On the linear distance to the giant component in the Poisson Boolean percolation model.

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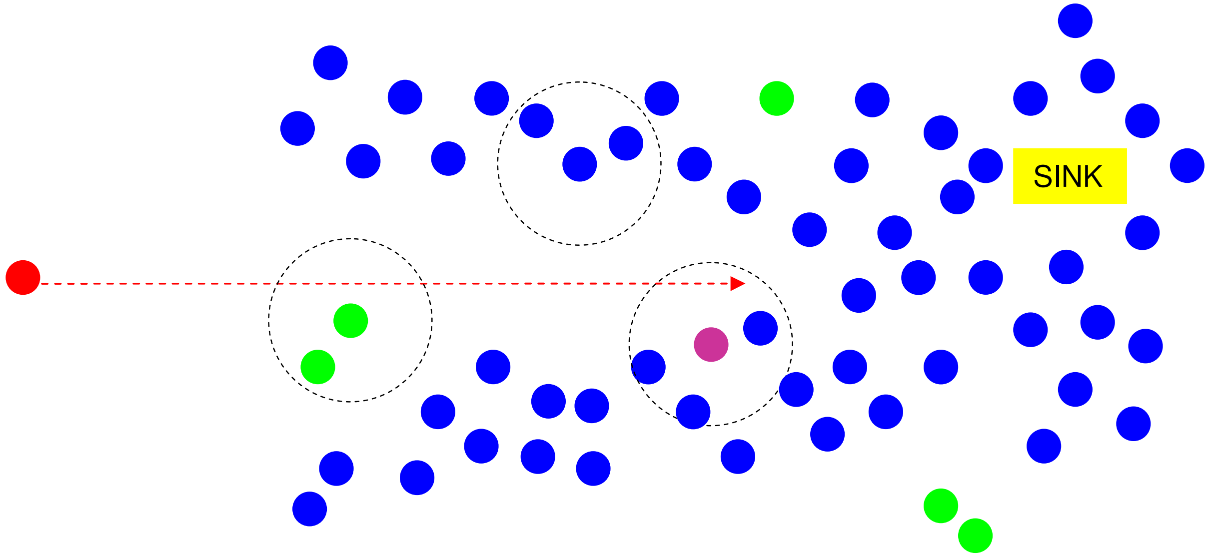
Motivation

Wireless sensor networks (WSNs)

- Target detection
  - Bounding time to detection of intruder with successful alarm relay.
- Connectivity under mobility
  - Connecting to the giant cluster when initially isolated, with no prior knowledge of relative cluster locations.



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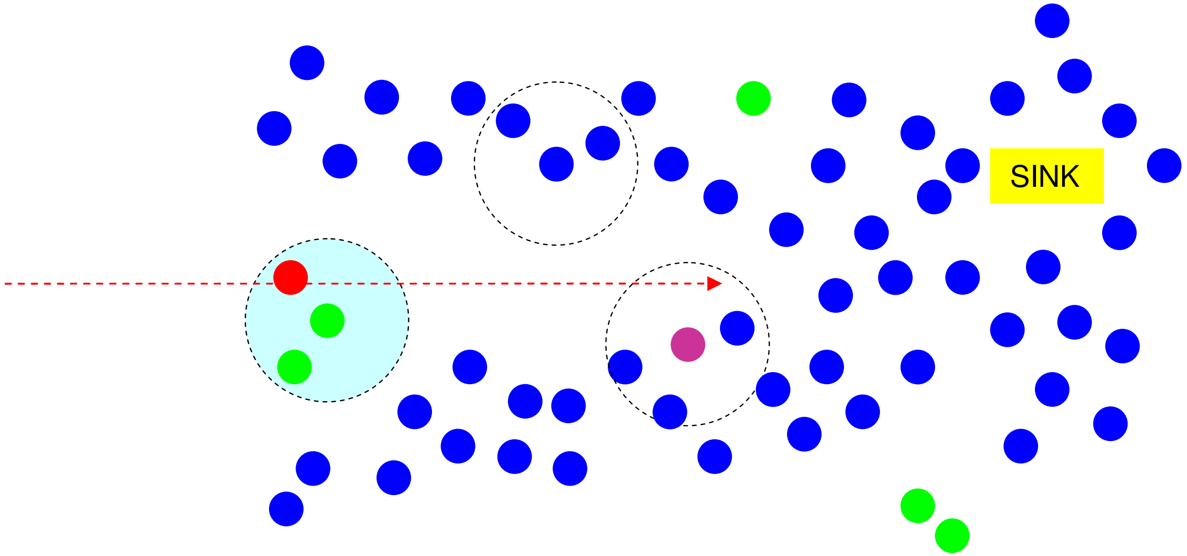

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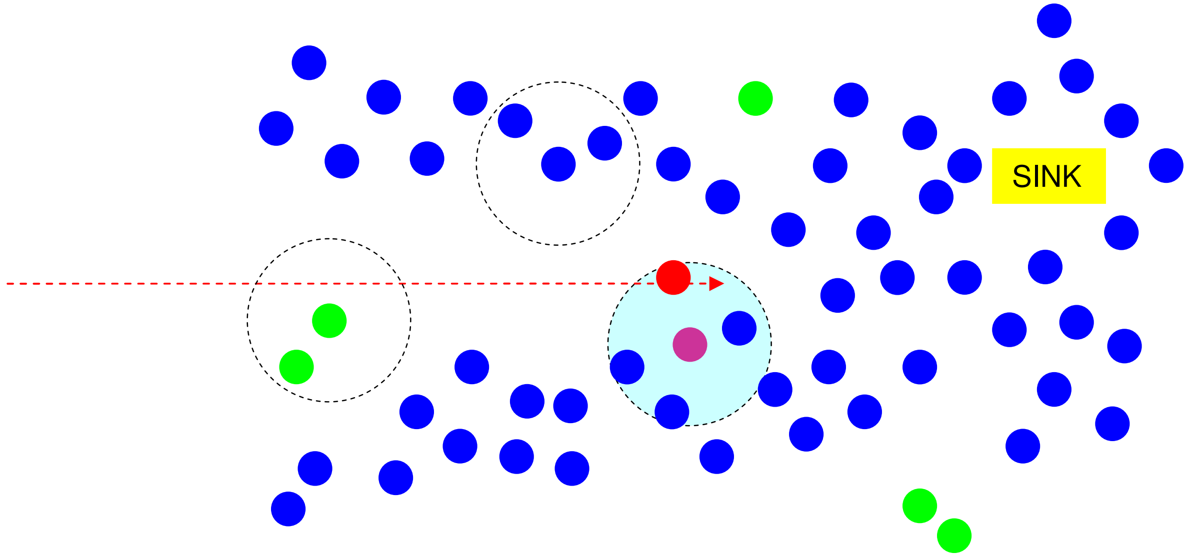

