Critical node lifetimes in random networks via the Chen-Stein method

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Abstract

This paper considers networks where nodes are connected randomly and can fail at random times. It provides scaling laws that allow to find the critical time at which isolated nodes begin to appear in the system as its size tends to infinity. Applications are in the areas of sensor and ad-hoc networks where nodes are subject to battery drainage and 'blind spots' formation becomes a primary concern. The techniques adopted are based on the Chen-Stein method of *Poisson approximation*, which allows to obtain elegant derivations that are shown to improve upon and simplify previous related results that appeared in the literature. Since blind spots are strongly related to full connectivity, we also obtain some scaling results about the latter.

1 Introduction

In this paper we present results for networks where nodes are connected randomly and can fail at random times. We consider two percolation-theory based models of random networks. The first one finds its original formulation in a paper by Broadbent and Hammersley [3], who considered grid networks made of edges that are drawn independently with probability p. Today, applications of this simple model range from random electrical networks to reliability theory, statistical mechanics and epidemics, to name a few. We refer to [11] for extensive references and for a more detailed account of these applications. The second model is a natural extension of the first one to the continuum plane and was proposed by Gilbert [10]. He considered nodes randomly located on the plane, according to a two-dimensional Poisson point process, and edges connecting pairs of nodes that are within range $2r$ of each other. This gives a family of random graphs, parametrised by the density λ of the point process, or equivalently by the radius r of the discs. Gilbert's model, as well as more general random connection models, are extensively discussed in [15]. Recently, there has been a

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growing interest within the information theory and networking community in applications and extensions of these two models to the emerging field of ad-hoc and sensor networks. In this setting, percolation theory can provide the tools necessary to derive the scaling laws of these systems as their size grows. Indeed, results obtained under both Gilbert's continuum formulation and Broadbent and Hammersley's discrete one, have revealed to be key tools to derive connectivity, capacity, and latency results of random communication networks. The reader is referred for example to [2, 6, 7, 12, 13]. More recently, [8] exploited results from discrete percolation to derive information theoretic lower bounds on the capacity of random wireless networks, and [9] used results from continuum percolation to derive bounds on the capacity of random wireless relay networks. These works show that connectivity results in both the discrete and the continuum setting can be used as building blocks to provide an answer to more complex questions.

In this paper we are concerned with the effect of an additional source of randomness present in the system besides connectivity: due to power consumption nodes may become inactive over time. That is, given a random network connectivity graph G_n formed by n nodes, we let T_i be the (random) failure time of node i, and for all $i \in G_n$ we let $P_n(T_i \leq t) = q_n(t)$. We are then interested in discovering the critical time scale t_n at which "blind spots" (i.e. nodes with no active neighbours) begin to appear in the network. As we shall see, before this phenomenon is observed, the network essentially appears as being fully connected. To estimate such a critical time, we first derive the precise asymptotics for the number of isolated nodes in a random network where nodes are always active, and then modify these results to tackle the node failure scenario. Throughout the paper we make use of the powerful Chen-Stein method to establish convergence to a Poisson distribution. This is named after work of Chen [5] and Stein [17] and is the subject of the monograph in [4]. Where necessary, we also make reference to results of Penrose [16] obtained in the continuum setting using the same Poisson approximation techniques.

We also refer the reader to two additional related papers. The authors in [18], using different methods, derived the asymptotic distribution of the number of isolated nodes in the continuum model in the special case when $q_n = q$, uniformly in n and t. Our corresponding results reduce to the ones in [18] when q_n is taken to be a constant, and in this case the entire derivation of the results in [18] follows almost trivially by applying scaling relations to the work in [16]. The authors in [14] considered the node isolation problem and the critical time for the emergence of singletons, for given $q_n(t)$. However, their results do not reduce to the sharp ones of [18], due to the incomplete characterization of some additional order terms. We also point out that the Chen-Stein method is relatively new and this work serves as an opportunity to introduce this approach into the networking community.

We summarize our contribution as follows. By using the Chen-Stein machinery, first we derive the asymptotic distribution of the number of isolated nodes in a random grid network where nodes are always active. We show how this is related to the notion of full connectivity of the entire network. Then, we compare these results with the corresponding ones derived by Penrose [16] in the continuum setting. Finally, we move on to the case when nodes in the random network have a common random lifetime distribution $q_n(t)$ and we show how to derive the critical threshold time t_n at which blind spots begin to arise in the two settings.

