies

Cryptocurrencies

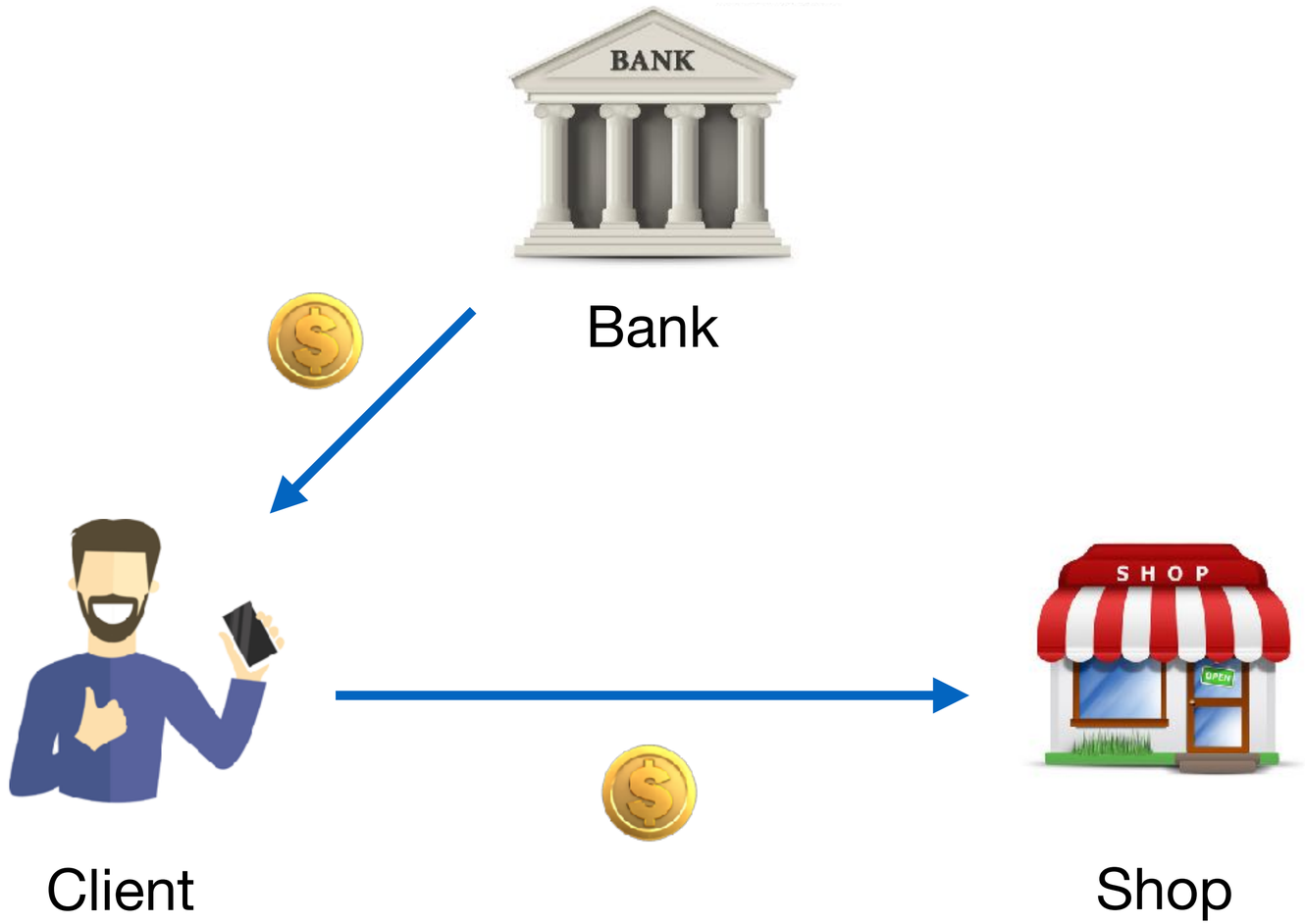
Electronic Money: credit cards etc. *Traceable* + *Central authority* (bank).

≠ Electronic Cash: not traceable.