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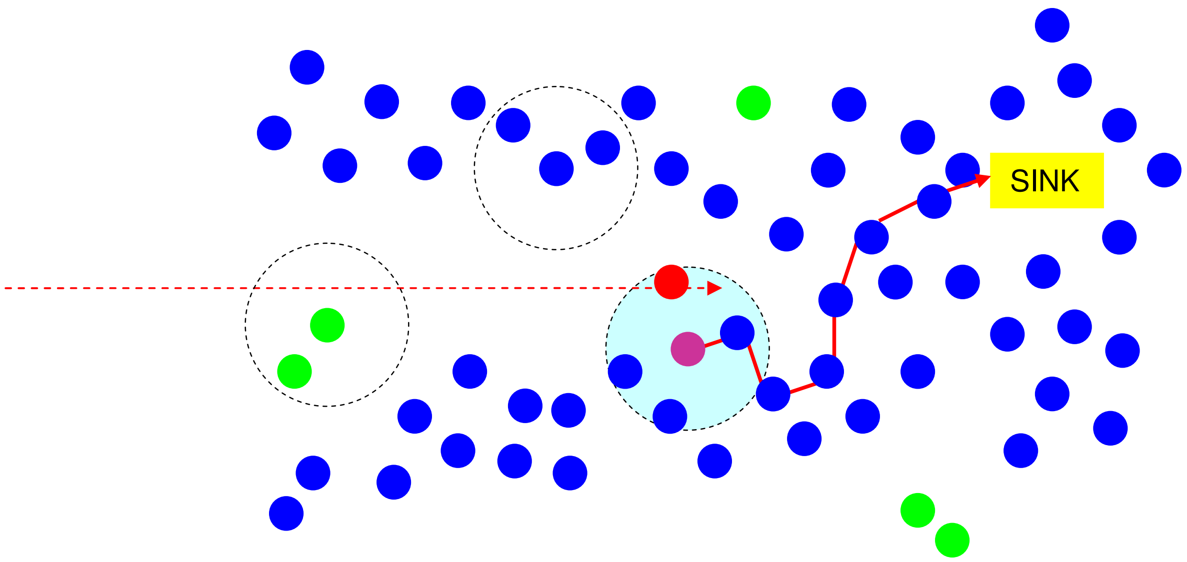

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- Contact distribution function with dynamic connectivity resulting from alternating on/off behavior. [Gui04]
- Contact distribution function in (mobile) Brownian Boolean model. [Kesidis03]

Problem statement

- Poisson Boolean model
  - Nodes are distributed according to a 2-dimensional Poisson point process with intensity λ.
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Question: How long does it take for the target to hit the *biggest* cluster?

Outline of approach

1. Consider bond percolation model on a square lattice.
2. Construct algorithm to upper bound the linear distance for the bond percolation model.
3. Resolve a mapping from the continuous Poisson Boolean model to the discrete bond percolation model.
4. Apply the result for the discrete case to the continuous case.

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Bond percolation model

- Bond percolation on the square lattice $\mathbb{L}^2$.
- Each edge open with probability p and closed with probability $(1-p)$ independently of all other edges.
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Theorem 1. *Let N be the first coordinate on the positive x-axis that belongs to the giant cluster. (When the origin is part of the giant cluster, $N = 0$.) When $p > p_c$, there exist constants c_1 and $c_2 \in \mathbb{R}^+$ such that*

$$P_p(N > n) \leq c_1 \exp(-c_2 n).$$

Some Variables

- $(N, 0)$ is the first point on the positive x-axis that belongs to the giant cluster.

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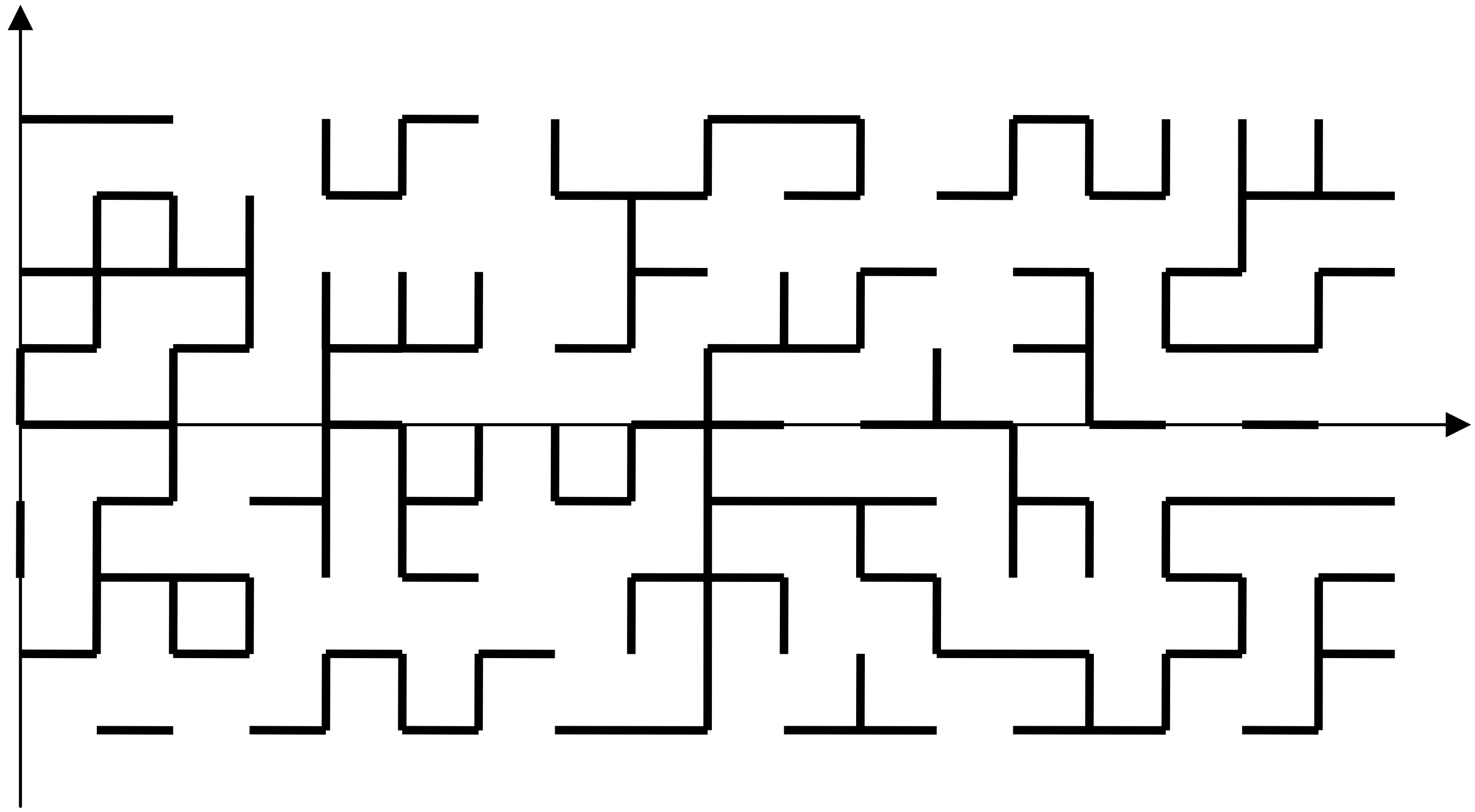
- M is the radius of a finite component.
- Define box $B^+(n) = [0, n][-n, n]$ with surface $\partial B^+(n)$.
- $\theta^+(p)$ is the probability that the origin is part of the infinite component on the half plane $\mathbb{H} = \mathbb{Z}^+ \times \mathbb{Z}$.

