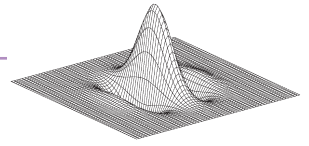


Wavelets for compressing images



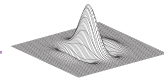
*Whether they are stored digitally in computer memories,
or they travel over the Internet, images take up a lot of space.
Fortunately, it is possible to "condense"
them without changing the quality!*



Figure 1. These three images illustrate the power of current compression methods. The original image (A) consists of 512×512 points, each of which has a certain level of gray taken from a palette of 256 levels. Image (B) is the result of a compression by a factor 8, obtained by reducing the levels of gray to 2 possible values only (black or white). Image (C) was obtained from (A) upon compressing by a factor 32 by using a wavelet basis. The difference in quality from the initial image is hardly perceptible. (Illustration by the author)

a digital image can be compressed, just as orange juice can be reduced to a few grams of concentrated powder. This is not by sleight of hand, but by mathematical and computer science techniques making it possible to reduce the amount of space occupied by an image in computer memory, or in communication cables. Nowadays, these techniques are essential for storing information, or for transmitting it by Internet, telephone, satellite or any other means.

The compression of an image amounts to representing it using a smaller number of parameters, by eliminating redundancies. An exaggerated example will help in understanding the basic principle: in the case of a uniformly white image, it is unnecessary to explicitly specify for each one of its points the grey level at that point; that would take much more space than to simply state: "all the points of the image are white". The problem of representation is central in mathematics, and its



applications go well beyond data compression. During the last ten years, considerable progress has been made thanks to the development of the theory of wavelets. In the field of image processing, this progress has led to the adoption of the new standard of compression JPEG-2000. This is a meandering tale, which well illustrates the role of mathematics in the modern scientific, or technological landscape.

Thirty-two times less space thanks to wavelets

Let us consider an image such as that of Figure 1A. It consists of 512×512 points, whose levels of grey can vary from 0 (black) to 255 (white). Each of the 256 possible levels of grey can be represented by a byte, i.e., a binary number made up of 8 bits (a byte is thus simply a succession of 8 bits 0 or 1, for example 11010001).

One thus needs $512 \times 512 \times 8 = 2097152$ bits to encode a single image of this kind - which is a lot! The first idea which comes to mind to reduce the number of bits is to decrease the number of levels of grey, for example, by limiting oneself to white and black, as in Figure 1B. The two possible values of the level of grey are encoded with only one bit (either 0 or 1), and one has thus decreased the number of bits by a factor of 8. Obviously, the quality of the image has deteriorated quite a bit. Now look at the image of Figure 1C.

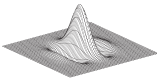
It has been encoded with 32 times fewer bits than the original image, by a method using the theory of wavelets; the deterioration is hardly perceptible! Why? Because instead of reducing the precision, it is the manner of representing the information which was changed.

It all started with the analysis of Joseph Fourier...

As we have said, a digital image is defined by 512×512 numbers which specify the light intensity at each point. One can thus think of this image as a point in a space of 512×512 dimensions - in the same way that a point on a surface, a two dimensional space, can be located by two co-ordinates - and to ask which co-ordinate axes are best adapted for representing such a point. A system of axes (of a more abstract nature here than the familiar axes of elementary geometry) defines what one calls a basis.

A first fundamental advance was made by the mathematician-physicist Joseph Fourier in 1802, in his report to the the Académie des Sciences on the propagation of heat, a subject which is a priori unrelated to our problem. Fourier showed, in particular, that to represent in a compact and convenient way the function $f(x)$ (from a mathematical point of view, such a function is a point in a space having infinitely many dimensions) one can use "axes" made up of an infinite set of sinusoidal functions. More precisely: Fourier showed that one can represent a function $f(x)$ as a sum of infinitely many sine and cosine functions of the form $\sin(ax)$ or $\cos(ax)$, each one carrying a certain coefficient.