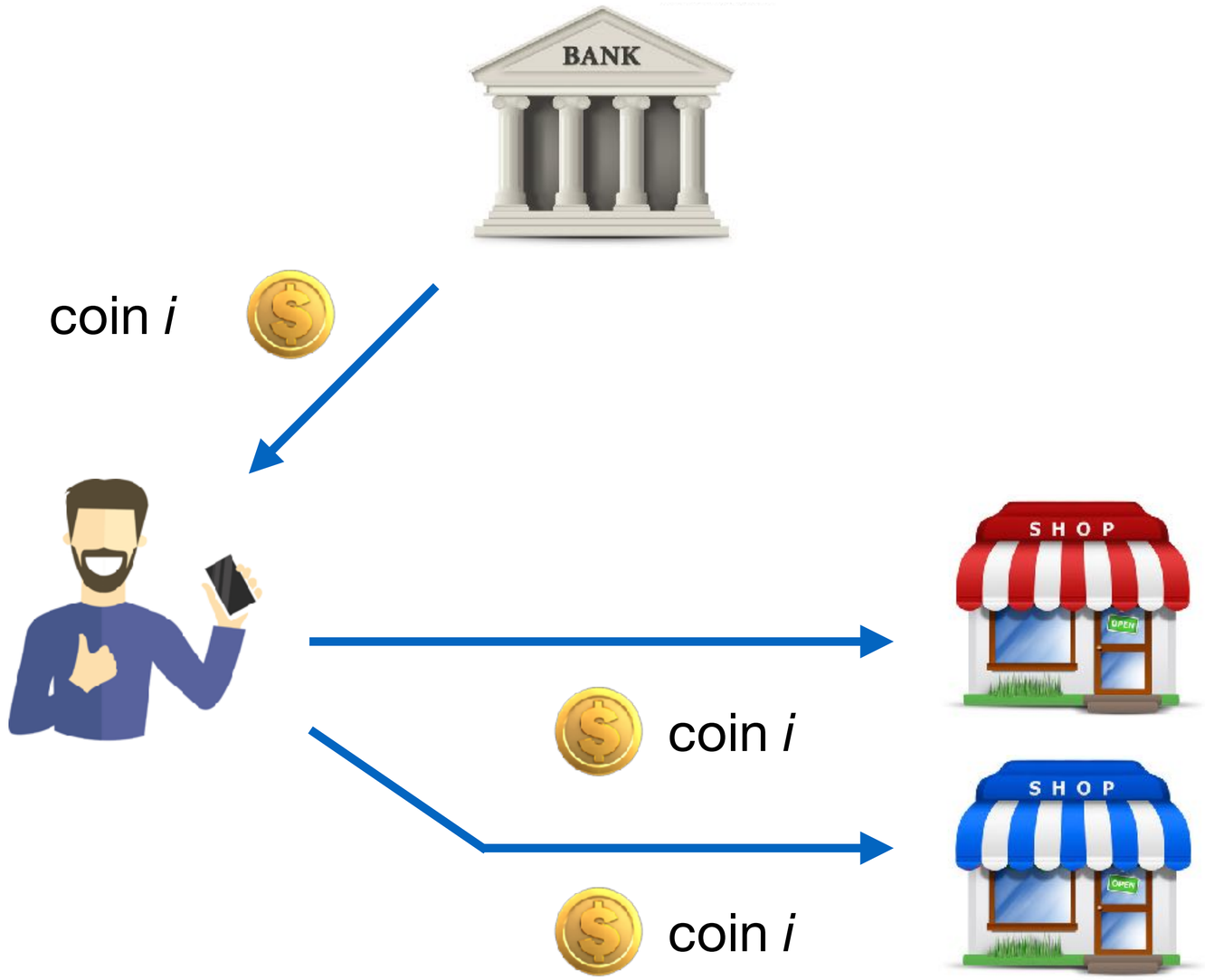
≠ Cryptocurrencies: no central authority*.

*in principle.