The rest of the paper is organized as follows. In the next section we present results for node isolation and full connectivity when the only randomness is due to random connections and there are no node failures. In Section 3 we derive the critical time scales for isolated nodes to emerge in the network in the presence of node failures. Finally, Section 4 concludes the paper.

2 Poisson approximation, node isolation and full connectivity

In this section we consider node isolation events in random networks where all nodes are active all the time, we determine their asymptotic distribution, and show how their disappearance corresponds to reaching full connectivity of the network.

We begin by introducing our technical machinery. The Poisson distribution naturally arises as the limiting distribution of the sum of n independent, low probability, indicator random variables. The idea behind Chen-Stein convergence is that this situation generalises to dependent, low probability random variables, as long as dependencies are negligible as n tends to infinity. More formally, we define the distance between two probability distributions as follows.

Definition 2.1 *The total variation distance between two probability distributions p and q over $\mathbb{N}$ is defined by*

$$d_{TV}(p,q) = \sup\{|p(A) - q(A)| : A \subset \mathbb{N}\} \tag{1}$$

We refer the reader to [4], where the Chen-Stein method is applied to obtain many bounds on the total variation distance between the Poisson distribution of parameter λ and the distribution of the sum of n dependent indicator random variables of expectations p_i. One bound that we use in this context holds when the indicator variables are increasing functions of independent random variables. We introduce the following notation. Let $\mathcal{I}$ be an arbitrary index set, and for $\alpha \in \mathcal{I}$, let I_α be the indicator random variable with expectation $E(I_\alpha) = p_\alpha$. Let $W = \sum_{\alpha\in\mathcal{I}} I_\alpha$ and $Po(\lambda)$ be a Poisson random variable of parameter λ. We shall use the following bound (Corollary 2.E.1 in [4]).

Theorem 2.2 *If the I_α's are increasing functions of independent random variables, then we have*

$$d_{TV}\left(W, Po(\lambda)\right) \leq \frac{1-e^{-\lambda}}{\lambda}\left(\mathrm{Var}\ W - \lambda + 2\sum_{\alpha\in\mathcal{I}} p_\alpha^2\right). \tag{2}$$

The application we make of this bound is by considering indicator random variables of node isolation events in random networks, whose probability decays with n. In this case we shall see that the bound converges to zero and the sum W of the indicators converges to the Poisson distribution of parameter $\lambda = E(W)$.

Another bound we use is given in [1], and makes use of the notion of *neighbourhood of dependence*, as defined below.

Definition 2.3 *For each $\alpha \in \mathcal{I}$, $B_\alpha \subset \mathcal{I}$ is a neighbourhood of dependence for α, if I_α is independent of all indices I_β, for $\beta \notin B_\alpha$.*

Theorem 2.4 *Let B_α be a neighbourhood of dependence for $\alpha \in \mathcal{I}$. Let*

$$\begin{aligned} b_1 &\equiv \sum_{\alpha\in\mathcal{I}} \sum_{\beta\in B_\alpha} E(I_\alpha)E(I_\beta), \\ b_2 &\equiv \sum_{\alpha\in\mathcal{I}} \sum_{\beta\in B_\alpha, \beta\neq\alpha} E(I_\alpha I_\beta). \end{aligned} \tag{3}$$

It is the case that

$$d_{TV}(W, Po(\lambda)) \leq 2(b_1 + b_2). \tag{4}$$

We start by looking at the random grid, where each nearest neighbour edge is present with probability p, independent of all other edges. We shall see that the notion of blind spots is intimately related to the notion of *full connectivity* of the network. We are interested in discovering *scaling laws* of sequences of finite networks G_n contained in a box of grid size $n \times n$. These scaling laws are events that occur asymptotically almost surely (a.a.s.), meaning with probability tending to one as as $n \to \infty$. We use the terminology with high probability (w.h.p.), to mean the same thing. To obtain a *fully* connected network inside the finite box, the value of p must be close to one, and we shall give the precise rate by which p must approach one as $n \to \infty$ in order to obtain full connectivity w.h.p. Note this question is is not very meaningful if we assign the probability p to the vertices of the grid rather than to the edges, as in this case each vertex can be disconnected with probability $(1-p)$, so the network is connected if and only if all sites are occupied with probability one. Hence we only deal with the edge model, that is, with bond percolation.

The first theorem shows the correct scaling of p_n required for the number of isolated vertices to converge to a Poisson distribution.

