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PRODUCT-FORM QUEUEING NETWORKS WITH NEGATIVE AND POSITIVE CUSTOMERS

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Abstract

We introduce a new class of queueing networks in which customers are either 'negative' or 'positive'. A negative customer arriving to a queue reduces the total customer count in that queue by 1 if the queue length is positive; it has no effect at all if the queue length is empty. Negative customers do not receive service. Customers leaving a queue for another one can either become negative or remain positive. Positive customers behave as ordinary queueing network customers and receive service. We show that this model with exponential service times, Poisson external arrivals, with the usual independence assumptions for service times, and Markovian customer movements between queues, has product form. It is quasi-reversible in the usual sense, but not in a broader sense which includes all destructions of customers in the set of departures. The existence and uniqueness of the solutions to the (non-linear) customer flow equations, and hence of the product form solution, is discussed.

WORK CANCELLATION; NEGATIVE CUSTOMERS

1. Introduction

Consider an open network of queues with n servers which have mutually independent, and i.i.d. exonential service times of rates $r(1),\cdots,r(n)$. Two types of customers circulate in the network: 'positive' and 'negative' customers. External arrivals to the network can either be positive customers which arrive to the ith queue according to a Poisson process of rate $\Lambda(i)$, or negative customers which constitute a Poisson arrival process of rate $\lambda(i)$ to the ith queue. A customer which leaves queue i (after finishing service) heads for queue j with probability $p^+(i,j)$ as a positive (or normal) customer, or as a negative customer with probability $p^-(i,j)$, or it will depart from the network with probability $d(i)$. Let $p(i,j)=p^+(i,j)+p^-(i,j)$; it is the transition probability of a Markov chain representing the movement of customers between servers. Clearly we shall have $\Sigma_j\, p(i,j)+d(i)=1$ for $1\leqq i\leqq n$. Positive and negative customers have different roles in the network. A negative customer *reduces by 1* the length of the

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queue to which it arrives (i.e. it 'cancels' an existing customer) or has no effect on the queue length if the queue is empty, while a positive or 'normal' customer *adds 1* to the queue length. Thus negative customers do not require service. The queue length is constituted only by positive customers and their service is carried out in the usual manner.

The model is equivalent to the following queueing network with FCFS service centers. Service times are exponential of rate $(\lambda(i) + r(i))$ at queue i. A customer which finishes service at queue i leaves the network by itself with probability $(r(i)d(i) + \lambda(i))/(\lambda(i) + r(i))$, or it leaves the network *together* with a customer from queue j with probability $r(i)p^-(j, i)/(\lambda(i) + r(i))$, or it joins queue j with probability $r(i)p^+(j, i)/(\lambda(i) + r(i))$. External (normal) customer arrivals occur according to a Poisson process of rate $\Lambda(i)$ at queue i.

The model we introduce is a new generalisation of standard [3], [4] queueing network models widely used in computer and communication system performance modeling and in operations research, which have only 'positive' customers. We show that it has a specific kind of product form: for the open network the stationary probability distribution of its state can be written as the product of the marginal probabilities of the state of each queue. Positive customers can be considered to be resource requests, while negative customers can correspond to decisions to cancel requests for resources. Our study has been motivated by the analogy to neural networks where each queue represents a neuron. Positive customers moving from one queue to another represent excitation signals, while negative customers going from one queue to another represent inhibition signals.

2. Results

Theorem 1. Let

$$q_i \equiv \lambda^+(i)/[r(i) + \lambda^-(i)], \tag{1}$$

where the $\lambda^+(i)$, $\lambda^-(i)$ for $i = 1, \cdots, n$ satisfy the following system of non-linear simultaneous equations:

$$\lambda^+(i) = \sum_j q_j r(j) p^+(j, i) + \Lambda(i), \quad \lambda^-(i) = \sum_j q_j r(j) p^-(j, i) + \lambda(i). \tag{2}$$

Let $k(t)$ be the vector of queue lengths at time t, and $k = (k_1, \cdots, k_n)$ be a particular value of the vector; let $p(k)$ denote the stationary probability distribution $p(k) = \lim_{t\to\infty} \Pr[k(t) = k]$.