Example



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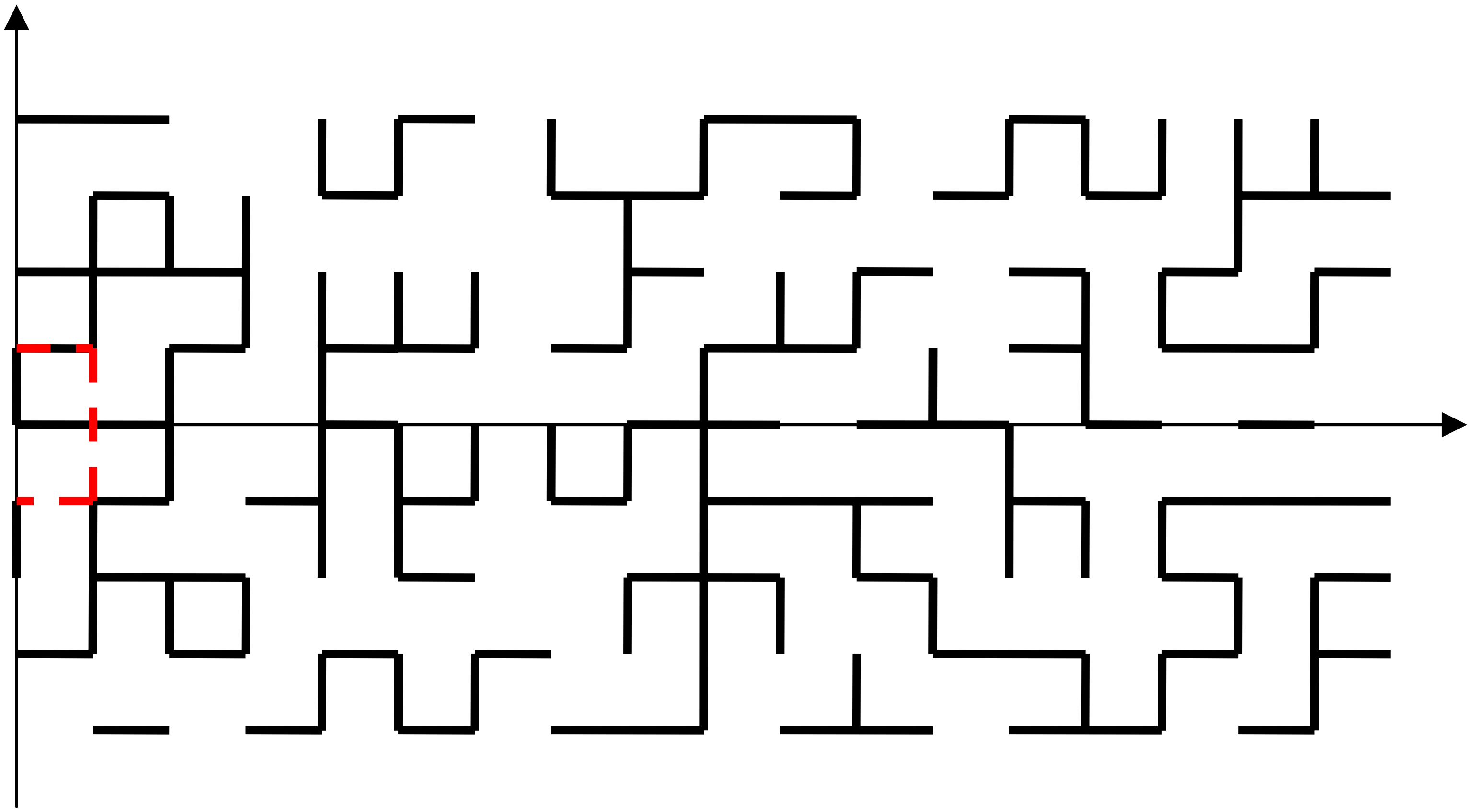

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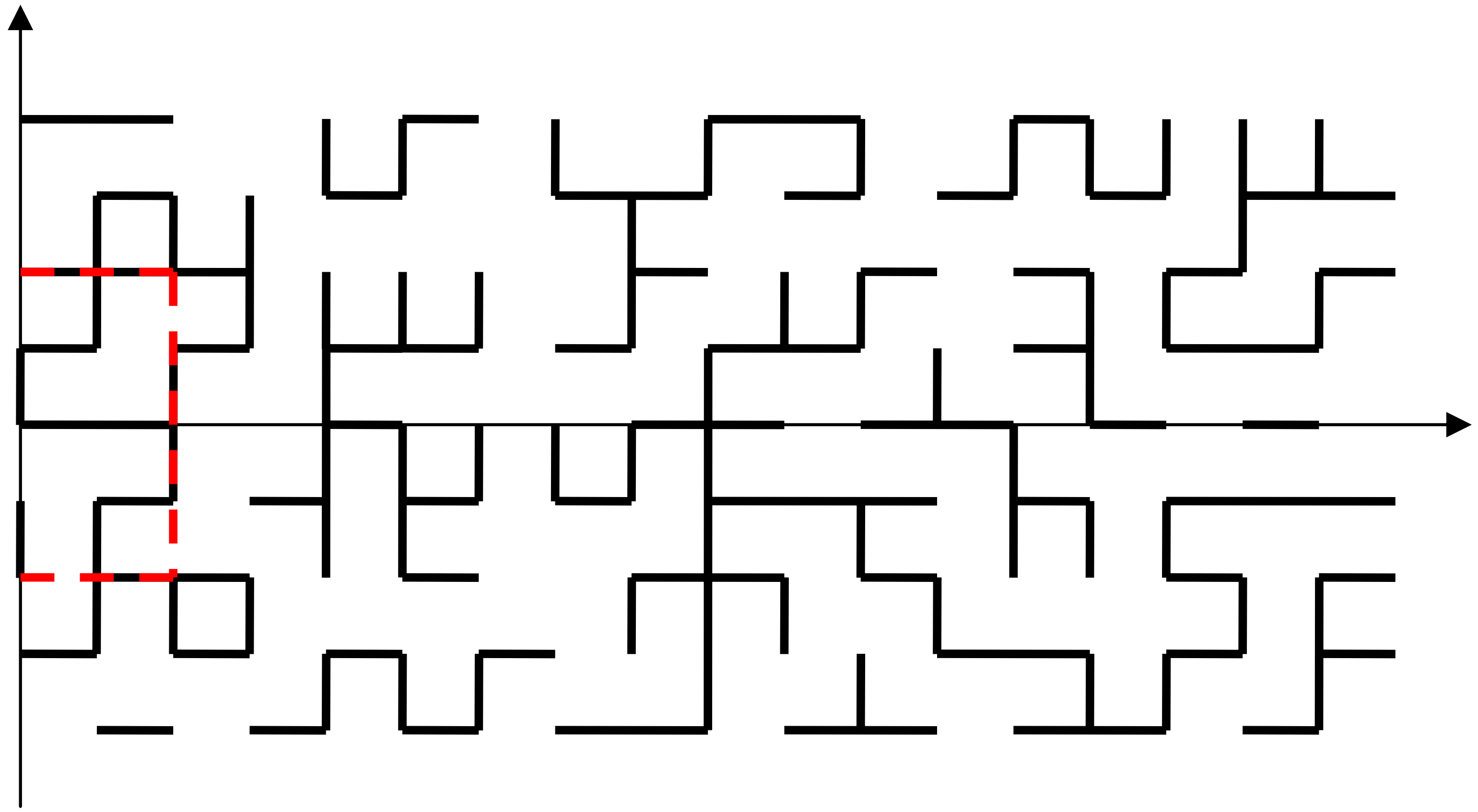

