Convex Optimization M2

Semidefinite Programming Applications

Distortion, embedding problems, . . .

Distortion, embedding problems, . . .

We cannot hope to always get low rank solutions to SDPs, unless we are willing to admit some **distortion**. . . The following result from [Ben-Tal, Nemirovski, and Roos, 2003] gives some guarantees.

Theorem

Approximate $\mathcal{S}$-lemma. *Let $A_1, \ldots, A_N \in$ **S**$_n$, $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$ and a matrix $X \in$ **S**$_n$ such that*

$$A_i,\, X \succeq 0, \quad \mathbf{Tr}(A_i X) = \alpha_i, \quad i = 1, \ldots, N$$

Let $\epsilon > 0$, there exists a matrix X_0 such that

$$\alpha_i(1-\epsilon) \leq \mathbf{Tr}(A_i X_0) \leq \alpha_i(1+\epsilon) \quad \textit{and} \quad \mathbf{Rank}(X_0) \leq 8\frac{\log 4N}{\epsilon^2}$$

Proof. Randomization, concentration results on Gaussian quadratic forms.

See [Barvinok, 2002, Ben-Tal, El Ghaoui, and Nemirovski, 2009] for more details.

Distortion, embedding problems, . . .

A particular case: Given N vectors $v_i \in \mathbb{R}^d$, construct their Gram matrix $X \in \mathbf{S}_N$, with

$$X \succeq 0, \quad X_{ii} - 2X_{ij} + X_{jj} = \|v_i - v_j\|_2^2, \quad i,j = 1, \ldots, N.$$

The matrices $D_{ij} \in \mathbf{S}_n$ such that

$$\mathbf{Tr}(D_{ij}X) = X_{ii} - 2X_{ij} + X_{jj}, \quad i,j = 1, \ldots, N$$

satisfy $D_{ij} \succeq 0$. Let $\epsilon > 0$, there exists a matrix X_0 with

$$m = \mathbf{Rank}(X_0) \leq 16 \frac{\log 2N}{\epsilon^2},$$

from which we can extract vectors $u_i \in \mathbb{R}^m$ such that

$$\|v_i - v_j\|_2^2 \, (1 - \epsilon) \leq \|u_i - u_j\|_2^2 \leq \|v_i - v_j\|_2^2 \, (1 + \epsilon).$$

In this setting, the **Johnson-Lindenstrauss** lemma is a particular case of the approximate $\mathcal{S}$ lemma. . .

Distortion, embedding problems, ...

- The problem of reconstructing an N-point Euclidean metric, given **partial** information on pairwise distances between points $v_i,\ i=1,\ldots,N$ can also be cast as an SDP, known as and **Euclidean Distance Matrix Completion** problem.

$$\begin{array}{ll} \mbox{find} & D \\ \mbox{subject to} & \mathbf{1}v^T + v\mathbf{1}^T - D \succeq 0 \\ & D_{ij} = \|v_i - v_j\|_2^2, \quad (i,j) \in S \\ & v \geq 0 \end{array}$$

  in the variables $D \in \mathbf{S}_n$ and $v \in \mathbb{R}^n$, on a subset $S \subset [1,N]^2$.

- We can add further constraints to this problem given additional structural info on the configuration.
- Applications in sensor networks, molecular conformation reconstruction etc...

Distortion, embedding problems, . . .

[Dattorro, 2005] 3D map of the USA reconstructed from pairwise distances on 5000 points. Distances reconstructed from Latitude/Longitude data.

Mixing rates for Markov chains & maximum variance unfolding

Mixing rates for Markov chains & unfolding

[Sun, Boyd, Xiao, and Diaconis, 2006]

- Let $G = (V, E)$ be an **undirected graph** with n vertices and m edges.
- We define a **Markov chain** on this graph, and let $w_{ij} \geq 0$ be the transition rate for edge $(i, j) \in V$.
- Let $\pi(t)$ be the state distribution at time t, its evolution is governed by the heat equation

$$d\pi(t) = -L\pi(t)dt$$

  with

$$L_{ij} = \begin{cases} -w_{ij} & \text{if } i \neq j,\ (i,j) \in V \\ 0 & \text{if } (i,j) \notin V \\ \sum_{(i,k)\in V} w_{ik} & \text{if } i = j \end{cases}$$

  the **graph Laplacian** matrix, which means

$$\pi(t) = e^{-Lt}\pi(0).$$

- The matrix $L \in \mathbf{S}_n$ satisfies $L \succeq 0$ and its smallest eigenvalue is zero.