If a unique non-negative solution $\{\lambda^+(i), \lambda^-(i)\}$ *exists to Equations* (2) *such that each* $q_i < 1$, then

$$p(k) = \prod_{i=1}^{n} [1 - q_i] q_i^{k_i}. \tag{3}$$

We omit the proof, which follows standard techniques [3], [4]. Since $\{k(t): t \geqq 0\}$ is a continuous-time Markov chain it satisfies the usual Chapman–Kolmogorov equations;

thus in steady state it can be seen that $p(k)$ satisfies the following global balance equations:

$$
\begin{aligned}
&p(k)\sum_i [\Lambda(i)+(\lambda(i)+r(i))\mathbf{1}[k_i>0]] \\
&\quad = \sum_i [\,p(k_i^+)r(i)d(i)+p(k_i^-)\Lambda(i)\mathbf{1}[k_i>0]+p(k_i^+)\lambda(i) \\
&\qquad + \sum_j \{\,p(k_{ij}^{+-})r(i)\,p^+(i,j)\mathbf{1}[k_j>0]+p(k_{ij}^{++})r(i)\,p^-(i,j)) \\
&\qquad + p(k_i^+)r(i)\,p^-(i,j)\mathbf{1}[k_j=0]\}]
\end{aligned} \tag{4}
$$

where the vectors used in (4) are defined as follows:

$$
\begin{aligned}
k_i^+ &= (k_1,\cdots,k_i+1,\cdots,k_n),\\
k_i^- &= (k_1,\cdots,k_i-1,\cdots,k_n),\\
k_{ij}^{+-} &= (k_1,\cdots,k_i+1,\cdots,k_j-1,\cdots,k_n),\\
k_{ij}^{++} &= (k_1,\cdots,k_i+1,\cdots,k_j+1,\cdots,k_n),
\end{aligned}
$$

and $\mathbf{1}[X]$ is the usual characteristic function which takes the value 1 if X is true and 0 otherwise. Theorem 1 is proved by showing that (3) satisfies this system of equations.

Remark 1. Consider a *closed* network for which $P=\{p(i,j)\}_{1\leqq i,j\leqq n}$ is the transition probability matrix of an *ergodic* Markov chain. Then if there exists some $p^-(u,v)>0$, it follows that $p(k)=0$ for all vectors k except for the null vector: $p(\mathbf{0})=1$. This statement is obvious, since in the long run the network will be empty. Indeed, let $K(t)=\Sigma_i k_i(t)$; if there exists a $p^-(u,v)>0$, then $P[K(t+\tau)<K(t)]>0$ for each $\tau>0$, so that $\lim_{t\to\infty} P[K(t)=0]=1$. As a consequence, we see that the only closed networks which are of interest are those for which either all $p^-(u,v)=0$, so that they reduce to standard closed Jackson networks, or those for which P is not ergodic so that positive customers may remain in one part of the network.

2.1. *Feedforward networks*. Let us now turn to the existence and uniqueness of the solutions $\lambda^+(i)$, $\lambda^-(i)$, $1\leqq i\leqq n$ to Equations (1) and (2) which represent the average arrival rate of positive and negative customers to each queue for the open model, and the average number of visits of a customer to each queue for the closed system. An important class of models is covered by the following result concerning *feedforward networks*. A network is said to be *feedforward* if for any sequence $i_1,\cdots,i_s,\cdots,i_r,\cdots,i_m$ of queues, $i_s=i_r$ for some $r>s$ implies that

$$\prod_{v=1}^{m-1} p(i_v,i_{v+1})=0.$$

Theorem 2. **If the network is feedforward, then the solutions $\lambda^+(i)$, $\lambda^-(i)$ to Equations (1) and (2) exist and are unique.**

Proof. For any feedforward network, we may construct an isomorphic network by renumbering the queues so that queue 1 has no predecessors (i.e. $p(i, 1) = 0$ for any i), queue n has no successors (i.e. $p(n, i) = 0$ for any i) and for any i we have $p(i, j) = 0$ if $j < i$. Thus in the isomorphic network, a customer can possibly (but not necessarily) go directly from queue i to queue j only if j is larger than i. For such a network, the $\lambda^+(i)$ and $\lambda^-(i)$ can be calculated recursively as follows:

- *first compute* $\lambda^+(1) = \Lambda(1)$, $\lambda^-(1) = \lambda(2)$.
- *then each successive value of* i such that $\lambda^+(i)$, $\lambda^-(i)$ have not yet been calculated proceed as follows: since the q_j for each $j < i$ are known, we compute

$$\lambda^+(i) = \sum_{j<i} q_j r(j) p^+(j, i) + \Lambda(i), \quad \lambda^-(i) = \sum_{j<i} q_j r(j) p^-(j, i) + \lambda(i).$$

This completes the proof since we have provided a procedure for calculating in a unique manner the solution to (1) and (2) for a feedforward network.