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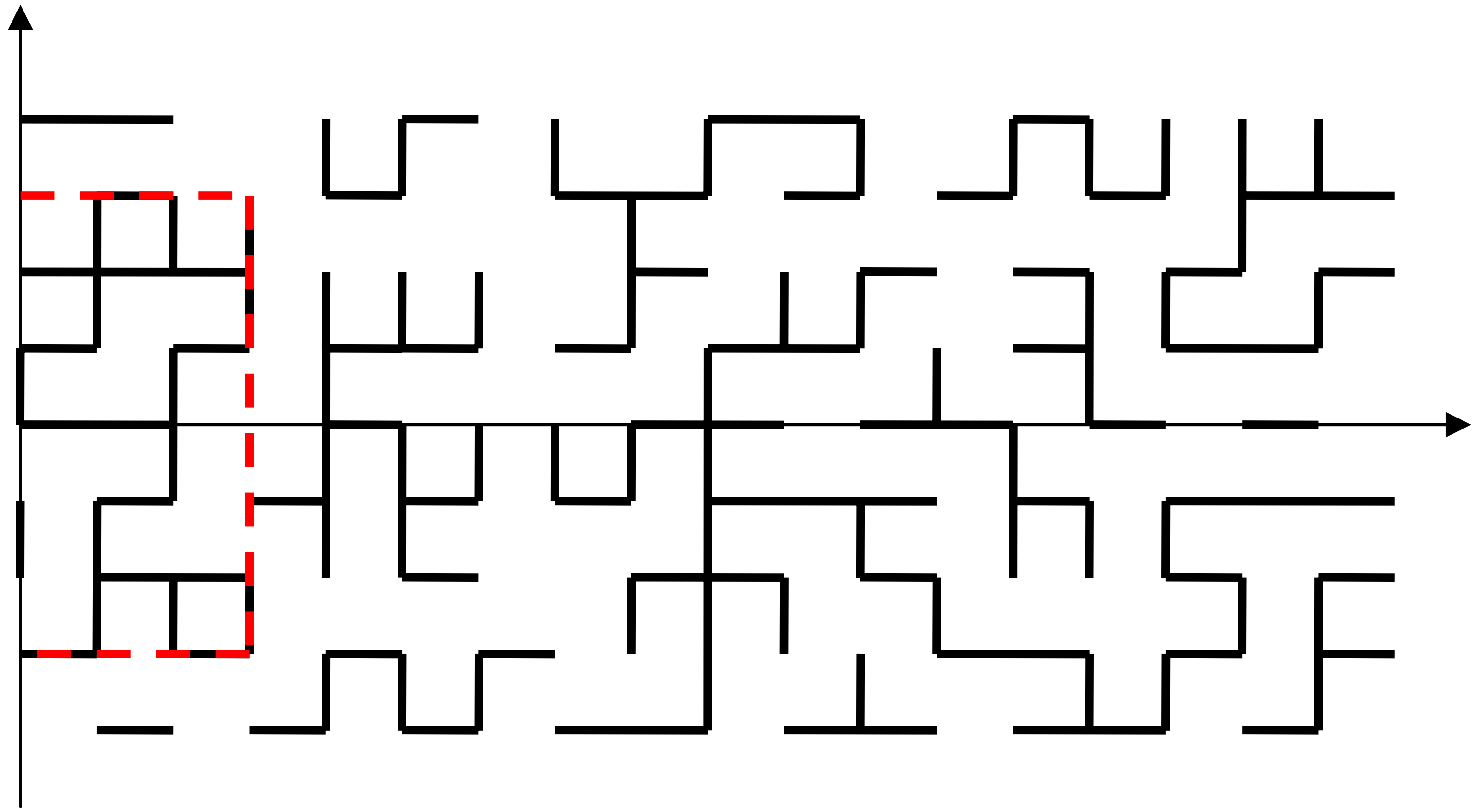