Theorem 2.5 *Let N_n be the number of isolated vertices in G_n. If, for some constant $\lambda > 0$, p_n is such that*

$$\lambda = n^2(1-p_n)^4, \tag{5}$$

then as $n \to \infty$, N_n converges in distribution to a Poisson random variable with parameter λ.

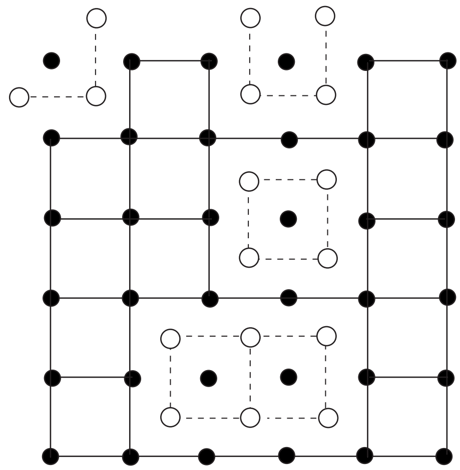

Figure 1: Some configurations of isolated nodes in the random grid G_n. Dual paths are indicated with a dashed line.

Proof of Theorem 2.5 The main idea behind this proof is to use the Chen-Stein upper bound (2) on the total variation distance between the distribution of the number of isolated vertices in G_n and the Poisson distribution of parameter λ. First, to ensure we can apply such bound we need to show that isolated vertices are functions of independent random variables. This follows from the *dual graph construction*, which is often used in percolation theory. We refer to Figure 1. Let the dual graph of the $n \times n$ grid G_n be obtained by placing a vertex in each square of the grid (and also along the boundary) and joining two such vertices by an edge whenever the corresponding squares share a side. We draw an edge of the dual if it crosses an edge of the original random grid, and delete it otherwise. It should be clear now that a node is isolated if and only if it is surrounded by a closed circuit in the dual graph, and hence node isolation events are increasing functions of independent random variables corresponding to the edges of the dual graph.

We can then proceed applying the Chen-Stein bound. Note that since isolated vertices are rare events, and most of them are independent, as $n \to \infty$, we expect this bound to tend to zero, which immediately implies convergence in distribution. Next we spell out the details; similar computations are necessary later in this paper, and it pays to see the details once[1].

Let I_i be the indicator random variable of node i being isolated, for $i = 1, \ldots, n^2$. Let ∂G_n be the vertices on the boundary of G_n. We denote the four corner vertices by $\angle G_n$; the boundary vertices excluding corners by $\| G_n \equiv$

[1]In the following we make use of Landau's order notation: with x_0 possibly being ∞ we write $f(x) = O(g(x))$ as $x \to x_0$ if $\limsup_{x \to x_0} \frac{f(x)}{g(x)} < \infty$; we also write $f(x) = o(g(x))$ as $x \to x_0$ if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$.

$\partial G_n \setminus \angle G_n$; and the interior vertices by $\Box G_n \equiv G_n \setminus \partial G_n$. Finally, we let

$$W = \sum_{i=1}^{n^2} I_i. \tag{6}$$

We want to bound $d_{TV}(W, Po(\lambda))$. We start by computing some probabilities. For this computation, examining the corresponding dual lattice configurations depicted in Figure 1 might be helpful.

$$\begin{aligned} P(I_i = 1) &= (1-p_n)^4, \text{ for } i \in \Box G_n, \\ P(I_i = 1) &= (1-p_n)^3, \text{ for } i \in \| G_n, \\ P(I_i = 1) &= (1-p_n)^2, \text{ for } i \in \angle G_n. \end{aligned} \tag{7}$$

Note now that by (5) we have that $1 - p_n = O(1/\sqrt{n})$. This, in conjunction with (7) and a counting argument, gives

$$\begin{aligned} E(W) &= \sum_{i=1}^{n^2} E(I_i) = \sum_{i=1}^{n^2} P(I_i = 1) \\ &= (n-2)^2(1-p_n)^4 + (4n-8)(1-p_n)^3 + 4(1-p_n)^2 \\ &= n^2(1-p_n)^4 + o(1/\sqrt{n}) \\ &\to \lambda, \end{aligned} \tag{8}$$

as $n \to \infty$. Similarly, we have

$$\begin{aligned} \sum_{i=1}^{n^2} [P(I_i = 1)]^2 &= n^2(1-p_n)^8 + o(1/\sqrt{n}) \\ &= \lambda(1-p_n)^4 + o(1/\sqrt{n}) \\ &\to 0, \end{aligned} \tag{9}$$

