Convex Optimization M2

Lecture 6

Large Scale Optimization

Outline

- First-order methods: introduction
- Exploiting structure
- First order algorithms
  - Subgradient methods
  - Gradient methods
  - Accelerated gradient methods
- Other algorithms
  - Coordinate descent methods
  - Localization methods
  - Franke-Wolfe
  - Dykstra, alternating projection
  - Stochastic optimization

First-order methods: introduction

- Most of these methods are very old (1950-. . .)
- Very large catalog of algorithms, no unifying theory as in IPM
- Many variations around a few key algorithmic templates
- Better scaling, worst dependence on precision target
- In practice: algorithmic choices are dictated by **problem structure**.

What subproblem (projection, etc...) can you solve efficiently?

First Order Algorithms

First-order methods: introduction

$$\begin{array}{ll}\text{minimize} & f(x)\\ \text{subject to} & x \in C\end{array}$$

In theory:

- The theoretical convergence speed of gradient based methods is mostly controlled by the smoothness of the objective.
- Obviously, the geometry of the (convex) feasible set also has an impact.

Convex objective $f(x)$	**Iterations...**
Nondifferentiable	$O(1/\epsilon^2)$
Differentiable	$O(1/\epsilon^2)$
Smooth (Lipschitz gradient)	$O(1/\sqrt{\epsilon})$
Strongly convex	$O(\log(1/\epsilon))$

In practice:

- Compared to IPM, much larger gap between theoretical complexity guarantees and empirical performance.
- Conditioning, well-posedness, etc. also have a very strong impact.

First-order methods: introduction

Solve

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in C \end{array}$$

in $x \in \mathbb{R}^n$, with $C \subset \mathbb{R}^n$ convex.

Main assumptions in the subgradient/gradient methods that follow:

- The gradient $\nabla f(x)$ or a subgradient can be computed efficiently.
- If C is not $\mathbb{R}^n$, for any $y \in \mathbb{R}^n$, the following **subproblem can be solved efficiently**

  $$\begin{array}{ll} \text{minimize} & y^T x + d(x) \\ \text{subject to} & x \in C \end{array}$$

  in the variable $x \in \mathbb{R}^n$, where $d(x)$ is a **strongly convex** function.

Typically, $d(x) = \|x\|_2$ and this is an Euclidean projection.

Subgradient Method

Subgradient

- Suppose that f is a convex function with $\mathbf{dom} f = \mathbb{R}^n$, and that there is a vector $g \in \mathbb{R}^n$ such that:

$$f(y) \geq f(x) + g^T(y - x), \quad \text{for all } y \in \mathbb{R}^n$$

- The vector g is called a **subgradient** of f at x, we write $g \in \partial f$.
- Of course, if f is differentiable, the gradient of f at x satisfies this condition
- The subgradient defines a **supporting hyperplane** for f at the point x

Subgradient Methods

Subgradient method:

- Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex
- We update the current point x_k according to:

$$x_{k+1} = x_k + \alpha_k g_k$$

  where g_k is a subgradient of f at x_k
- α_k is the step size sequence
- Similar to gradient descent but, not a descent method . . .
- Instead: use the best point and the minimum function value found so far

Subgradient Methods

Step size strategies:

- Constant step size: $\alpha_k = h$ for all $k \geq 0$
- Constant step length: $\alpha_k / \|g_k\| = h$ for all $k \geq 0$
- Square summable but not summable:

$$\sum_{k=0}^{\infty} \alpha_k = \infty \quad \text{and} \quad \sum_{k=0}^{\infty} \alpha_k^2 < \infty$$

- Nonsummable diminishing:

$$\sum_{k=0}^{\infty} \alpha_k = \infty \quad \text{and} \quad \lim_{k \to \infty} \alpha_k = 0$$

Subgradient Methods

Convergence:

Assuming $\|g\|_2 \leq G$, for all $g \in \partial f$, we can show

$$f_{\text{best}} - f^\star \leq \frac{\textbf{dist}(x_1, x^*) + G^2 \sum_{i=1}^k \alpha_i^2}{2 \sum_{i=1}^k \alpha_i}$$