Normal life

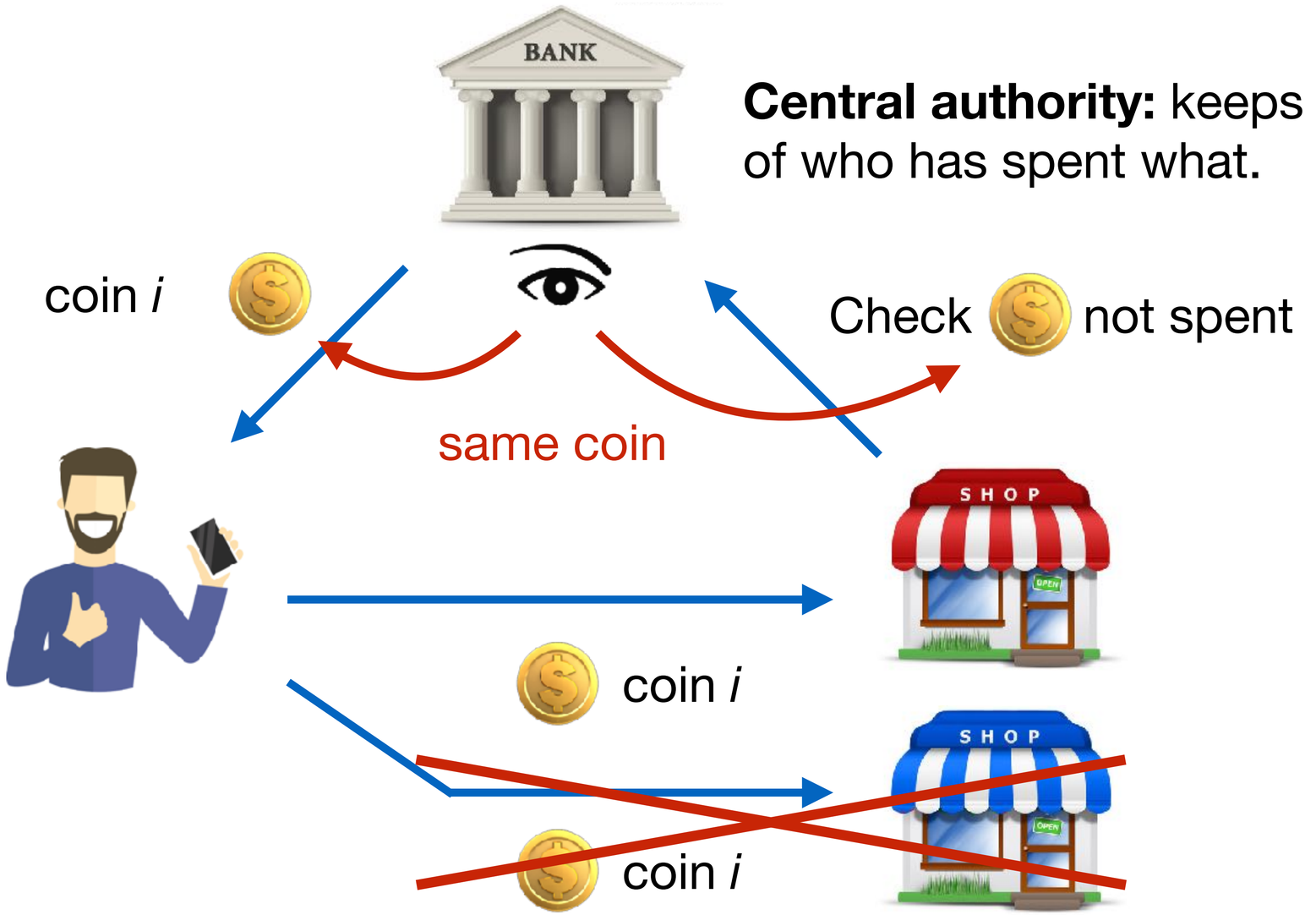


Problem: double spending



Fundamental problem with electronic money.

Central Authority solution



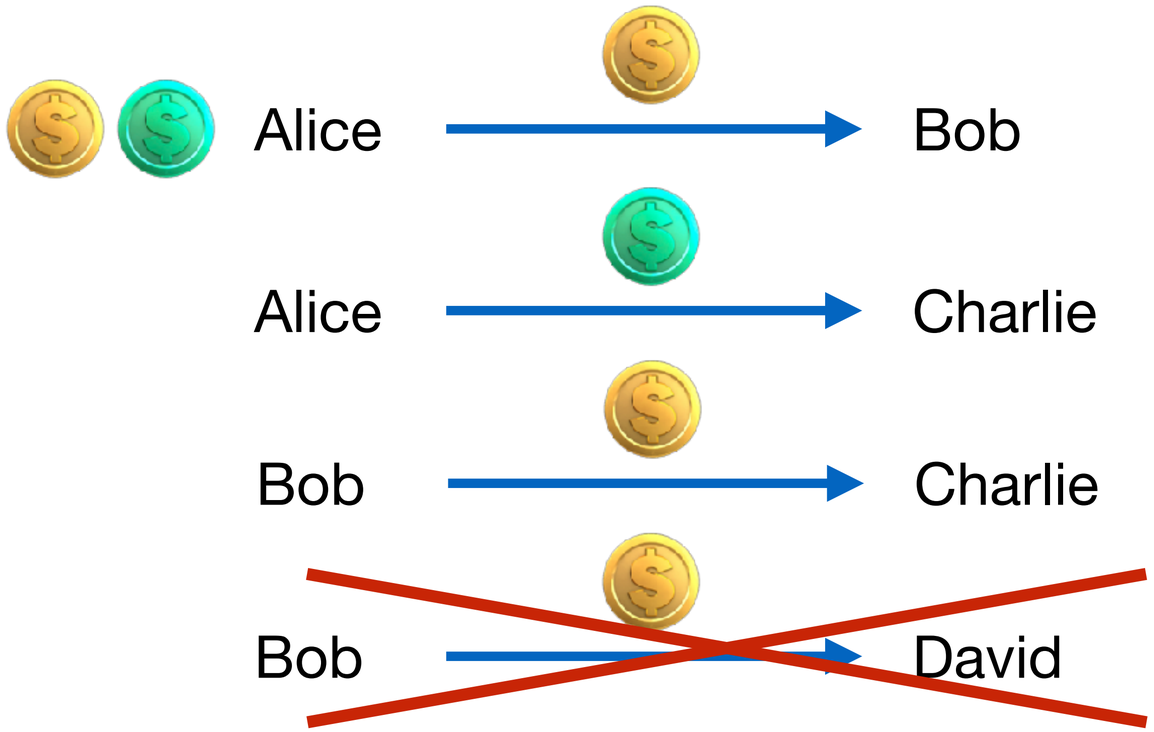
Central authority: keeps ledger of who has spent what.

Cryptocurrency solution: public ledger

No bank → who checks validity of transactions? (no double spending)

Idea: just publish all transactions! Everybody can check.

Public ledger:



Public ledger

How to prevent people from writing any transaction they want?

An account is a (public key, secret key) pair for signature scheme.

Pseudo-anonymity: account is just a key.

Alice: (pk_A, sk_A).

Bob: (pk_B, sk_B).