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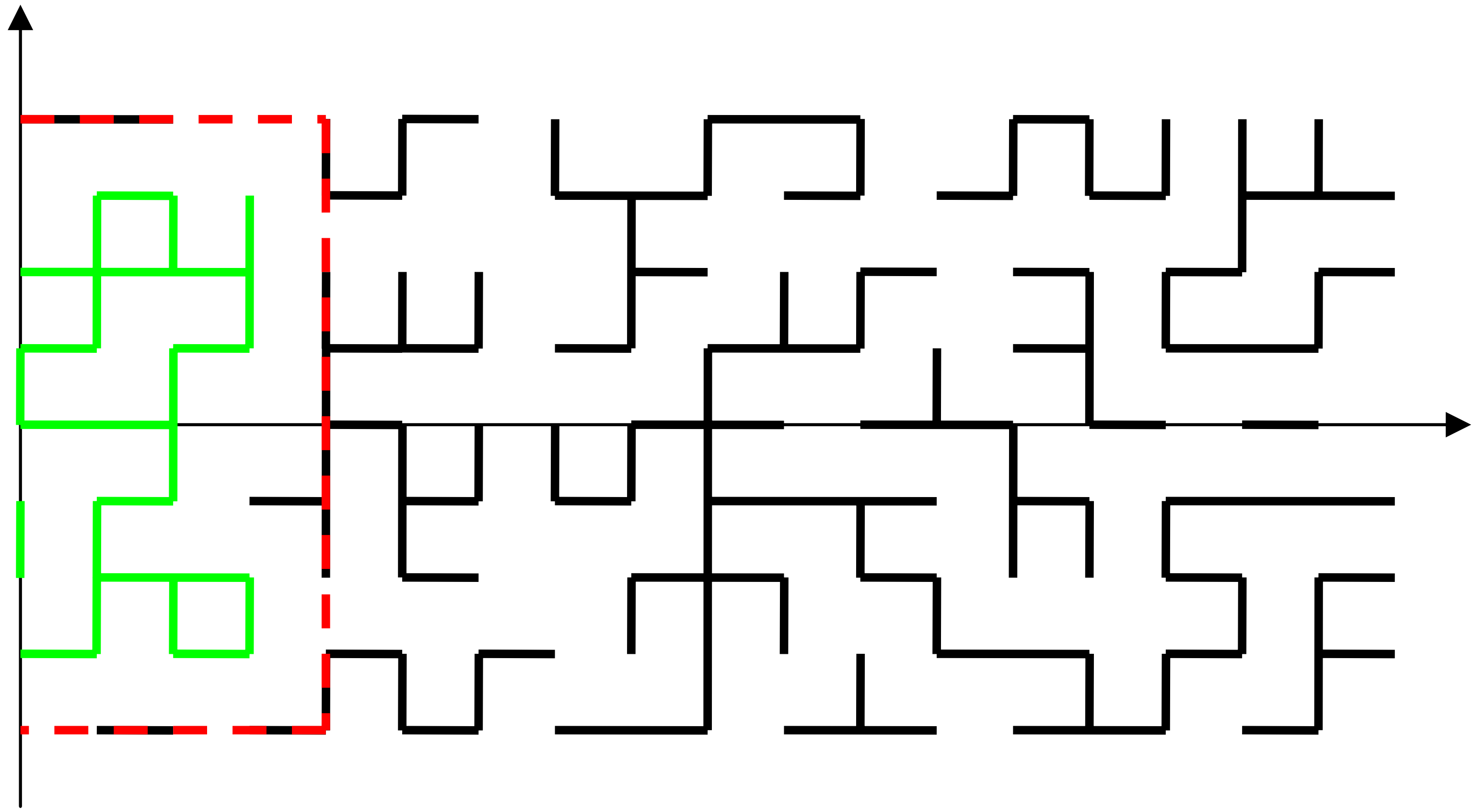
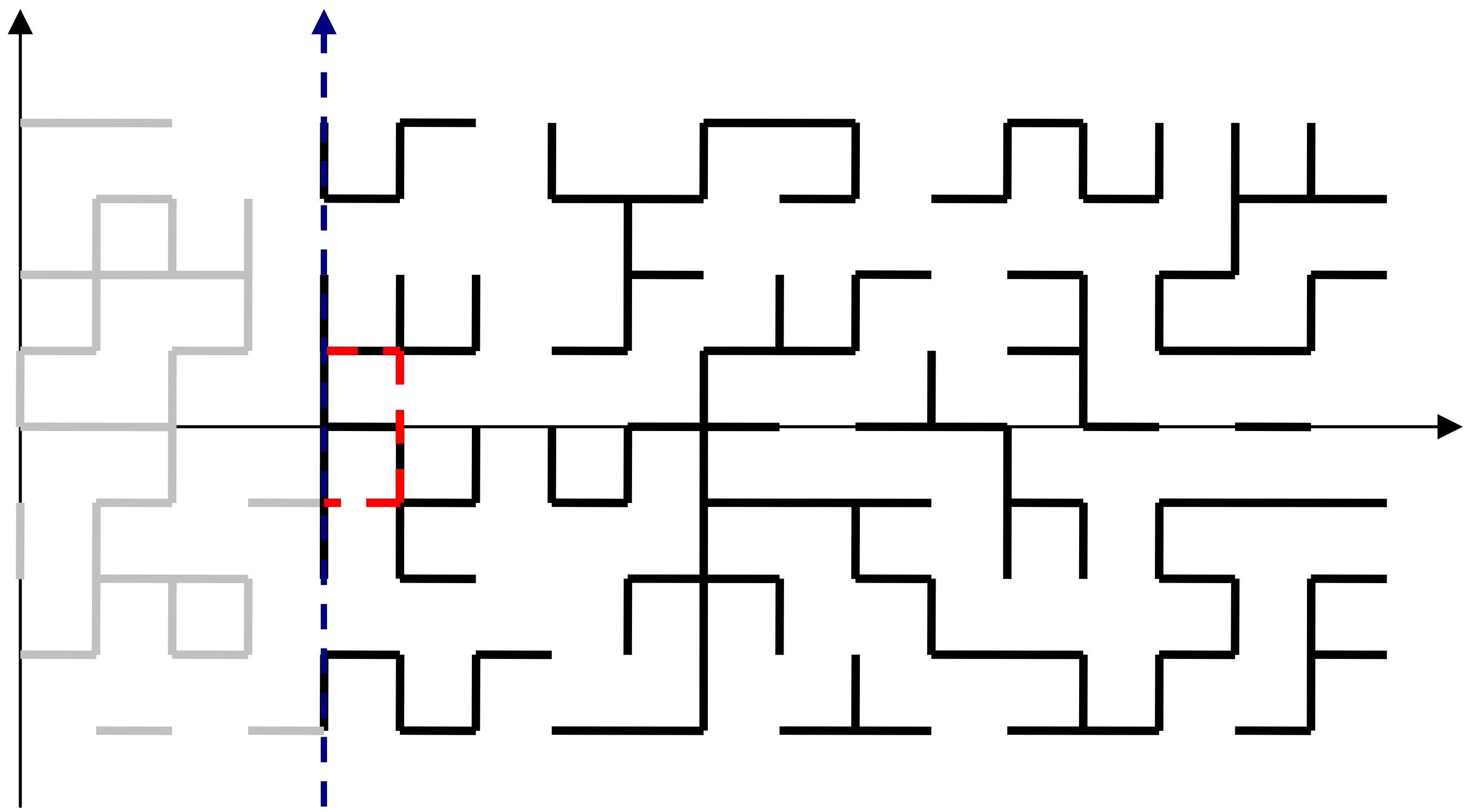


