Convex Optimization

Convex Functions

Duality: applications in finance

Duality in finance & economics

- Shadow prices: an economic interpretation of duality
- Duality and arbitrage

Shadow prices

- Consider resource assignment problem
- We can form its dual. . .
- The dual gives information on the sensitivity of the solution
- This has a particular interpretation here

Shadow prices

Suppose that we want to solve a minimum cost **resource allocation** problem:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \\ & x \geq 0 \end{array}$$

- Here c is a vector of **costs**
- The variables x_i represent goods, calories, etc
- The constraints $a_i^T x \leq b_i$ account for limiting factors in the problem (warehouse space, labor, CO_2 emissions, etc)

Shadow prices

Suppose there is a twist:

- The constraints can be violated
- The cost of violating constraint i is linear in the amount of violation and given by $-\lambda_i(a_i^T x - b_i)$
- If the constraint is not tight (some resources are left) then a payment to the firm is made $-\lambda_i(a_i^T x - b_i)$
- Of course, we assume that the prices must be positive $\lambda_i \geq 0$

As an example, suppose $a_i^T x - b_i$ represents a warehouse space constraint. The firm can rent additional space at a price of λ_i per square foot. It can also rent out the unused space at the same rate.

Shadow prices

The total cost of the firm, when the constraints can be violated is given by:

$$c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

And the firm will operate to minimize that cost, the final cost being given by:

$$g(\lambda) = \inf_x c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

So the Lagrange dual function gives the production cost as a function of the constraint price vector λ.

Shadow prices

The dual problem is written:

$$\begin{array}{ll} \text{maximize} & g(\lambda) \\ \text{subject to} & x \geq 0 \end{array}$$

- This can be interpreted as the computing the production costs under the least favorable set of prices. At equilibrium
  - the resources you would need (saturated constraints) are too expensive,
  - the resources you have too much of are worthless.
- This is just another (pessimistic) way to describe complementary slackness (in the KKT conditions).

Shadow prices

An economic interpretation for **weak duality**:

Weak duality simply means that the production costs when the constraints cannot be violated is always higher than the cost when constraints can be violated

This is very intuitive:

- Suppose we start from the optimal solution $x^\star$ without violation
- Suppose now that we can buy and sell resources
- Even when the resources we need are too expensive, we can directly improve our existing solution by selling those resources for which $a_i^T x^\star - b_i$ is not saturated.

Shadow prices

Suppose now that **strong duality** holds and the dual objective is attained.

- We then have $d^\star = p^\star$
- The production cost for the resource prices $\lambda^\star$ is the same when the constraints cannot be violated
- This means that, at these prices, the firm is indifferent between violating the constraints or not
- In other words, $\lambda^\star$ are **fair prices** for the additional resources i.

Direct interpretation for sensitivity analysis. . .

Duality and Arbitrage

Duality and Arbitrage

- Another economic interpretation of duality
- Due to the same crowd: Arrow, Debreu, in the 50's. . .
- Used **every day** on financial markets (sometimes unknowingly)
- Simple LP duality result, but underpins most of modern finance theory. . .

One period model

- Basic discrete, **one period** model on an asset x.
- The asset takes the following values

$$x = \{x_1, \ldots, x_n\}$$

at a maturity date T, with probabilities

$$p = \{p_1, \ldots, p_n\}$$

- We can only trade **today** and at **maturity**
- There is a **cash** security worth $\$1$ today, that pays $\$1$ at maturity

One period model

A few other prices are available for products that are actively traded in the market:

- The **forward** contract, $i.e.$ the price of getting the security x at maturity is tradeable.
- No interest rates means this is equal to the price q_1 of the asset today.
- There are also $m-1$ other securities with payoffs at maturity given by:

$$h_k(x_i) \quad \text{if } x = x_i \text{ at time } T$$

- We denote by q_k the price of security with payoff $h_k(x)$.

All these securities are tradeable, can we use them to get information on the price of **another security** with payoff $h_0(x)$?