For constant step $\alpha_i = h$, this becomes

$$f_{\text{best}} - f^\star \leq \frac{\textbf{dist}(x_1, x^*)}{2hk} + G^2 h/2$$

to get an ϵ solution, we set $h = 2\epsilon/G^2$ and

$$\frac{\textbf{dist}(x_1, x^*)}{2hk} \leq \epsilon$$

hence

$$k \geq \frac{\textbf{dist}(x_1, x^*) G^2}{4\epsilon^2}.$$

Subgradient Methods

- If the problem has constraints:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in C \end{array}$$

  where $C \subset \mathbb{R}^n$ is a convex set

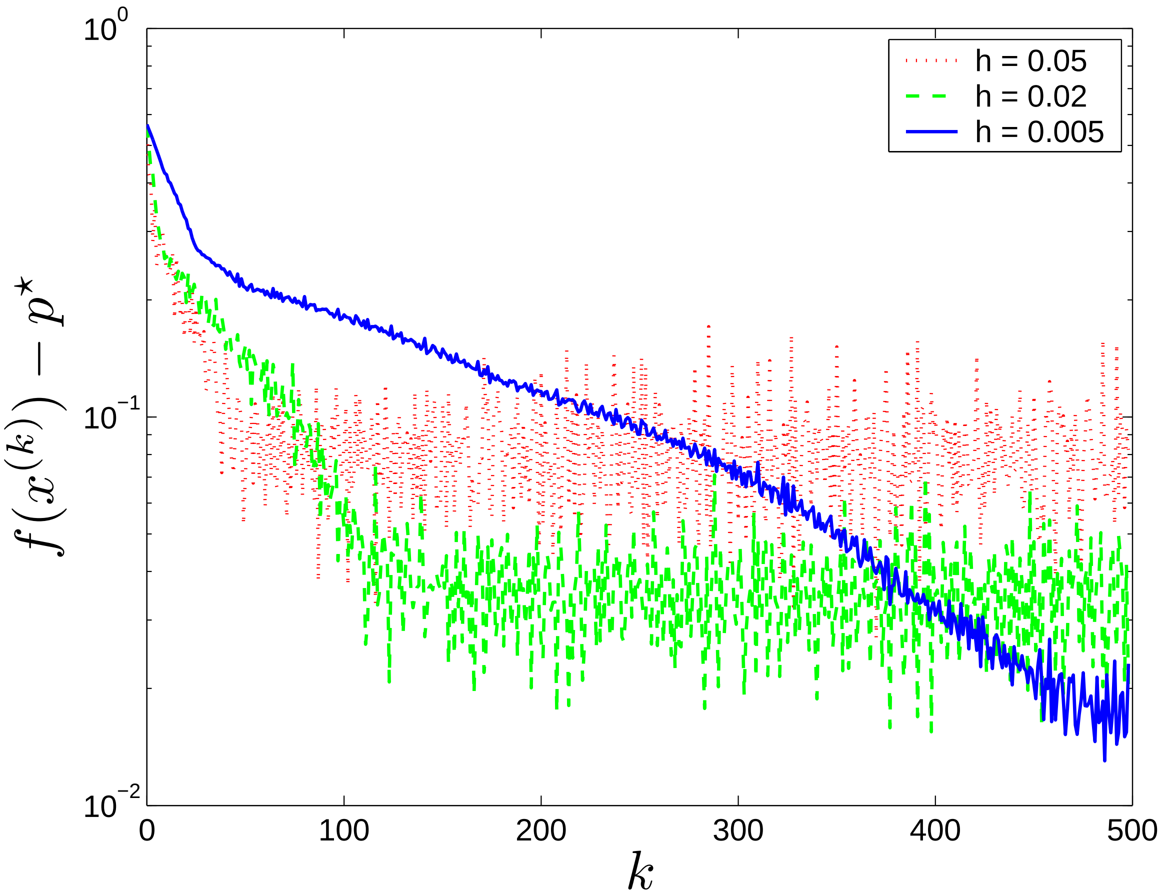
- Use the Euclidean projection $p_C(\cdot)$