Mixing rates for Markov chains & unfolding

- With
$$\pi(t) = e^{-Lt}\pi(0)$$
the **mixing rate** is controlled by the second smallest eigenvalue $\lambda_2(L)$.
- Since the smallest eigenvalue of L is zero, with eigenvector $\mathbf{1}$, we have
$$\lambda_2(L) \geq t \quad \Longleftrightarrow \quad L(w) \succeq t(\mathbf{I} - (1/n)\mathbf{1}\mathbf{1}^T),$$
- Maximizing the mixing rate of the Markov chain means solving
$$\begin{array}{ll} \text{maximize} & t \\ \text{subject to} & L(w) \succeq t(\mathbf{I} - (1/n)\mathbf{1}\mathbf{1}^T) \\ & \sum_{(i,j)\in V} d_{ij}^2 w_{ij} \leq 1 \\ & w \geq 0 \end{array}$$
in the variable $w \in \mathbb{R}^m$, with (normalization) parameters $d_{ij}^2 \geq 0$.
- Since $L(w)$ is an affine function of the variable $w \in \mathbb{R}^m$, this is a semidefinite program in $w \in \mathbb{R}^m$.
- Numerical solution usually performs better than **Metropolis-Hastings**.

Mixing rates for Markov chains & unfolding

- We can also form the **dual** of the maximum MC mixing rate problem.
- The dual means solving

$$\begin{array}{ll}\mbox{maximize} & \mathbf{Tr}(X(\mathbf{I}-(1/n)\mathbf{1}\mathbf{1}^T))\\ \mbox{subject to} & X_{ii}-2X_{ij}+X_{jj}\leq d_{ij}^2, \quad (i,j)\in V\\ & X\succeq 0,\end{array}$$

  in the variable $X \in \mathbf{S}_n$.
- Here too, we can interpret X as the gram matrix of a set of n vectors $v_i \in \mathbb{R}^d$. The program above maximizes the variance of the vectors v_i

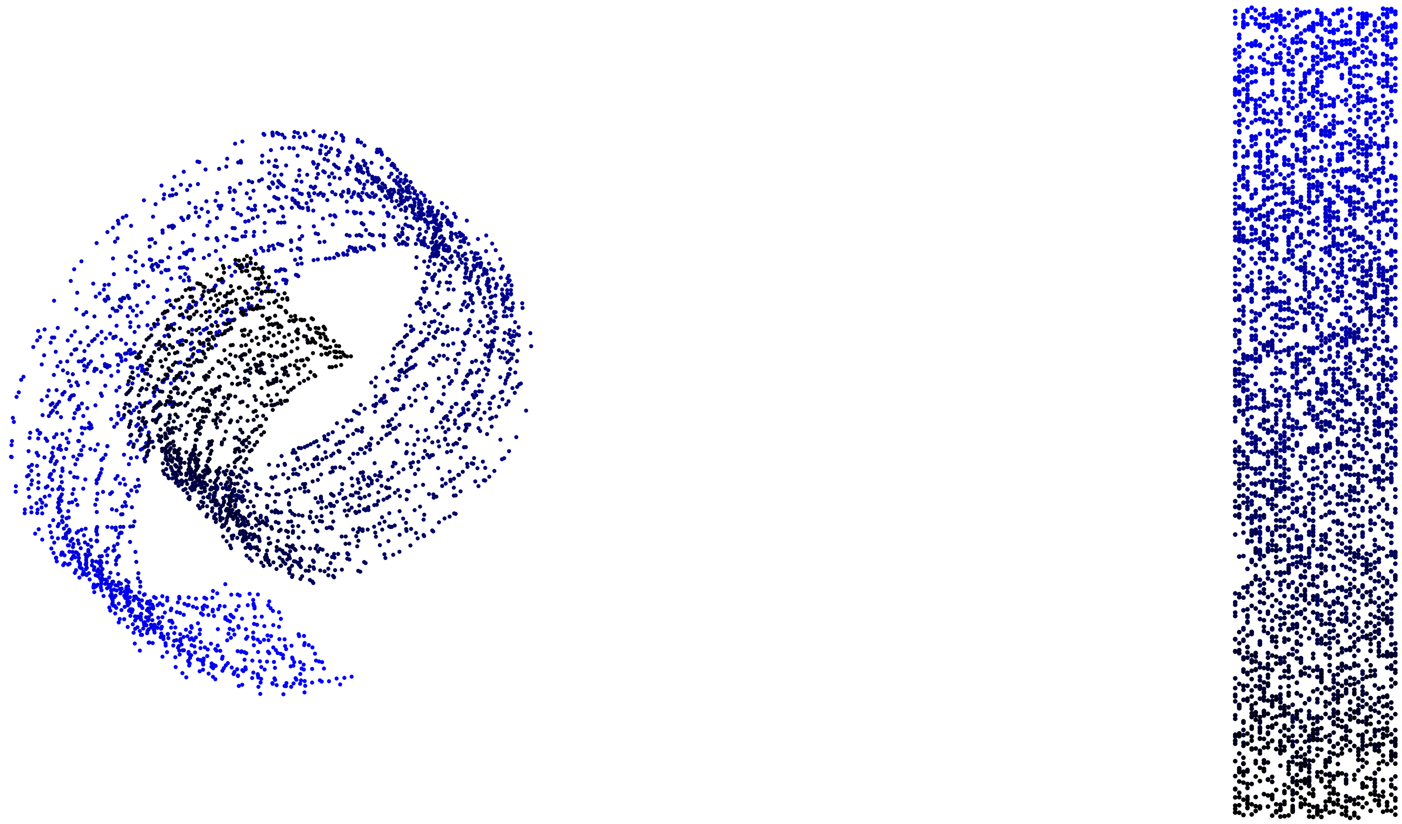
$$\mathbf{Tr}(X(\mathbf{I}-(1/n)\mathbf{1}\mathbf{1}^T)) = \textstyle\sum_i \|v_i\|_2^2 - \|\sum_i v_i\|_2^2$$