Ledger: $pk_A \rightarrow pk_B$

$+ \text{sign}_{sk_A}(pk_A \rightarrow pk_B)$

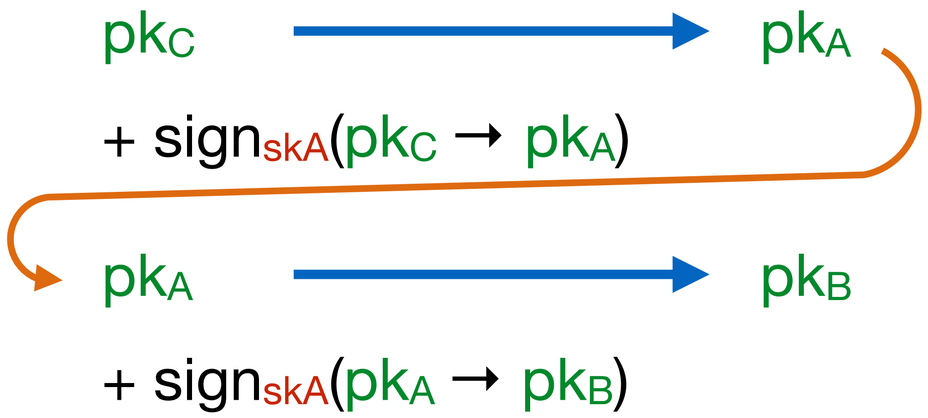
Accounts

Ledger: $pk_A \rightarrow pk_B$

$+ \text{sign}_{skA}(pk_A \rightarrow pk_B)$

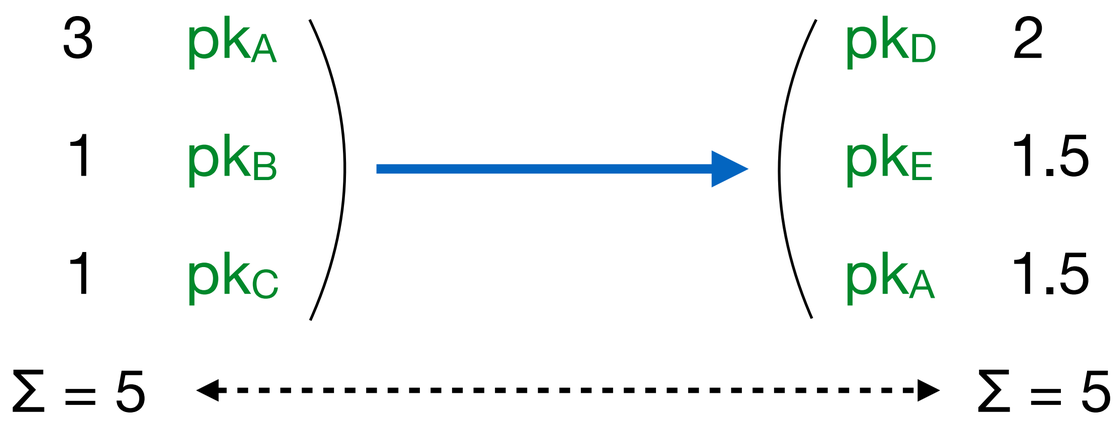
How do you know pk_A has the money?

Comes from previous transaction (**tx**) in the ledger (chain).



Fungibility

One transaction:

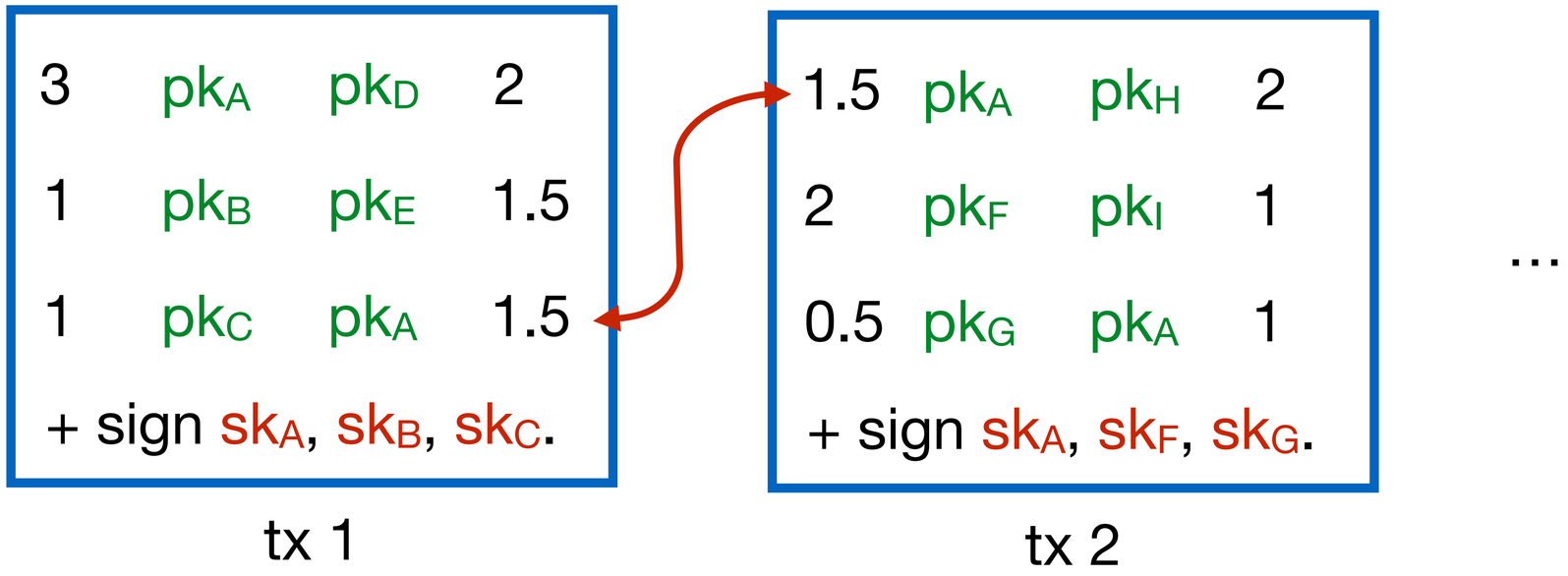


+ signatures with sk_A, sk_B, sk_C.

Payback: pk_A is giving the change back to itself.

Public ledger, revisited

Ledger is a chain of transactions.