$$x_{k+1} = p_C(x_k + \alpha_k g_k)$$

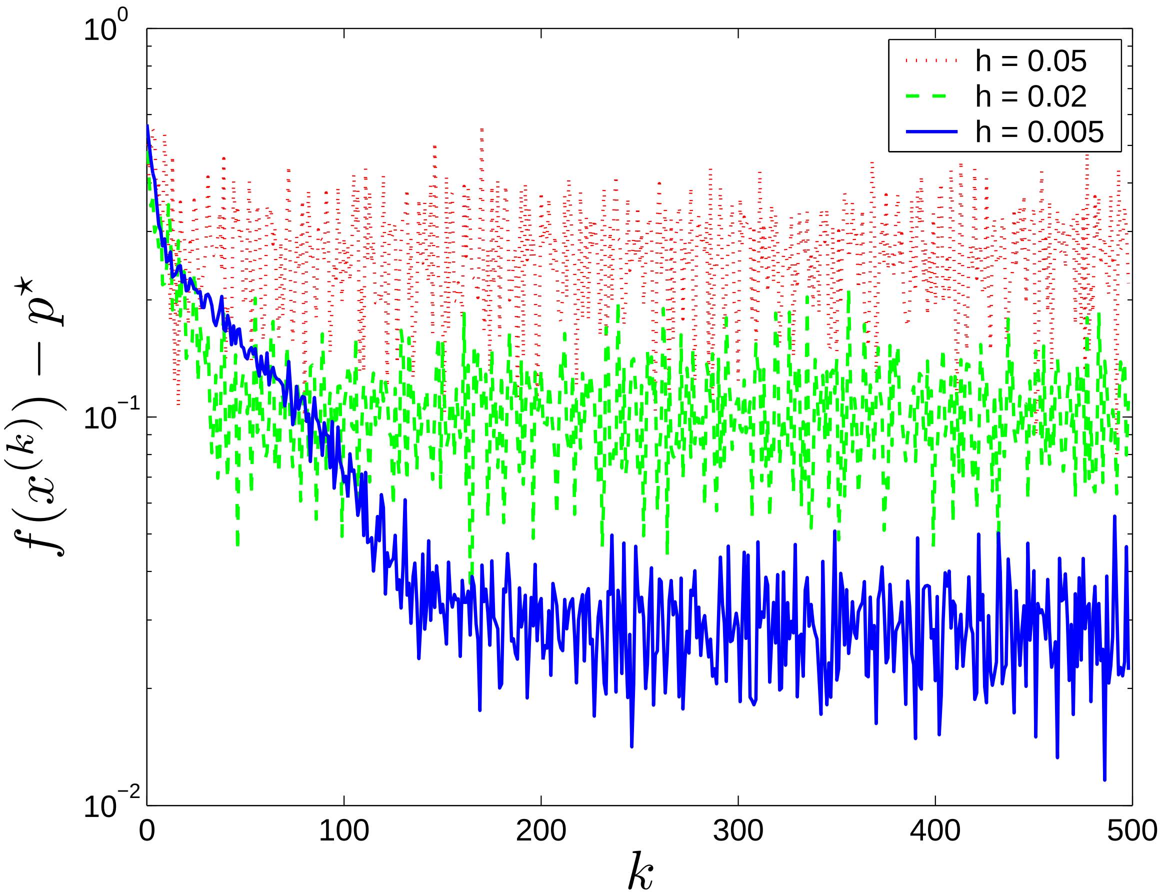
- Similar complexity analysis
- Some numerical examples on piecewise linear minimization. . . Problem instance with $n = 10$ variables, $m = 100$ terms