as $n \to \infty$. Finally, we also need to compute

$$\begin{aligned} E(W^2) &= E\left(\sum_\alpha I_\alpha \sum_\beta I_\beta\right) \\ &= E\left(\sum_\alpha I_\alpha + \sum_{\alpha \not\sim \beta} I_\alpha I_\beta + \sum_{\alpha \sim \beta} I_\alpha I_\beta\right), \end{aligned} \tag{10}$$

where we have indicated with $\alpha \sim \beta$ the indices corresponding to neighbouring vertices, and $\alpha \not\sim \beta$ the indices corresponding to vertices that are not neighbouring neither the same. We proceed by evaluating the three sums in (10).

$$E \sum_\alpha I_\alpha = E(W) \to \lambda, \tag{11}$$

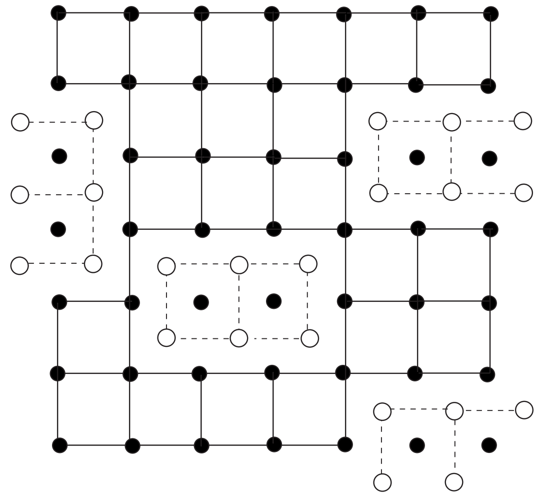

Figure 2: Configurations of adjacent isolated nodes in the random grid G_n. Dual paths are indicated with a dashed line.

$$
\begin{aligned}
E\sum_{\alpha\sim\beta} I_\alpha I_\beta &= \big\{O(n^2)(1-p_n)^7 + O(n)[(1-p_n)^6 + (1-p_n)^5] \\
&+ O(1)(1-p_n)^4\big\} \\
&= o(1/n) \to 0, \qquad (12)
\end{aligned}
$$

where the different possible configurations of adjacent isolated nodes are depicted in Figure 2. Finally, the third sum yields

$$
\begin{aligned}
E\sum_{\alpha\not\sim\beta} I_\alpha I_\beta &= 2\left[\binom{n^2-4n-4}{2} + O(n^2)\right](1-p_n)^8 + o(1/n) \\
&= n^4(1-p_n)^8 + o(1/n) \to \lambda^2, \qquad (13)
\end{aligned}
$$

where the dominant term corresponds to the first order term of the configuration of all $\binom{n^4-4n-4}{2}$ isolated pairs not lying on the boundary, excluding the $O(n^2)$ isolated pairs that are adjacent to each other. Substituting (11), (12), (13) into (10) it follows that

$$\lim_{n\to\infty} \text{Var } W = \lim_{n\to\infty} E(W^2) - E^2(W) = \lambda + \lambda^2 - \lambda^2 = \lambda. \qquad (14)$$

By substituting (9) and (14) into (2), we finally obtain

$$\lim_{n\to\infty} d_{TV}(W, Po(\lambda)) = 0, \qquad (15)$$

and the proof is complete. □

The previous theorem tells us that for some constant $c > 0$, $p_n = 1 - \frac{c}{\sqrt{n}}$ is the critical regime for convergence of the number of isolated nodes to a non-trivial distribution. The next proposition explores this a little further and articulates the relation between full connectivity and isolated nodes.

Proposition 2.6 *For any $0 < c < \infty$, if $p_n = 1 - \frac{c}{\sqrt{n}}$, then w.h.p. G_n is either fully connected, or it contains at least one isolated vertex.*

Proof of Proposition 2.6. We use a counting argument in conjunction with the dual graph construction. First, we note that in order for G_n to be disconnected *and* not to contain isolated vertices, there must be either a self-avoiding path of length at least three in the dual graph starting at the boundary of the dual graph, or a self-avoiding path of length at least six starting in the interior of the dual graph, see Figure 2.