These "Fourier bases" became an essential tool, frequently used in science, because they can be used to represent many types of functions, therefore many physical quantities. In particular, they are also used to represent sounds and images. And yet, engineers know well that these sinusoids are far from being ideal for signals as complex as images: they do not efficiently represent transitory structures such as contours of the image.



...then came the wavelet transform

Specialists in signal processing were not the only ones to become aware of the limitations of Fourier bases. In the 1970's, a French engineer-geophysicist, Jean Morlet, realised that they were not the best mathematical tool for underground exploration; this led to one of the discoveries—in a laboratory of Elf-Aquitaine—of wavelet transform. This mathematical method, based on a set of basic functions different from the sinusoidal functions used in Fourier's method, advantageously replaces the Fourier transform in certain situations. In addition, already in the 1930's, physicists had realised that the Fourier bases were not well-adapted for analysing the states of an atom. This spurred much work, which later on contributed much to the theory of wavelets. It was also in the 1930's that mathematicians started trying to improve the Fourier bases for analysing localised singular structures, which opened an important research program still very much alive today.

In other words, a multitude of scientific communities developed modifications of Fourier bases with the means at their disposal.

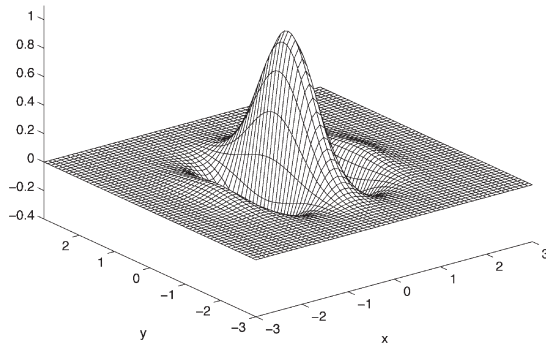
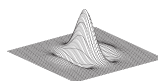


Figure 2. The graph of a wavelet used in the compression of images.

In the 1980's, Yves Meyer, a French mathematician, discovered the first orthogonal wavelet bases (orthogonality is a property which considerably simplifies reasoning and calculations; Fourier bases are also orthogonal). This discovery, followed by some unexpected meetings around photocopiers or coffee tables, started a vast multi-disciplinary scientific movement in France, which has had a considerable impact internationally. The applications of the theory and algorithms of wavelets have made their way not only into many scientific and technological disciplines, but have also led to the creation of several companies in the United States.

Mathematics of wavelets has played a pivotal role in a number of fields

Mathematics has played a fundamental role here as a catalyst, and in the clarification and deepening of ideas. By isolating the fundamental concepts from specific applications, it allowed scientists from very diverse fields of physics, such as signal processing, computer science, etc. to realise that they were working with the same tool. Modern mathematical work on Fourier analysis has now permitted us to go further and to refine these tools, and to control their performance. Finally, this theory provides a standard technique for scientific computation (the fast wavelet transform), thanks to a collaboration between mathematicians and specialists in signal processing. The image of Figure 1C was thus obtained thanks to the same wavelet bases as those used in sta-



tistics, seismology, or scientific computation, with the same fast algorithm. And through the international standard JPEG-2000 for the compression of images, these wavelets have currently invaded all fields of imaging, from the Internet to digital cameras, and is moving towards satellites.

A bridge remains to be built between the world of wavelets and the world of geometry

Fourier bases were not well-adapted to the analysis of transitory phenomena, whereas wavelet bases are. Is this the end of the story? No. In image processing, as in all other fields where wavelets have become a basic tool, everyone currently confronts the same type of problem: how to exploit geometrical regularities. Indeed, we know that an image, however complex, is remarkably well represented by a simple drawing made up of relatively few strokes, and one can often think of the contours of the objects appearing in the image as being made up of rather simple geometrical curves. Using profitably these curves and their regularity should make it possible to improve considerably the results obtained up until now; but wavelet theory is not at present capable of this. To build this bridge with the world of geometry poses difficult mathematical problems. However, the scientific and industrial stakes being high, one can expect that it will be built in the coming ten years. In France?

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