Subgradient Methods: Numerical Examples

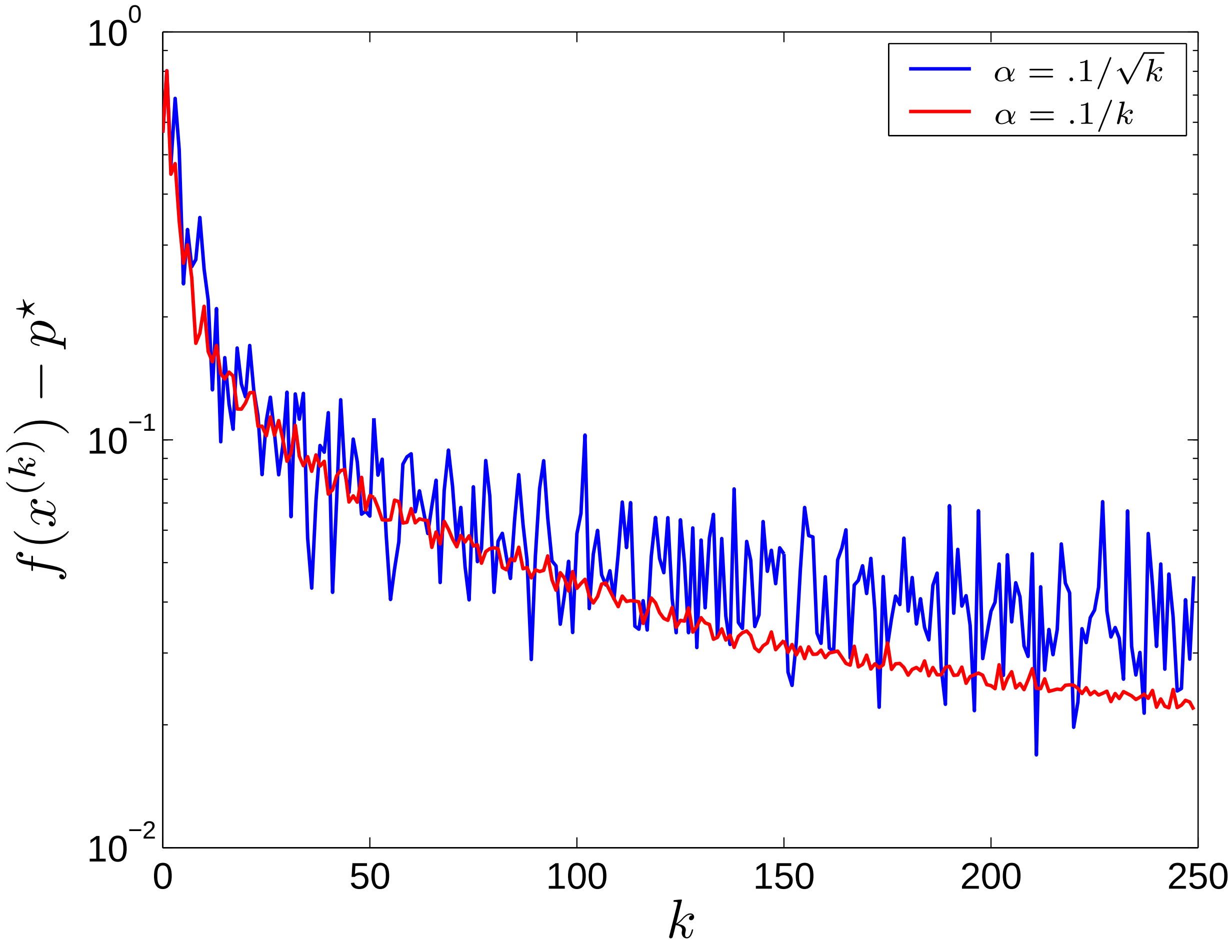
Constant step length, $h = 0.05,\ 0.02, 0.005$