$$M_1 = 4.$$

Example



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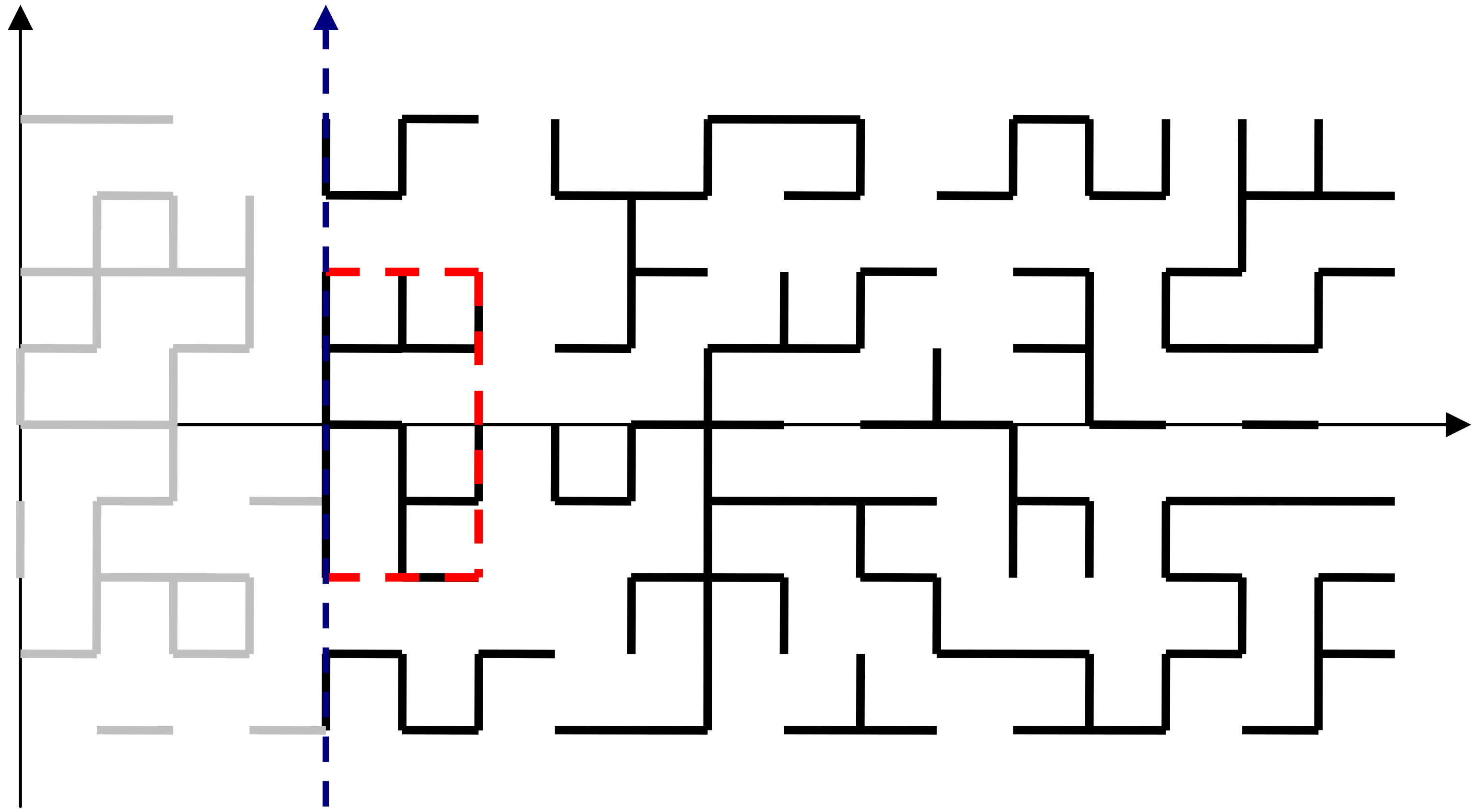