Static Arbitrage

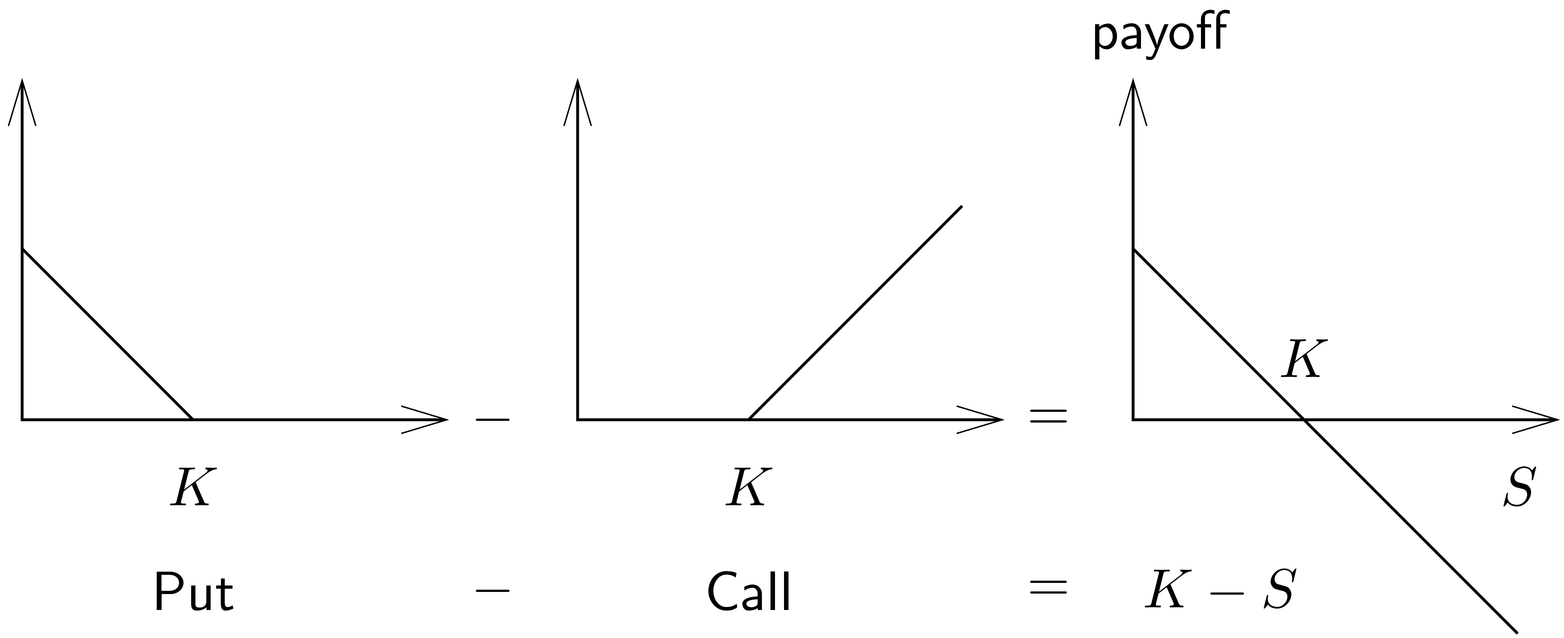
Remember:

- We can only trade today and at maturity.
- We can only trade in securities which are priced by the market.

We want to exclude **arbitrage strategies**

- If the payoff of a portfolio A is always larger than that of a portfolio B then Price(A) $\geq$ Price(B).
- The price of the sum of two products is equal to the sum of the prices.

Simplest Example: Put Call Parity



Price bounds

Suppose that we form a portfolio of cash, stocks and securities $h_k(x)$ with coefficients λ_k:

$$\begin{array}{ll} \lambda_0 & \text{in cash} \\ \lambda_1 & \text{in stock} \\ \lambda_k & \text{in security } h_k(x) \end{array}$$

- All portfolios that satisfy

$$\lambda_0 + \lambda_1 x_i + \sum_{k=2}^{m} \lambda_k h_k(x_i) \geq h_0(x_i) \quad \text{i=1,...,n}$$

  must be **more expensive** than the security $h_0(x)$
- All portfolios that satisfy the **opposite** inequality must be **cheaper**
- For portfolios that satisfy neither of these, **nothing** can be said. . .

Price bounds

- For each of these portfolios, we get an upper/lower bound on the price today of the security $h_0(x)$.
- We can look for optimal bounds. . .

We can solve:

$$\begin{array}{ll} \text{minimize} & \lambda_0 + \lambda_1 q_1 + \sum_{k=1}^{m} \lambda_k q_k \\ \text{subject to} & \lambda_0 + \lambda_1 x_i + \sum_{k=2}^{m} \lambda_k h_k(x_i) \geq h_0(x_i), \quad i = 1, \ldots, n \end{array}$$

- Linear program in the variable $\lambda \in \mathbf{R}^{(m+1)}$
- Produces an optimal upper bound on the price today of the security $h_0(x)$

Linear Programming Duality

The original linear program looks like:

$$\begin{array}{ll} \text{minimize} & c^T\lambda \\ \text{subject to} & A\lambda \succeq b \end{array}$$

which is a linear program in the variable $\lambda \in \mathbf{R}^m$.