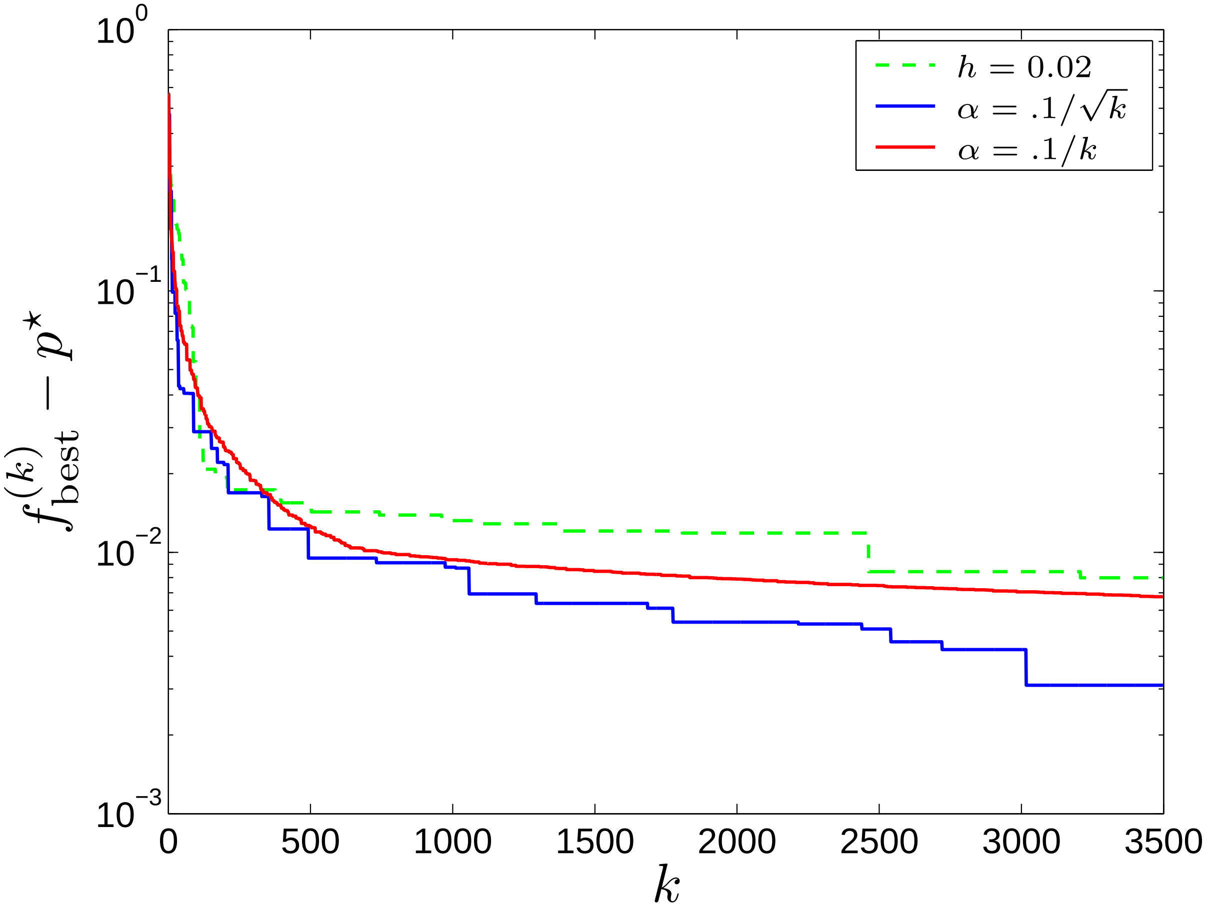
Constant step size $h = 0.05,\ 0.02,\ 0.005$



Diminishing step rule $\alpha = 0.1/\sqrt{k}$ and square summable step size rule $\alpha = 0.1/k$.



Constant step length $h = 0.02$, diminishing step size rule $\alpha = 0.1/\sqrt{k}$, and square summable step rule $\alpha = 0.1/k$



Gradient Descent

Gradient descent method

general descent method with $\Delta x = -\nabla f(x)$

> **given** a starting point $x \in \textbf{dom}\, f$.
> **repeat**
> 1. $\Delta x := -\nabla f(x)$.
> 2. *Line search.* Choose step size t via exact or backtracking line search.
> 3. *Update.* $x := x + t\Delta x$.
>
> **until** stopping criterion is satisfied.

- stopping criterion usually of the form $\|\nabla f(x)\|_2 \leq \epsilon$
- convergence result: for **strongly convex** f,

$$f(x^{(k)}) - p^\star \leq c^k (f(x^{(0)}) - p^\star)$$

  $c \in (0,1)$ depends on m, $x^{(0)}$, line search type.