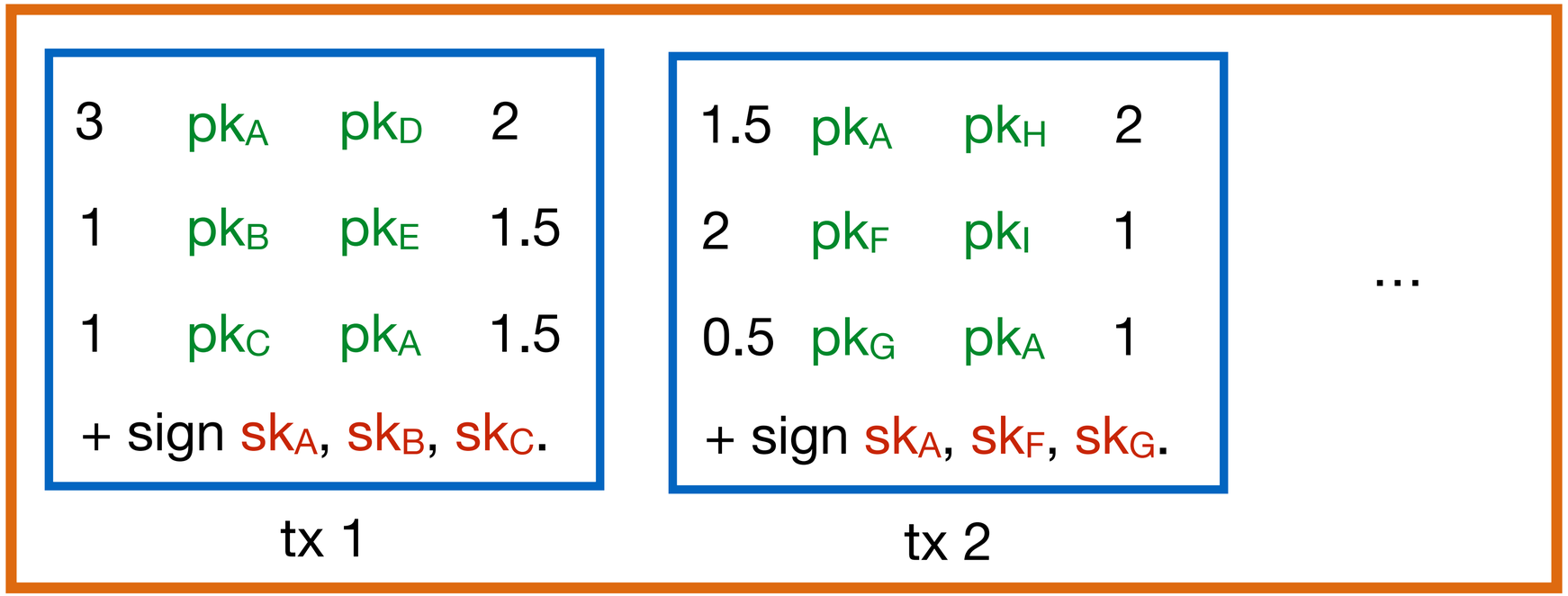


No real notion of account: every tx input links to previous unspent tx output (utxo).

To receive money, user can create new “account” (pk, sk) as destination, for every tx.

The blockchain

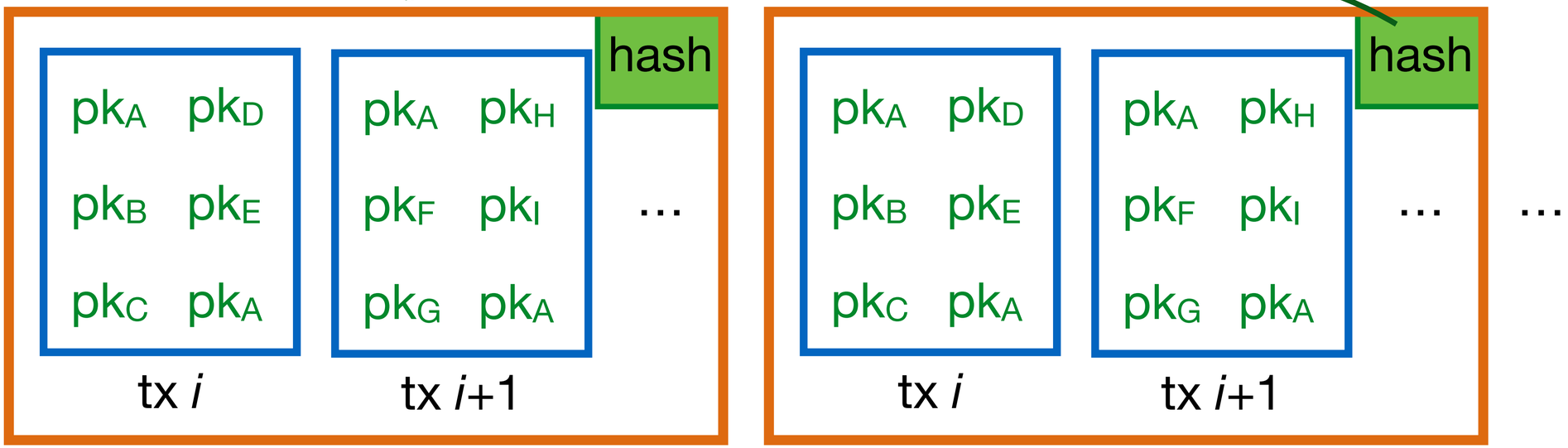
Transactions are arranged into blocks.



One block

The blockchain

Blocks are arranged into a chain.