  while the constraints bound pairwise distances

$$X_{ii}-2X_{ij}+X_{jj}\leq d_{ij}^2 \quad\Longleftrightarrow\quad \|v_i-v_j\|_2^2 \leq d_{ij}^2$$

- This is a **maximum variance unfolding problem** [Weinberger and Saul, 2006, Sun et al., 2006].

Mixing rates for Markov chains & unfolding



From [Sun et al., 2006]: we are given pairwise 3D distances for k-nearest neighbors in the point set on the right. We plot the maximum variance point set satisfying these pairwise distance bounds on the right.

Moment problems & positive polynomials

Moment problems & positive polynomials

[Nesterov, 2000]. Hilbert's 17^{th} problem has a positive answer for univariate polynomials: a polynomial is nonnegative iff it is a **sum of squares**

$$p(x) = x^{2d} + \alpha_{2d-1}x^{2d-1} + \ldots + \alpha_0 \geq 0, \text{ for all } x \quad \Longleftrightarrow \quad p(x) = \sum_{i=1}^{N} q_i(x)^2$$

We can formulate this as a linear matrix inequality, let $v(x)$ be the moment vector

$$v(x) = (1, x, \ldots, x^d)^T$$

we have

$$\sum_i \lambda_i u_i u_i^T = M \succeq 0 \quad \Longleftrightarrow \quad p(x) = v(x)^T M v(x) = \sum_i \lambda_i (u_i^T v(x))^2$$

where (λ_i, u_i) are the eigenpairs of M.

Moment problems & positive polynomials

- The dual to the cone of Sum-of-Squares polynomials is the cone of moment matrices