- this means $O(\log 1/\epsilon)$ iterations to get ϵ solution.
- very simple, but often very slow; rarely used in practice

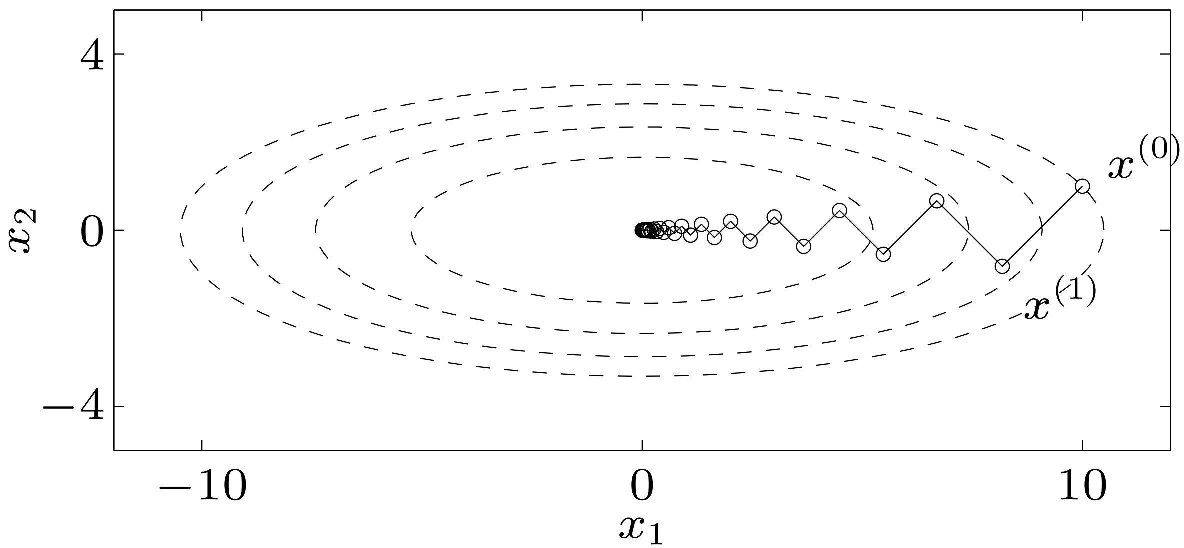
quadratic problem in $\mathbb{R}^2$

$$f(x) = (1/2)(x_1^2 + \gamma x_2^2) \qquad\qquad (\gamma > 0)$$

with exact line search, starting at $x^{(0)} = (\gamma, 1)$:

$$x_1^{(k)} = \gamma \left(\frac{\gamma - 1}{\gamma + 1}\right)^k, \qquad x_2^{(k)} = \left(-\frac{\gamma - 1}{\gamma + 1}\right)^k$$

- very slow if $\gamma \gg 1$ or $\gamma \ll 1$
- example for $\gamma = 10$:



Accelerated Gradient Methods

Accelerated Gradient Methods

Solve

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in C \end{array}$$

in $x \in \mathbb{R}^n$, with $C \subset \mathbb{R}^n$ convex.

- Additional **smoothness** assumption: the gradient is Lipschitz continuous

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \quad \text{for all } x, y \in C$$

where $\|\cdot\|$ is the Euclidean norm (to simplify).

Accelerated Gradient Methods

- Under this new smoothness assumption, we can improve the complexity bound for the most basic gradient method

$$x_{k+1} = x_k - h\nabla f(x_k)$$

for some $h > 0$. We get

$$f(x_k) - f(x^*) \leq \frac{2L(f(x_0) - f(x^*))\|x_0 - x^*\|^2}{2L\|x_0 - x^*\|^2 + k(f(x_0) - f(x^*))}$$

having set $h = 1/L$.

- Roughly $O(1/\epsilon)$ iterations to get ϵ-solution. This is suboptimal as the lower complexity bound is $O(1/\sqrt{\epsilon})$. In what follows, we will see how to reach this optimal complexity.

Accelerated Gradient Methods

The fact that the gradient $\nabla f(x)$ is Lipschitz continuous

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \quad \text{for all } x, y \in C$$

has important algorithmic consequences:

- For any $x, y \in \mathbb{R}^n$,

$$f(y) \leq f(x) + \nabla f(x)^T(y - x) + \frac{L}{2}\|y - x\|^2$$

  and we get a quadratic upper bound on the function $f(x)$.