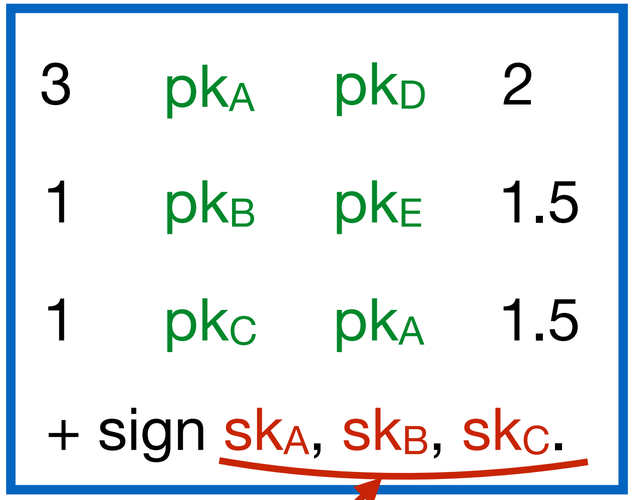


Each new block contains hash(previous block).

Bitcoin and anonymity

Whole transaction graph is public!

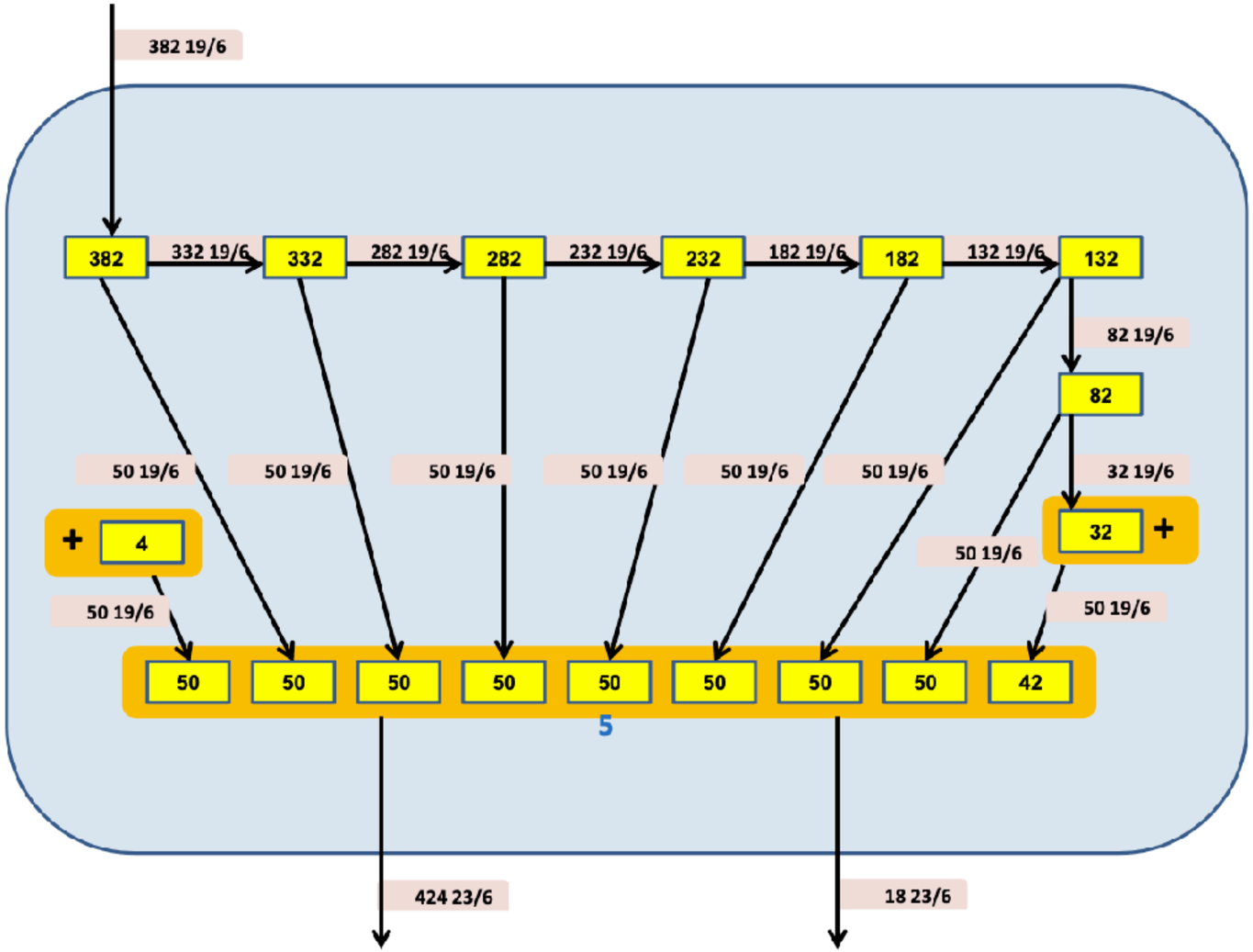
Can trace transactions. See e.g. Ron and Shamir 2012.



Probably same person

Bitcoin and anonymity

Suspicious activity.



Normal transaction

3	pk_A	pk_D	2
1	pk_B	pk_E	1.5
1	pk_C	pk_A	1.5

+ sign sk_A, sk_B, sk_C.

Encrypt is public: identities (public keys), amounts.