2.2. *Balanced networks.* We now consider a special class of networks, with *feedback*, whose customer flow equations have an unique solution. We shall say that *a network with negative and positive customers is balanced* if the ratio

$$\left[\sum_j q_j r(j) p^+(j, i) + \Lambda(i)\right] \Big/ \left[\sum_j q_j r(j) p^-(j, i) + \lambda(i) + r(i)\right] \tag{5}$$

is identical for any $i = 1, \cdots, n$. This in effect means that all the q_i are identical.

Theorem 3. The customer flow equations (1) and (2) have an unique solution if the network is balanced.

Proof. From (1) and (2) we write

$$q_i = \left[\sum_j q_j r(j) p^+(j, i) + \Lambda(i)\right] \Big/ \left[\sum_j q_j r(j) p^-(j, i) + \lambda(i) + r(i)\right]. \tag{6}$$

If the system is balanced, $q_i = q_j$ for all i, j. From (10) we then have that the common $q = q_i$ satisfies the quadratic equation:

$$q^2 R^-(i) + q[\lambda(i) + r(i) - R^+(i)] - \Lambda(i) = 0 \tag{7}$$

where $R^-(i) = \Sigma_j r(j) p^-(j, i)$, $R^+(i) = \Sigma_j r(j) p^+(j, i)$. The positive root of this quadratic equation, which will be independent of i, is the solution of interest:

$$q = \{(R^+(i) - \lambda(i) - r(i)) + [(R^+(i) - \lambda(i) - r(i))^2 + 4R^-(i)\Lambda(i)]^{1/2}\}/2R^-.$$

2.3. *Quasi-reversibility.* The usual definition of quasi-reversibility (QR) [6] is the following. Let Q_A, Q_D be subsets of the set of state transition rates $q(k, k')$ (i.e. of the elements of the infinitesimal generator Q of the network): $Q = \{q(k, k')$: k, k' state vectors of the network$\}$, such that $q(k, k') \in Q_A$ iff the transition from k to k' occurs when a *positive customer* arrives to the network. Similarly $q(k, k') \in Q_D$ if the transition from k to k' occurs with a customer's real departure, i.e. a positive customer leaving the network from one of the servers towards the outside. Following [6], but with slightly different notation adapted to our problem, we shall say that a network is QR iff for some positive real numbers λ, μ the following two conditions are satisfied:

(i) $\Sigma_{k'} q(k, k')\mathbf{1}[q(k, k') \in Q_A] = \lambda$, for all k;

(ii) $\Sigma_k p(k)q(k, k')\mathbf{1}[q(k, k') \in Q_D] = \mu p(k')$ for all k'.

Theorem 4. The queueing network with negative and positive customers is QR, if there exists an unique solution to (1) and (2) with $q_i < 1$.

Proof. The property of the arrival instant transitions of positive customers is obvious, and λ is merely $\Sigma_i \Lambda(i)$. For the real departure transitions D, write for any k'

$$\sum_k p(k)q(k, k')\mathbf{1}[q(k, k') \in Q_D]/p(k') = \sum_i q_i r(i)d(i)$$

which establishes the result.

The definition given above of QR is not intuitively satisfactory for our model, since 'departures' now result also from the arrivals of negative customers to a queue. Unfortunately our network is not QR in the more general sense given below.

Denote by D' the set of all state transitions with *destruction of positive customers plus departures*: $D' = \{(k_{ij}^{++}, k), (k_i^+, k)$ for all $i, j\}$ and let $Q_{D'}$ be the corresponding subset of the state transition rates. We shall say that a network with negative and positive customers is QR in the sense of (a), (b) iff there exist positive real numbers λ, μ such that:

(a) $\Sigma_{k'} q(k, k')\mathbf{1}[q(k, k') \in Q_A] = \lambda$, for all k;

(b) $\Sigma_k \| k - k' \| \, p(k)q(k, k')\mathbf{1}[q(k, k') \in Q_{D'}] = \mu p(k')$ for all k', where $\| k - k' \|$ is the difference between the total number of customers in k' and k.