- This means in particular that if $y = x - \frac{1}{L}\nabla f(x)$, then

$$f(y) \leq f(x) - \frac{1}{2L}\|\nabla f(x)\|^2$$

  and we get a guaranteed decrease in the function value at each gradient step.

Accelerated Gradient Methods

We construct an **estimate sequence** $\phi_k(x)$ of the function $f(x)$, together with sequences $x_k \in \mathbb{R}^n$ and $\lambda_k \geq 0$, satisfying

$$\phi_k(x) \leq (1-\lambda_k)f(x) + \lambda_k \phi_0(x)$$

and

$$f(x_k) \leq \phi_k^* \triangleq \min_{x \in \mathbb{R}^n} \phi_k(x).$$

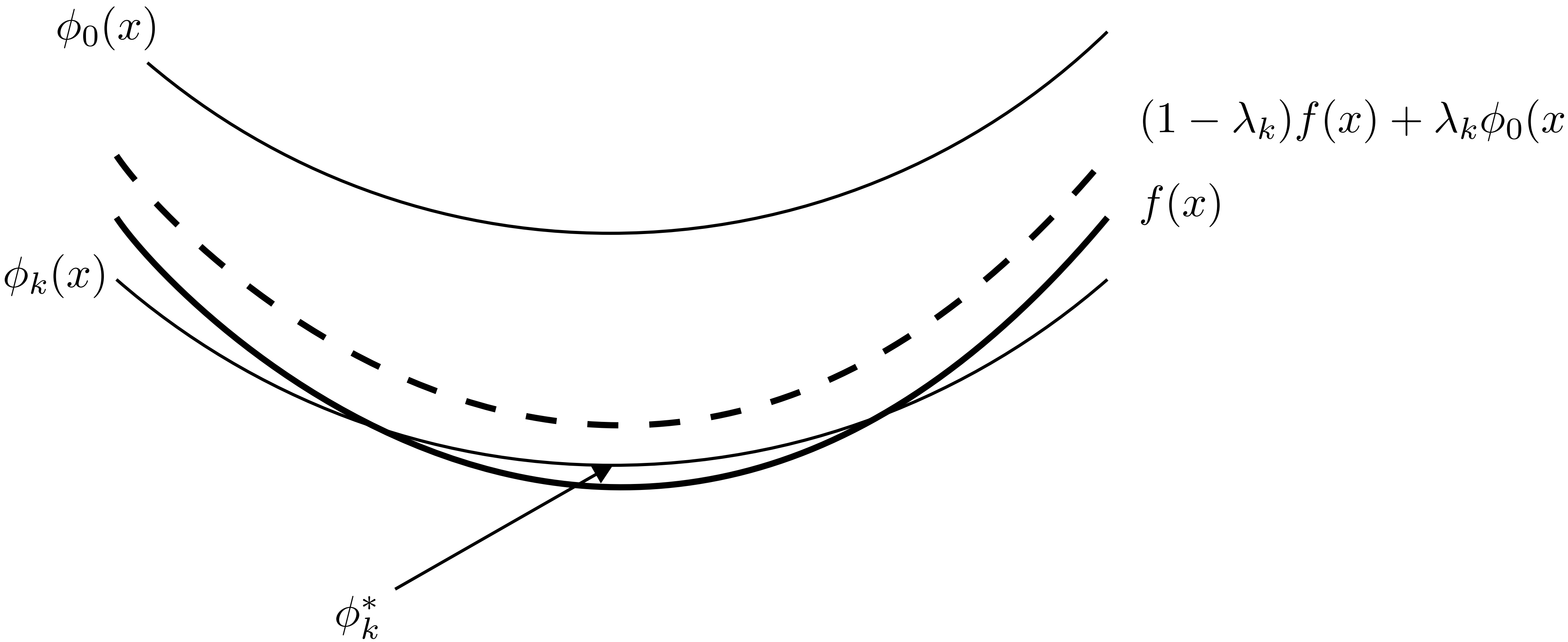
This means in particular that

$$f(x_k) - f^* \leq \lambda_k(\phi_0(x^*) - f^*)$$

(just plug x^* in the inequalities above) so we get convergence if $\lambda_k \to 0$.