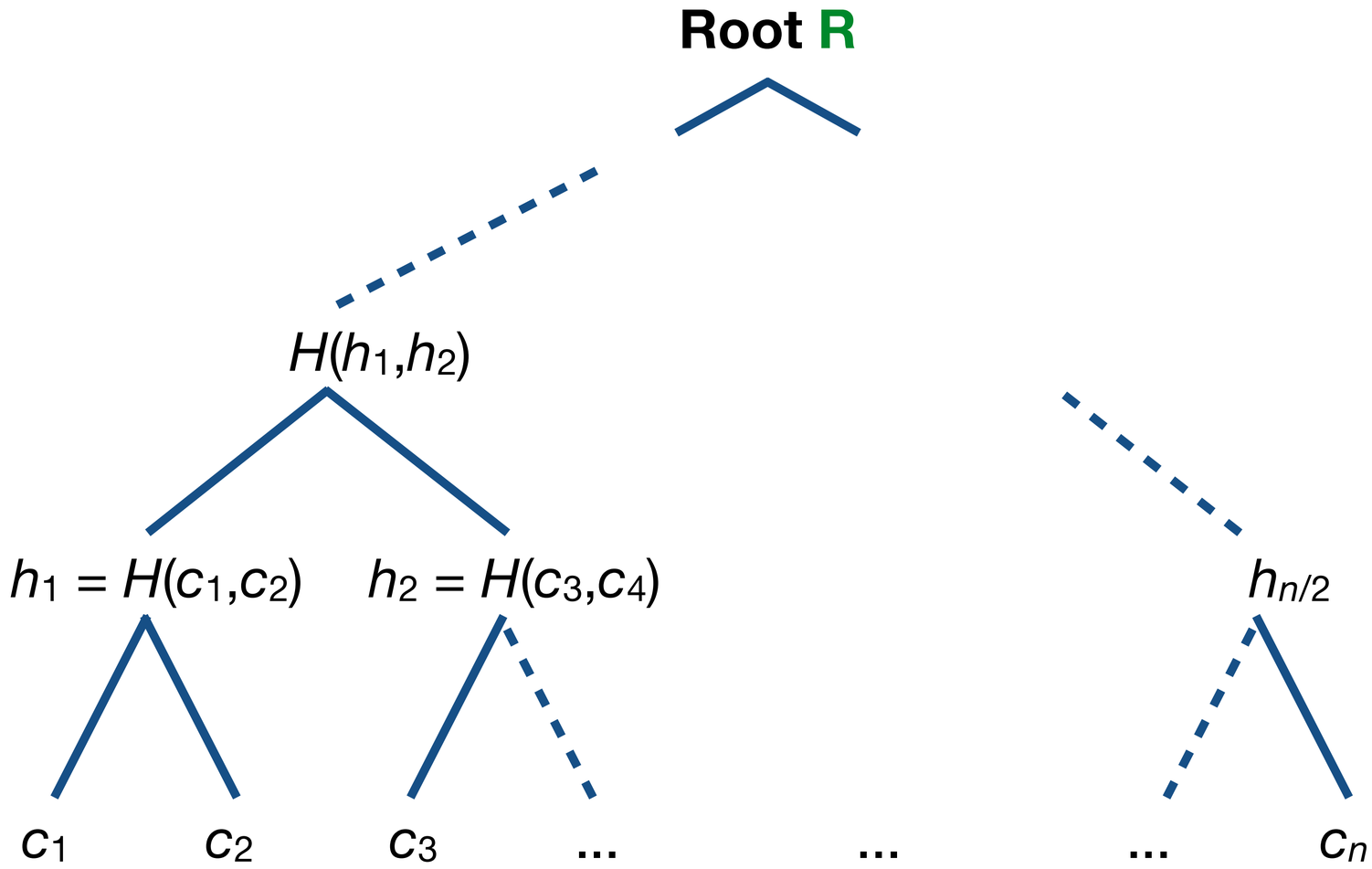
Hidden transaction (Zcash)

Everything is encrypted: identities, amounts.

Need to check:

- Sum of inputs = sum of outputs. ← **ZK proof (w/ homomorphism)** ✔
- Quantities are positive. ← **ZK proof (“range proof”)** ✔
- Spender knows secret key for each spender ID . ← **zk-SNARK** ✔
- spender IDs correspond to previous unspent tx's. ← **?**