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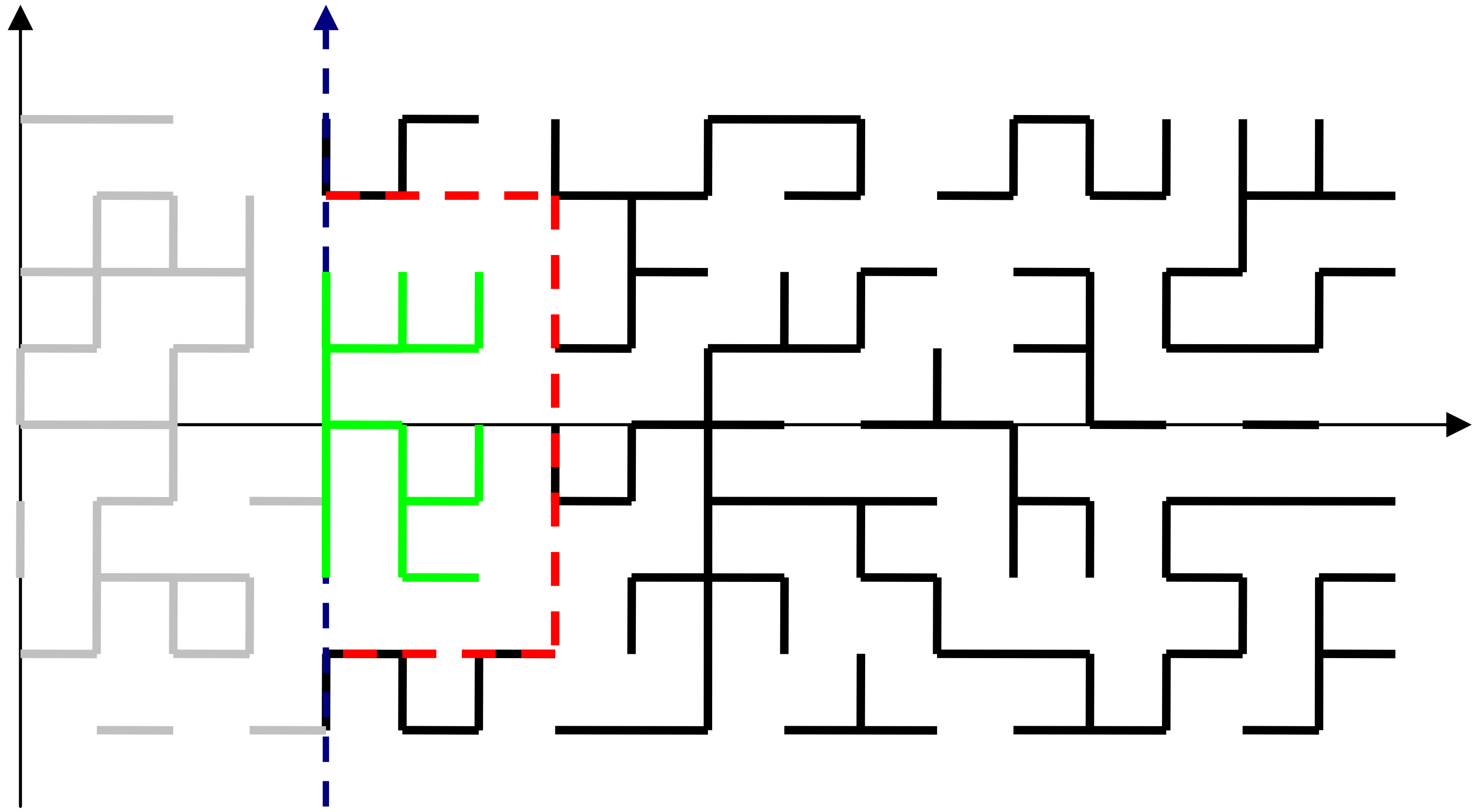
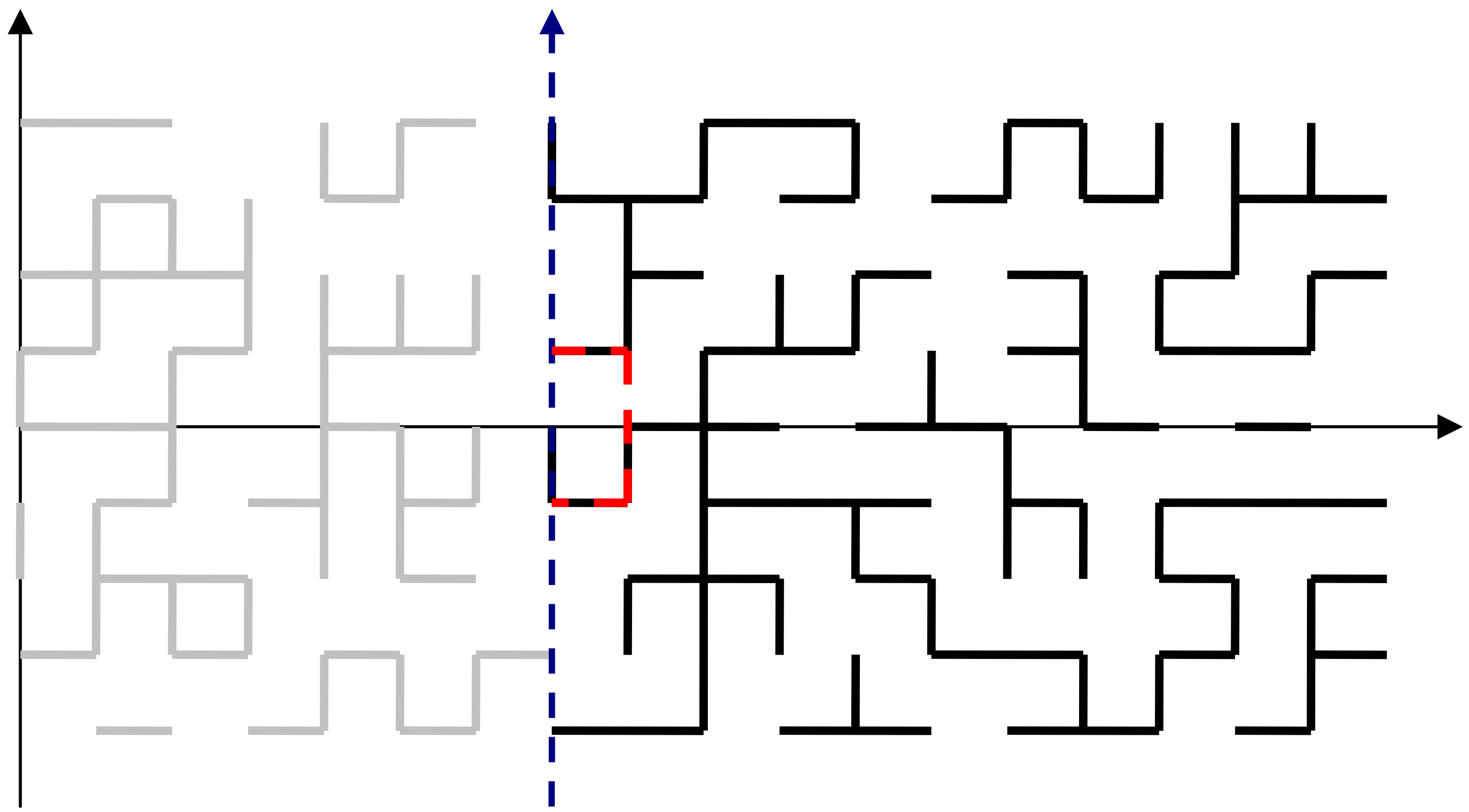


$$M_2 = 3.$$

Example



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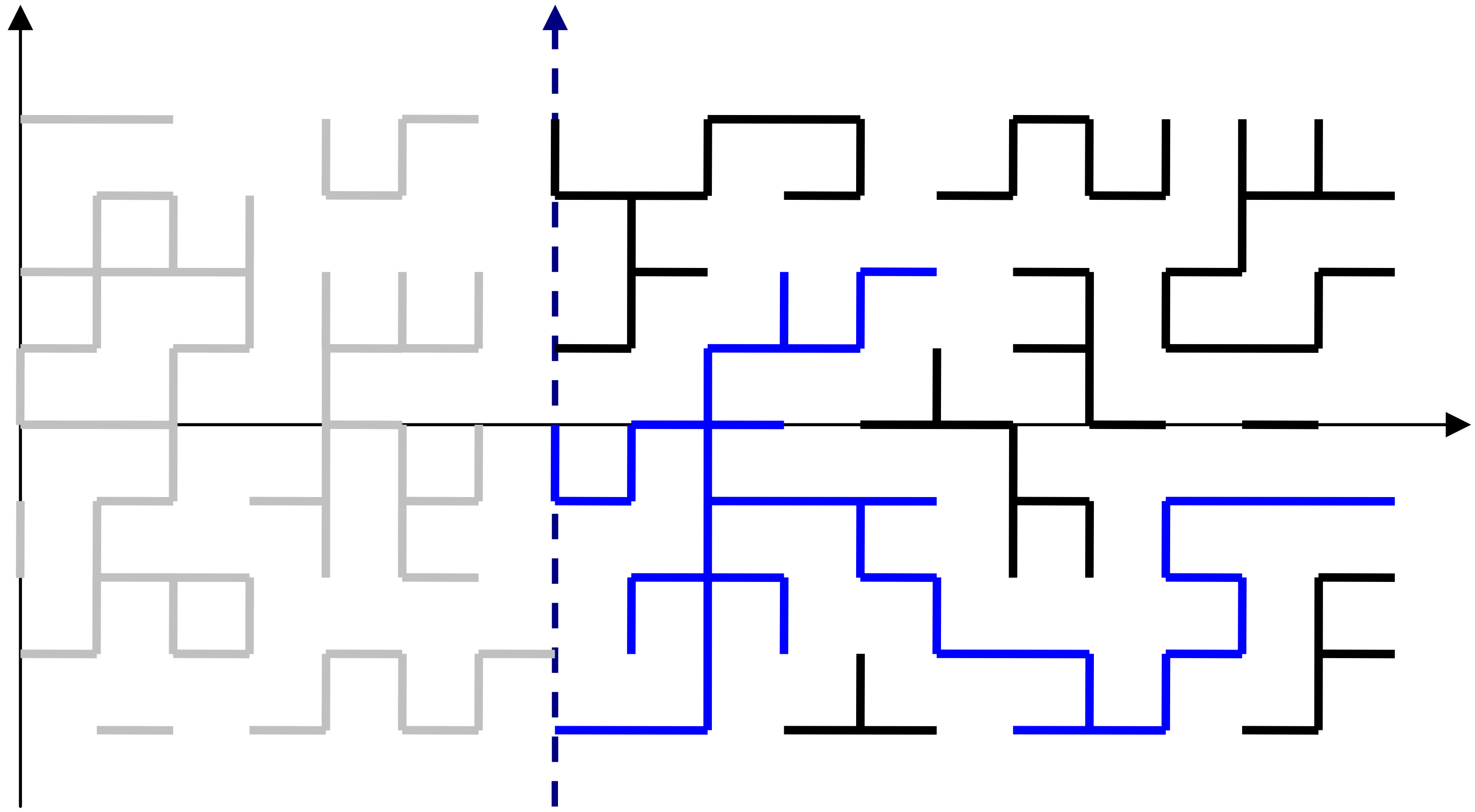