Accelerated Gradient Methods

The function $f(x)$ and its estimate functions $\phi_k(x)$:



The functions are $\phi_k(x)$ are increasingly precise approximations of $f(x)$ around the optimum and are easier to minimize.

Accelerated Gradient Methods

Intuition behind the method. Use the fact that the gradient is Lipschitz continuous.

- The inequality

$$f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{L}{2}\|y - x\|^2$$

  helps us build the bounds $\phi_k(x)$.

- In fact, we can pick

$$\phi_k(x) = \phi_k^* + \gamma_k \|x - v_k\|^2$$

  for some $\gamma_k \geq 0$ and $v_k \in \mathbb{R}^n$.

- We get the points x_{k+1} by making a gradient step starting around the minimum of $\phi_k(x)$ (easy to compute), using the guarantee

$$f(y) \leq f(x) - \frac{1}{2L}\|\nabla f(x)\|^2$$

Accelerated Gradient Methods

Also solves minimization problems over simple convex sets $C \subset \mathbb{R}^n$. Define the **gradient mapping**

$$g_C(y,\gamma) = \gamma(y - x_C(y,\gamma))$$

where

$$x_C(y,\gamma) = \operatorname*{argmin}_{x \in C} \left(f(y) + \nabla f(y)^T(x-y) + \frac{\gamma}{2}\|x-y\|^2 \right)$$

- Here, $g_C(y,\gamma)$ plays the role of the gradient for constrained problems, and satisfies

$$f(x) \geq f(x_C(y,\gamma)) + g_C(y,\gamma)^T(x-y) + \frac{1}{2\gamma}\|g_C(y,\gamma)\|^2 + \frac{\mu}{2}\|x-y\|^2$$

- This means in particular

$$f(x_C(y,\gamma)) \leq f(y) - \frac{1}{2\gamma}\|g_C(y,\gamma)\|^2$$

(just set $y = x$ in the previous inequality).

Accelerated Gradient Methods

Minimize $f(x)$ over $C \subset \mathbb{R}^n$. Assuming $\nabla f(x)$ is Lipschitz continuous with constant L and that $f(x)$ is strongly convex with parameter $\mu \geq 0$.

- Choose $x_0 \in \mathbb{R}^n$ and $\alpha_0 \in (0,1)$, set $y_0 = x_0$ and $q = \mu/L$.
- **For** $k = 1, \ldots, k^{max}$ **iterate**

1. Compute $\nabla f(y_k)$ and set

$$x_{k+1} = x_C(y_k, \gamma)$$

2. Compute $\alpha_{k+1} \in (0,1)$ by solving

$$\alpha_{k+1}^2 = (1 - \alpha_{k+1})\alpha_k^2 + q\alpha_{k+1}$$

3. Update the current point, with

$$y_{k+1} = x_{k+1} + \frac{\alpha_k(1-\alpha_k)}{\alpha_k^2 + \alpha_{k+1}}(x_{k+1} - x_k)$$

Accelerated Gradient Methods

Suppose we set $\alpha_0 \geq \sqrt{\mu/L}$, we have the following **complexity** bound

$$f(x_k) - f^* \leq \Delta_0 \min\left\{\left(1-\sqrt{\frac{\mu}{L}}\right)^k, \frac{4L}{(2\sqrt{L}+k\sqrt{\gamma_0})^2}\right\}$$

where

$$\Delta_0 = \left(f(x_0) - f^* + \frac{\gamma_0}{2}\|x_0 - x^*\|^2\right) \quad \text{and} \quad \gamma_0 = \frac{\alpha_0(\alpha_0 L - \mu)}{1-\alpha_0}.$$

When the strong convexity parameter $\mu = 0$, this means roughly $O(1/\sqrt{\epsilon})$ iterations to get an ϵ solution.

Remarks:

- The iterates y_k are not guaranteed to be feasible (in some case, $f(x)$ is not defined outside of C).
- The norm $\|\cdot\|$ is Euclidean. Using other norms is sometimes more efficient.

Both issues can be remedied using an extra minimization subproblem.