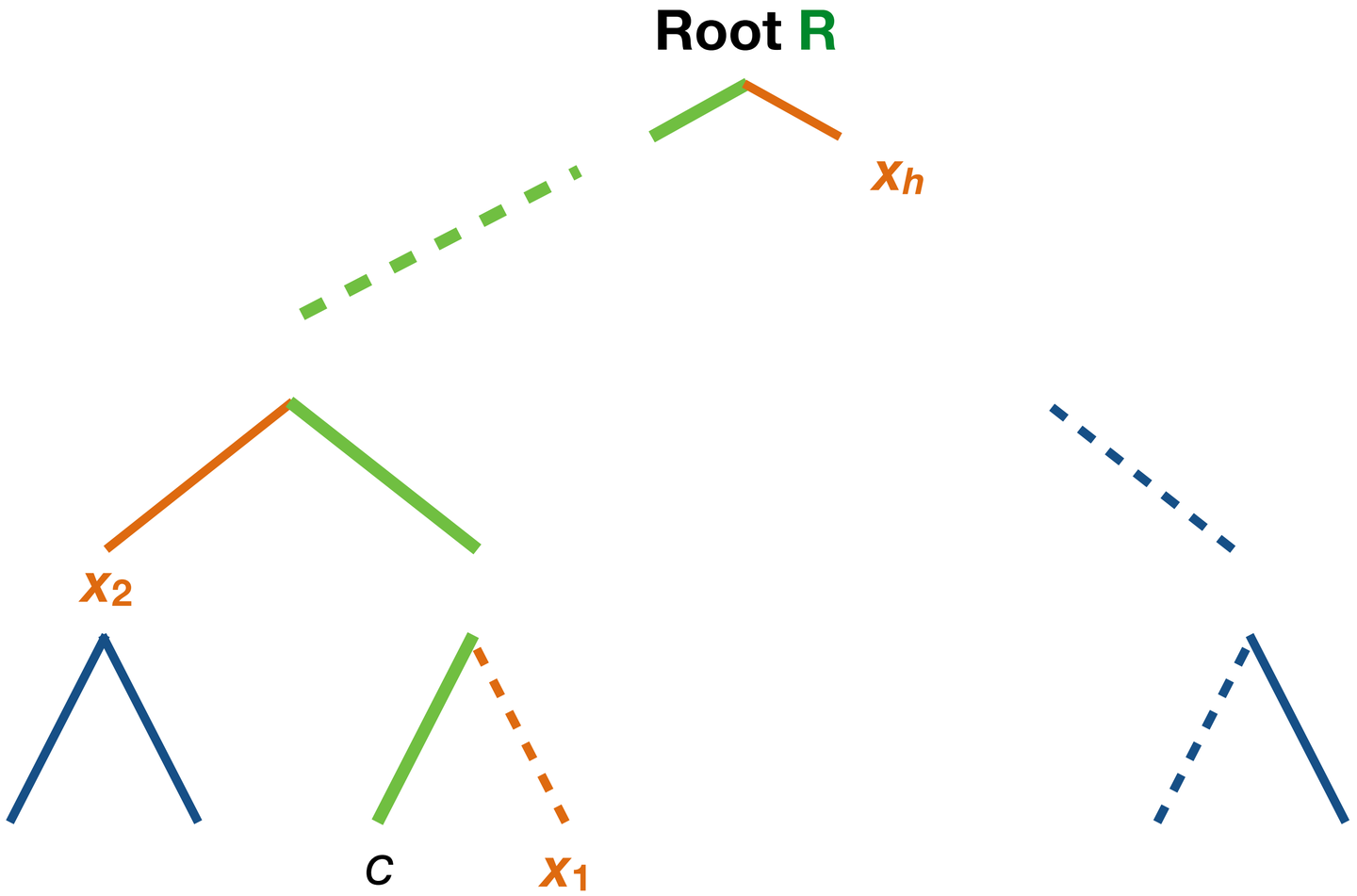
Interlude: Merkle trees



Leaves of the Merkle tree are (all) commits c_1, …, c_n.

Each **internal node** of the Merkle tree is a hash of its children.

Proving an element is in a set...



To prove *c* is a leaf, without revealing location, prove:

"There exist x_1, …, x_h such that **R** = H(x_h, …, H(x_2,H(x_1,c))…)."

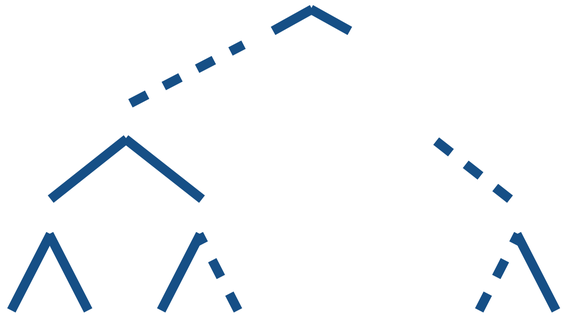
where H is collision-resistant (with symmetric inputs).

Linking transaction to unspent tx's

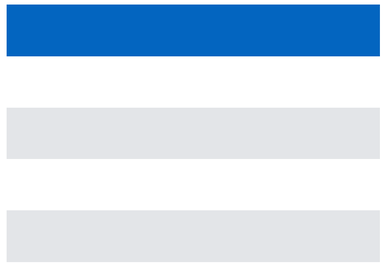
Scheme (simplified):

- Every tx contains **commit** = Hash(recipient, tx-ID, randomness).
- Every tx publishes **nullifier** = Hash(spending key, tx-ID) for the tx-ID of previous tx it spends from.

Blockchain stores:



T: Merkle tree of **commits**



L: List of **nullifiers**

When a new tx is issued:

- Check ZK proof (included in tx) that ∃ **commit** for spender in **T**.
- Check **nullifier** is new (not in **L**), then add it to **L**.
- Check ZK proof (included in tx) that **commit** and **nullifier** are for same tx-ID.

Non-application: Annoying mathematicians

Please don't.

Theorem proving vs NP

Let's encode theorem statements in some way: Encoding of T is $[T]$.

$$\mathcal{L} = \{([T], 1^B): \text{there exists a proof of } T \text{ of length at most B}\}$$

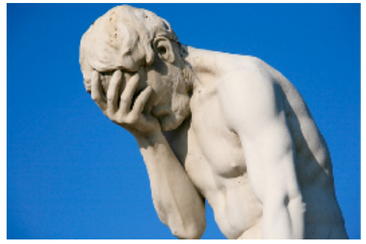
This language is NP-complete. (Exercise)

→ If you prove the Riemann hypothesis, can in theory publish a succint ZK proof of knowledge. Only information revealed:

- You know a (correct) proof.
- An upper bound on the size of your proof.
- No information about the proof whatsoever. 👍

Tip: To maximize annoyance, publish ZK proof of knowledge for “Riemann Hypothesis is true OR Syracuse conjecture is true”.

Fin



Bonus question: If you read a proof of “P = NP” using the zk-SNARK from this course, can you deduce that P = NP?