Note that in (b) we take into account the fact that transitions from some state k_{ij}^{++} to some other state k are caused by the departure of two customers, i.e. $\| k_{ij}^{++} - k \| = 2$. We call this quasi-reversibility 'QR in the sense of (a), (b)'.

We may also choose to define quasi-reversibility by identifying all departure instants, without counting the number of departures; (a) will not change, but (b) will be replaced simply by:

(c) $\Sigma_k p(k)q(k, k')\mathbf{1}[(k, k') \in D'] = \mu p(k')$ for all k',

which we call 'QR in the sense of (a) and (c)'.

Remark 2. The class of queueing networks with positive and negative customers is not QR in the sense of (a) and (b), or in the sense of (a) and (c).

Proof. It will suffice to show that (b) and (c) are not satisfied for the two-queue network with $p^+(1, 2) = p^+(2, 1) = d(1) = d(2) = 0$. Dividing both sides of (b) by $p(k')$, we have that for any $k' = (k_1', k_2')$, the left-hand side of the resulting equation can be written as:

$$\begin{aligned}
&\sum_k p(k)q(k, k')\mathbf{1}[(k, k') \in D']/p(k') \\
&= 2q_1q_2[r(1) + r(2)], && \text{if } k_1' > 0, k_2' > 0, \\
&= 2q_1q_2[r(1) + r(2)] + q_1 r(1), && \text{if } k_1' > 0, k_2' = 0, \\
&= 2q_1q_2[r(1) + r(2)] + q_2 r(2), && \text{if } k_1' = 0, k_2' > 0, \\
&= 2q_1q_2[r(1) + r(2)] + q_1 r(1) + q_2 r(2), && \text{if } k_1' = 0, k_2' = 0,
\end{aligned}$$

where $q_1 = \Lambda(1)/[r(1) + q_2r(2)]$, $q_2 = \Lambda(2)/[r(2) + q_1r(1)]$. Obviously each of these terms will be different as long as both $q_1r(1)$ and $q_2r(2)$ are non-zero, which can easily be shown. Hence (b) cannot be satisfied. The same can be written and shown for (c).

Remark 3. The class of *feedforward* queueing networks with positive and negative customers is not QR in the sense of (a) and (b) or (a) and (c).

Proof. Consider now the two-queue feedforward network with $p^-(1, 2) = 1$, $p^+(2, 1) = p^-(2, 1) = 0$, $d(1) = 0$, $d(2) = 1$. In order to verify (b) write

$$\begin{aligned}
&\sum_k p(k)q(k, k')\mathbf{1}[(k, k')\in D']/p(k') \\
&\quad = 2q_1q_2r(1), && \text{if } k_1' > 0,\ k_2' > 0, \\
&\quad = 2q_1q_2[r(1) + r(2)] + q_1r(1), && \text{if } k_1' > 0,\ k_2' = 0, \\
&\quad = 2q_1q_2[r(1) + r(2)] + q_2r(2), && \text{if } k_1' = 0,\ k_2' > 0, \\
&\quad = 2q_1q_2[r(1) + r(2)] + q_1r(1) + q_2r(2), && \text{if } k_1' = 0,\ k_2' = 0.
\end{aligned}$$

2.4. *Existence and uniqueness of the network solution*. We now address the issue of *uniqueness* of the product-form solution of the network, whenever it can be shown to exist. Then *existence* will be shown for the class of 'hyperstable' networks.

Remark 4. If the $0 < q_i < 1$ solutions to (1), (2) exist for $i = 1, \cdots, n$, then it is the unique solution.

Indeed, since $\{k(t){:}\ t \geqq 0\}$ is an irreducible and aperiodic Markov chain, if a positive stationary solution $p(k)$ exists, then it is unique. By Theorem 1, if the $0 < q_i < 1$ solution to (1), (2) exists for $i = 1, \cdots, n$, then $p(k)$ is given by (3) and is clearly positive for all k. Suppose now that for some i there are two different q_i, q_i' satisfying (1), (2). But this implies that for all k_i, $\lim_{t\to\infty} P[k_i(t) = 0]$ has two different values $[1 - q_i]$ and $[1 - q_i']$, which contradicts the uniqueness of $p(k)$; hence the result.