$$N' = 7.$$

Algorithm construction

1. Initialize $M = 0$.
2. From origin, consider the set of points located the right of the vertical axis in the half-plane $\mathbb{H} = \mathbb{Z}^+ \times \mathbb{Z}$ that are connected to it.
3. If this set is infinite, return $N' = M$.
   Otherwise, determine the size of the component by iteratively looking at the boxes $B^+(n) = [0, n] \times [-n, n]$, n=1,2,. . .
   - Let $\{0 \leftrightarrow \partial B^+(n)\}$ be the event that there is a path from the origin to the surface of box $B^+(n)$.
   - stop at n when this event first becomes false.
4. Let $M = n$ and repeat from step 2 treating position $(M, 0)$ as the origin and considering the half-plane $\mathbb{H} + M$.

Remarks

- The event $\{M = n\}$ only depends on the state of the edges inside the box $B^+(n) = [0, n] \times [-n, n]$.
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- We require that $\theta^+(p) > 0$ (i.e. that the probability that the origin is part of the infinite component on the half plane is greater than 0.)
  - Theorem in [Grimmett99] which tells us that $\theta(p) > 0 \Rightarrow \theta^+(p) > 0$. (The threshold probabilities are equal.)

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- N' is a worst case for N, $N' \geq N$.

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Theorem 2. *There exist finite constants $k_1, k_2 > 0$ for $p_c < p < 1$ such that*

$$P_p(0 \leftrightarrow \partial B^+(n), |C^+| < \infty) \leq k_1 n^2 e^{-k_2 n} \text{ for all } n.$$

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- Thus, $P_p(M > n) \leq k_1 n^2 e^{-k_2 n}$ for all n.

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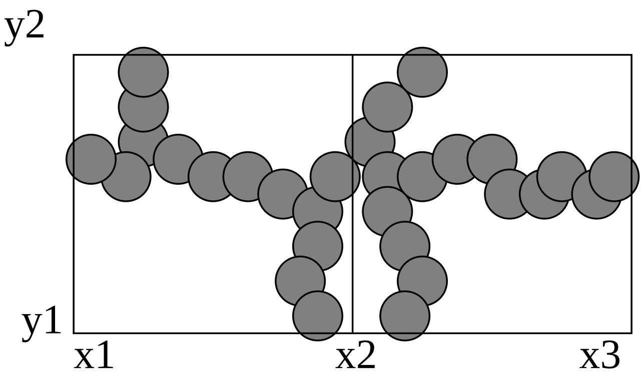
Stopping point N'

- The random variable N' is the sum of K independent variables M_i identically distributed to M, $N' = \sum_{i=1}^{K} M_i$.
- K is a geometric random variable, $P(K = n) = \theta^{+}(p)(1 - \theta^{+}(p))^{n}$.
- We have

$$
\begin{aligned}
P(N > n) &\leq P(N' > n) \\
&= P(\sum_{i=1}^{K} M_i > n) \\
&\leq c_1 \exp(-c_2 n)
\end{aligned}
$$

Mapping



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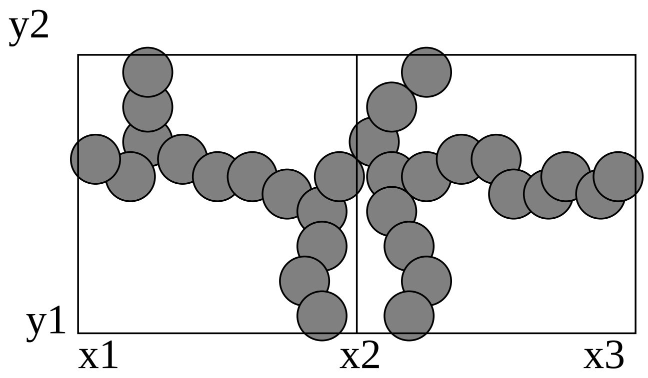

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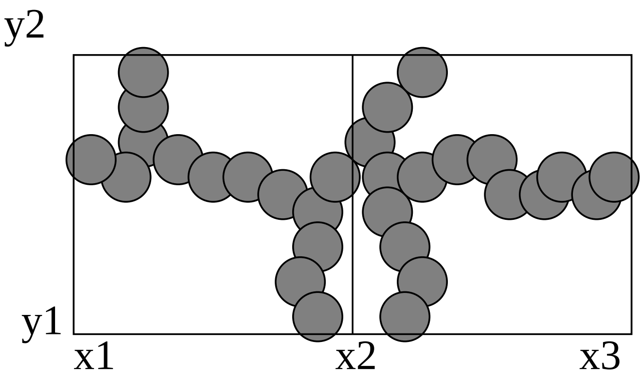

- $(N, 0)$ in the discrete model$\Rightarrow$ there exists a point of the Boolean model in the interval $[Nd, (N+1)d]$ that belongs to the infinite cluster.
- This mapping is for a dependent bond percolation model.
  - A k-dependent model stochastically dominates the independent bond percolation model. [Liggett1997]

$$P_{p'}(N \leq n) \geq P_p(N \leq n)$$

Poisson Boolean model

Theorem 3. *Let D denote the coordinate of the first point on the positive part of the horizontal axis that belongs to the unbounded component ($D = 0$ if the origin is already in the unbounded component). There exists c_3 and $c_4 \in \mathbb{R}^+$ such that*

$$P(D > x) \leq c_3 \exp(-c_4 x).$$

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Conclusions and Future Work

- We showed that the distance traveled by an object moving along a straight line in a supercritical percolation model until it hits the giant component is bounded from above by an exponential variable.

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  - Bounding time for an isolated mobile node to connect to the giant cluster.
- Our future work includes adding
  - Full mobility.
  - General mobility models.

Thank you!