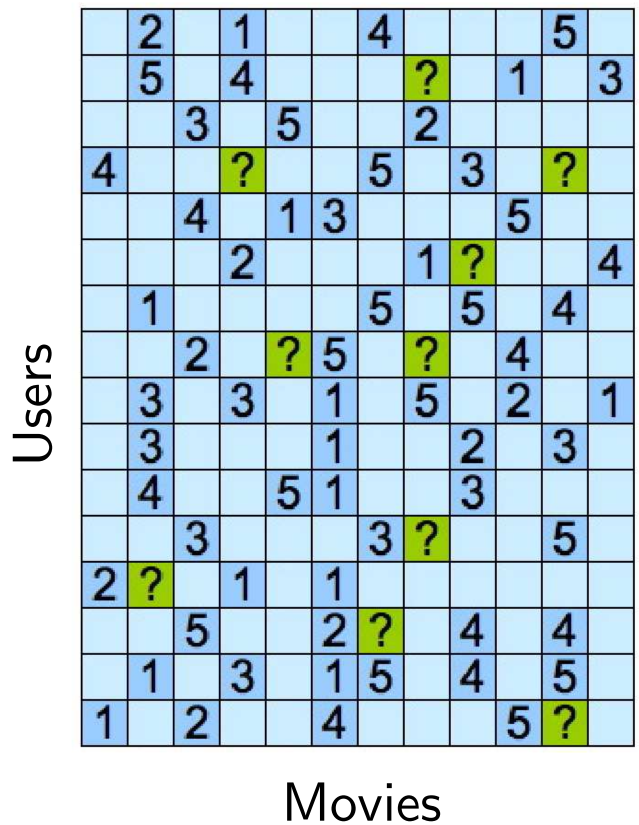
$$\mathbf{E}_\mu[x^i] = q_i,\ i = 0, \ldots, d \quad \Longleftrightarrow \quad \begin{pmatrix} q_0 & q_1 & \cdots & q_d \\ q_1 & q_2 & & q_{d+1} \\ \vdots & & \ddots & \vdots \\ q_d & q_{d+1} & \cdots & q_{2d} \end{pmatrix} \succeq 0$$

- [Putinar, 1993, Lasserre, 2001, Parrilo, 2000] These results can be extended to multivariate polynomial optimization problems over compact semi-algebraic sets.
- This forms exponentially large, ill-conditioned semidefinite programs however.

Collaborative prediction

Collaborative prediction

- Users assign **ratings** to a certain number of movies:



- Objective: make recommendations for other movies. . .

Collaborative prediction

- Infer **user preferences** and **movie features** from user ratings.
- We use a linear prediction model:

$$rating_{ij} = u_i^T v_j$$

  where u_i represents user characteristics and v_j movie features.
- This makes collaborative prediction a **matrix factorization** problem
- Overcomplete representation. . .

Collaborative prediction

- **Inputs**: a matrix of ratings $M_{ij} = \{-1, +1\}$ for $(i,j) \in S$, where S is a subset of all possible user/movies combinations.
- We look for a linear model by factorizing $M \in \mathbb{R}^{n \times m}$ as:

$$M = U^T V$$

  where $U \in \mathbb{R}^{n \times k}$ represents user characteristics and $V \in \mathbb{R}^{k \times m}$ movie features.
- **Parsimony**. . . We want k to be as small as possible.
- **Output**: a matrix $X \in \mathbb{R}^{n \times m}$ which is a low-rank approximation of the ratings matrix M.

Least-Squares

- Choose Means Squared Error as measure of discrepancy.
- Suppose S is the full set, our problem becomes:

$$\min_{\{X:\,\mathbf{Rank}(X)=k\}} \|X - M\|^2$$

- This is just a **singular value decomposition** (SVD)...

Problem: Not true when S is not the full set (partial observations). Also, MSE not a good measure of prediction performance...

Soft Margin

$$\text{minimize} \quad \mathbf{Rank}(X) + c \sum_{(i,j)\in S} \max(0, 1 - X_{ij}M_{ij})$$

non-convex and numerically hard. . .

- Relaxation result in Fazel et al. [2001]: replace $\mathbf{Rank}(X)$ by its convex envelope on the spectahedron to solve:

$$\text{minimize} \quad \|X\|_* + c \sum_{(i,j)\in S} \max(0, 1 - X_{ij}M_{ij})$$

  where $\|X\|_*$ is the **nuclear norm**, $i.e.$ sum of the singular values of X.

- Srebro [2004]: This relaxation also corresponds to multiple large margin SVM classifications.

Soft Margin

- The dual of this program:

$$\begin{array}{ll}\text{maximize} & \sum_{ij} Y_{ij}\\ \text{subject to} & \|Y \odot M\|_2 \leq 1\\ & 0 \leq Y_{ij} \leq c\end{array}$$

  in the variable $Y \in \mathbb{R}^{n \times m}$, where $Y \odot M$ is the Schur (componentwise) product of Y and M and $\|Y\|_2$ the largest singular value of Y.

- This problem is **sparse**: $Y_{ij}^* = c$ for $(i,j) \in S^c$

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