We shall say that a network is *hyperstable* if the following property holds:

$$0 < r(i) + \lambda(i) > \Lambda(i) + \sum_j r(j)\, p^+(j, i), \quad \text{for all } i = 1, \cdots, n. \tag{8}$$

Theorem 5. If the network is hyperstable then the customer flow equations (1) and (2) always have a unique solution with $q_i < 1$.

Proof. The proof is based on a method for proof of existence of non-linear equilibrium equations. It uses the n-dimensional vector homotopy [2] function $H(q, x)$ for a real number $0 \leqq x < 1$.

Let us define the following n-vectors:

$$q = (q_1, \cdots, q_n), \qquad F(q) = (F_1(q), \cdots, F_n(q))$$

where

$$F_i(q) = \left[\sum_j q_j r(j) p^+(j, i) + \Lambda(i)\right] \Big/ \left[\sum_j q_j r(j) p^-(j, i) + \lambda(i) + r(i)\right].$$

The equation we are interested in is $q = F(q)$, which is identical to (1) and (2). Let $D = [0, 1]^n$, and $D = D^0 \cup \delta D$ where δD stands for the boundary of D, and D^0 is the set of interior points. We shall prove that $q = F(q)$ has a solution in D^0 when the network is hyperstable.

We shall use Theorem 22.5.1 of [2] which states the following. If $F: D \to R^n$ is continuous, and D is the closure of an open bounded set, then F has a fixed point $F(q^*) = q^*$, $q^* \in D$, if the function

$$H(q, x) = (1 - x)(q - y) + x(q - F(q))$$

for some $y \in D^0$ is boundary-free, i.e. the solution of $H(q, x) = 0$ never touches the boundary δD as x varies from 0 to 1.

Notice that each $F_i(q)$ is the ratio of two first-degree polynomials, with non-negative coefficients, in the elements of q; $[\lambda(i) + r(i)] > 0$, so its denominator does not have any zeros in D. Clearly $F(q)$ is continuous. Since D is the closure of an open bounded set, the conditions of the theorem are satisfied if we can show that $H(q, x)$ is boundary-free.

Choose $y = (y_1, \cdots, y_n)$ where

$$y_i = \left[\sum_j r(j) p + (j, i) + \Lambda(i)\right] \Big/ [\lambda(i) + r(i)].$$

By assumption $0 < y_i < 1$ for all $i = 1, \cdots, n$, since the network is hyperstable. Clearly $H(q, 0) = q - y$ and $H(q, 1) = q - F(q)$. Now consider

$$H^{-1} = \{q : q \in D, H(q, x) = 0 \text{ and } 0 \leqq x < 1\}.$$

We can show that H^{-1} and δD have an empty intersection, i.e. as x is varied from 0 towards 1 the solution of $H(q, x)$ if it exists does not touch the boundary of D. To do this assume the contrary; this implies that for some $x = x^*$ there exists some $q = q^*$ for which $H(q^*, x^*) = 0$ and such that $q_i^* = 0$ or 1. If $q_i^* = 0$ we can write

$$-(1 - x^*) y_i - x^* F_i(q^*) = 0, \quad \text{or} \quad x^*/(1 - x^*) = -y_i / F_i(q^*) < 0,$$

which contradicts the assumption about x. If on the other hand $q_i^* = 1$, then we can write

$$(1 - x^*)(1 - y_i) + x^*(1 - F_i(q^*)) = 0,$$

or

$$x^*/(1 - x^*) = -(1 - y_i)/(1 - F_i(q^*)) < 0$$

because $(1 - y_i) > 0$ and $0 < F_i(q^*) < y_i$ so that $(1 - F_i(q^*)) > 0$, contradicting again the assumption about x. Thus $H(q, x) = 0$ cannot have a solution on the boundary δD for any $0 \leqq x < 1$. Thus we have established (ii), and proved the existence, and hence the uniqueness, of the stationary solution of a hyperstable network.

3. Conclusions

We conjecture that within the framework of the model studied in this paper, the customer flow equations (2) have an unique solution, under appropriate stability conditions, even for the most general case with feedback. However, this is not simple to establish in a direct manner and perhaps requires a more abstract approach. Extensions of this model to the usual state-dependent service disciplines, multiple classes, and general service time distributions with specific service disciplines [1] should be studied; they are not obvious because of the non-linear nature of the customer flow equations. Variants of this model for *a single-server queue with general service times* are discussed in [5].

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