We can form the Lagrangian

$$L(\lambda, p) = c^T\lambda + y^T(b - A\lambda)$$

in the variables $\lambda \in \mathbf{R}^m$ and $y \in \mathbf{R}^n$, with $y \succeq 0$.

Linear Programming Duality

We then minimize in λ to get the dual function

$$g(y) = \inf_{\lambda} c^T\lambda + y^T(b - A\lambda)$$

for $y \succeq 0$, which is again

$$g(y) = \inf_{\lambda} y^T b + \lambda^T(c - A^T y)$$

and we get:

$$g(y) = \begin{cases} y^T b & \text{if } c - A^T y = 0 \\ -\infty & \text{if not.} \end{cases}$$

Linear Programming Duality

With

$$g(y) = \begin{cases} y^T b & \text{if } c - A^T y = 0 \\ -\infty & \text{if not.} \end{cases}$$

we get the **dual linear program** as:

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y = c \\ & y \succeq 0 \end{array}$$

which is also a linear program in $x \in \mathbf{R}^n$.

LP duality: summary

The primal LP is the original linear program looks like:

$$\begin{array}{ll} \text{minimize} & c^T\lambda \\ \text{subject to} & A\lambda \succeq b \end{array}$$

its **dual** is then given by:

$$\begin{array}{ll} \text{maximize} & b^Ty \\ \text{subject to} & A^Ty = c \\ & y \succeq 0 \end{array}$$

Strong duality: both optimal values are **equal***

* Except in some pathological cases

LP duality & arbitrage

Let's look at what this produces for the portfolio problem. . .

- The **primal** problem in the variable $\lambda \in \mathbf{R}^m$ is given by:

$$p^{\max} := \quad \text{min.} \quad \lambda_0 + \lambda_1 q_1 + \sum_{k=2}^m \lambda_k q_k$$

$$\text{s.t.} \quad \lambda_0 + \lambda_1 x_i + \sum_{k=2}^m \lambda_k h_k(x_i) \geq h_0(x_i), \quad i = 1, \ldots, n$$

- The **dual** in the variable $y \in \mathbf{R}^n$ is then

$$p^{\max} := \quad \text{max.} \quad \sum_{i=1}^n y_i h_0(x_i)$$

$$\begin{array}{ll} \text{s.t.} & \sum_{i=1}^n y_i h_k(x_i) = q_k, \quad k = 2, \ldots, m \\ & \sum_{i=1}^n y_i x_i = q_1 \\ & \sum_{i=1}^n y_i = 1 \\ & y \succeq 0 \end{array}$$

LP duality & arbitrage

- The last two constraints $\{\sum_{i=1}^n y_i = 1,\ y \succeq 0\}$ mean that y is a **probability measure**.
- We can rewrite the previous program as:

$$
\begin{array}{rll}
p^{\max} := & \text{max.} & \mathbf{E}_y[h_0(x)] \\
 & & \\
 & \text{s.t.} & \mathbf{E}_y[h_k(x)] = q_k, \quad k = 2, \ldots, m \\
 & & \mathbf{E}_y[x] = q_1 \\
 & & y \text{ is a probability}
\end{array}
$$

- We can compute $p^{\min}$ by minimizing instead.

LP duality & arbitrage

What does this mean?

There are three ranges of prices for the security with payoff $h_0(x)$:

- Prices above $p^{\max}$: these are **not viable**, you can get a cheaper portfolio with a payoff that always dominates $h_0(x)$.
- Prices in $[p^{\min}, p^{\max}]$: prices are **viable**, $i.e.$ compatible with the absence of arbitrage.
- Prices below $p^{\min}$: these are **not viable**, you can get a portfolio that is more expensive than $h_0(x)$ with a payoff that is always dominated by $h_0(x)$.

Price bounds

Example:

- Suppose the product in the objective is a call option:

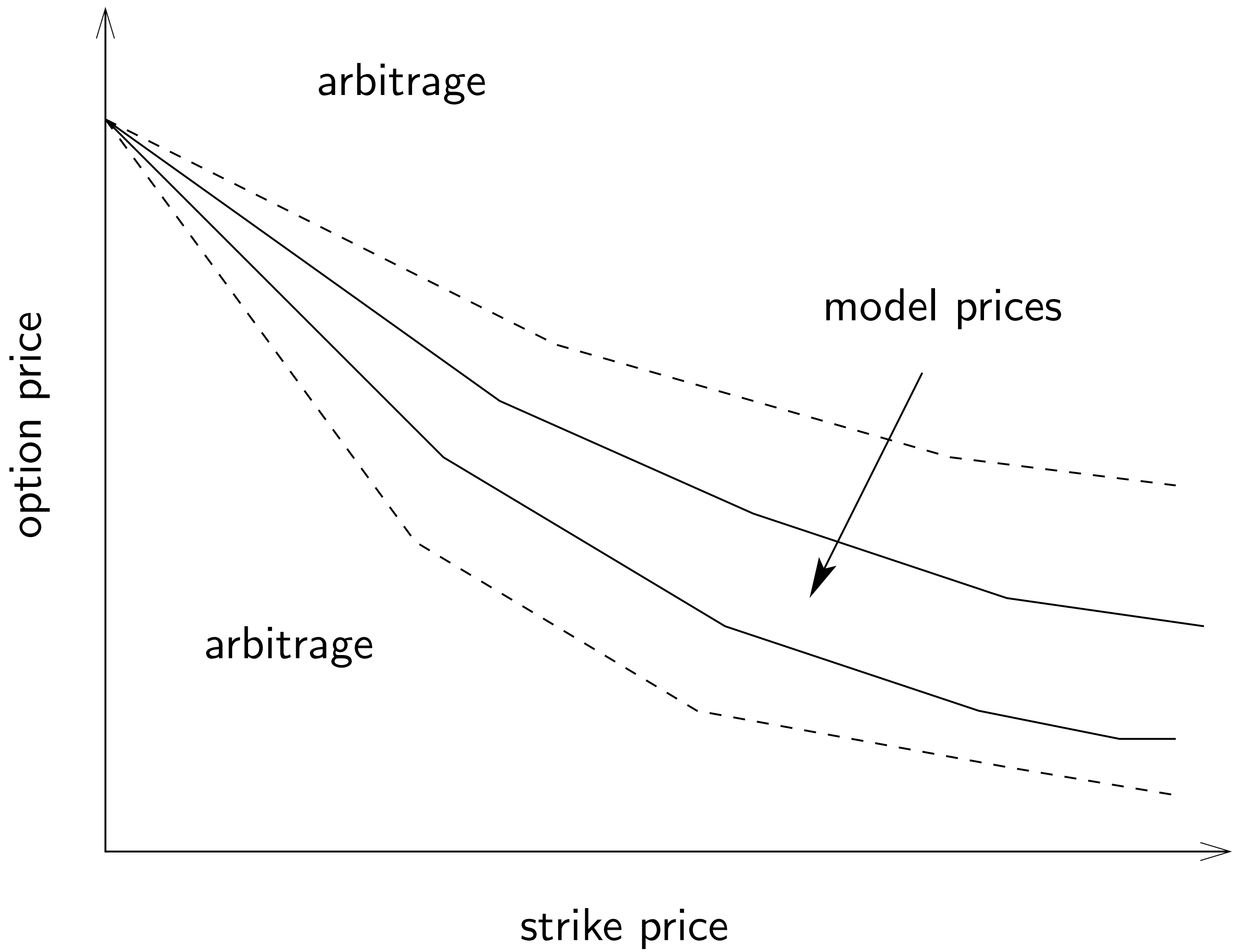
$$h_0(x) = (x - K)^+$$

where K is called the strike price.

- Suppose also that we know the prices of some other instruments
- We get upper and lower price bounds on the price of this call for each strike K

On a graphic. . .

Price Bounds



LP duality & arbitrage

What if there is no solution y and the linear program is infeasible?

- Then the original data set q must contain an arbitrage.
- Start with one product, stock and cash. . . and test.
- Increase the number of products. . .

LP duality & arbitrage

Fundamental theorem of asset pricing

In the one period model, there is not arbitrage between the prices $\{q_0, \ldots, q_m\}$ of securities with payoff at maturity $\{h_0(x), \ldots, h_m(x)\}$

$$\Updownarrow$$

There exists a probability y (with $\sum_{i=1}^n y_i = 1$ and $y \succeq 0$) such that

$$q_k = \mathbf{E}_y[h_k(x)], \quad k = 0, \ldots, m$$

LP duality & arbitrage

- Because prices are computed using expectations under y (and not expected utility/certain equivalent), we call the probability y **risk-neutral**.
- In particular, it satisfies $q_1 = \mathbf{E}_y[x]$
- If there are constant interest rates, simply use discounted values. . .
- This probability y has **nothing to do** with the observed distribution of the asset x or its past distribution! (Very common mistake)

LP duality & arbitrage

- Because you can trade in the asset and its derivatives to form portfolios to hedge/replicate other products, it is possible to evaluate these products using expected value under an **appropriate choice** of probability.
- This risk-neutral probability y is a **tool**, it has nothing to do with the statistical properties of the underlying asset x.
- Linear programming duality is interpreted as a duality between **portfolio** problems and **models** (probabilities)