Let $P_n(\ell)$ be the probability of existence of a self-avoiding path of length at least ℓ starting from a particular vertex G_n. By the union bound, and since the number of paths of length k starting at a given vertex is bounded by $4 \cdot 3^{k-1}$, we have

$$\begin{aligned} P_n(\ell) &\leq \sum_{k=\ell}^{\infty} 4 \cdot 3^{k-1}(1-p_n)^k = \frac{4}{3}\sum_{k=\ell}^{\infty}[3(1-p_n)]^k \\ &= \frac{4}{3}\sum_{k=\ell}^{\infty}\left(\frac{3c}{n^{\frac{1}{2}}}\right)^k = \frac{4}{3}\frac{\left(\frac{3c}{n^{1/2}}\right)^\ell}{1-\frac{3c}{n^{1/2}}} \\ &= \frac{4}{3}(3c)^\ell \frac{n^{-\frac{1}{2}(\ell-1)}}{n^{\frac{1}{2}} - 3c}. \end{aligned} \tag{16}$$

For boundary vertices of the dual graph, we are interested in $\ell = 3$, leading to an upper bound of $\frac{4}{3}(3c)^3 \frac{n^{-1}}{n^{\frac{1}{2}}-3c}$. Since the number of boundary vertices is $4n$, again applying the union bound it follows that the probability of a self-avoiding path of length three in the dual starting at the boundary, tends to 0 as $n \to \infty$. For the interior vertices we take $\ell = 6$, leading to an upper bound of $\frac{4}{3}(3c)^6 \frac{n^{-\frac{5}{2}}}{n^{\frac{1}{2}}-3c}$. Since the number of such interior vertices is of the order n^2, the probability of a path of length at least 6 tends to 0 as $n \to \infty$, completing the proof. $\square$

We also have the following result,

Theorem 2.7 *Let $p_n = 1 - \frac{c_n}{\sqrt{n}}$ and let A_n be the probability that there are no isolated nodes in G_n. We have that*

$$\lim_{n \to \infty} P(A_n) = e^{-c^4}, \tag{17}$$

if and only if $c_n \to c$ (where $c = \infty$ is allowed).

Proof of Theorem 2.7 We give the proof for $c < \infty$. The case for $c = \infty$ follows from similar arguments. We start by showing that $c_n \to c$ is a sufficient condition for (17) to hold. We have in this case that for any $\epsilon > 0$ there exists an $N_1 > 0$ such that for all $n > N_1$ $c_n \in [c-\epsilon, c+\epsilon]$. We also have by Theorem 2.5 that if $p_n = 1 - \frac{c}{\sqrt{n}}$, then

$$\lim_{n\to\infty} P(A_n) = e^{-c^4}. \tag{18}$$

Now note that since p_n is monotone in c, and A_n is a monotone event, then $P(A_n)$ is also monotone in c, for all n. It follows that there exists an $N_2 \geq N_1$ such that for all $n > N_2$ $P(A_n) \in [e^{-(c+\epsilon)^4} - \epsilon, e^{-(c-\epsilon)^4} + \epsilon]$, and since ϵ can be arbitrarily small the sufficient condition is proven.

We now have to show that if $c_n \not\to c$, then (17) does not hold. We have that if $c_n \not\to c$, then there exists a subsequence c_{n_k}, $k = 1, 2, \ldots$, such that $c_{n_k} \to c^* \neq c$ for $k \to \infty$. But in the first part of the proof we have shown that this is a sufficient condition for the subsequence $P(A_{n_k})$ to converge to $e^{-{c^*}^4} \neq e^{-c^4}$. It follows that $P(A_n)$ cannot converge to e^{-c^4}, because otherwise all subsequences of $P(A_n)$ would also converge to this value. □

Combining Theorem 2.7 and Proposition 2.6, the following Corollary immediately follows.

Corollary 2.8 *Let the edge probability* $p_n = 1 - \frac{c_n}{\sqrt{n}}$. *We have that* G_n *is connected w.h.p. if and only if* $c_n \to 0$.

Note that in the above corollary c_n is an arbitrary sequence that tends to zero. The corollary states that in order for the random grid to be fully connected, the edge probability must tend to one at a rate that scales slightly higher than the square root of the side length of the box. Where 'slightly higher' is quantified by the rate of convergence to zero of the sequence c_n, which can be arbitrarily slow.

Similar results appear in [16] for the following continuum model. Let X be a Poisson process of density $\lambda = 1$ on the plane. We consider the random network model $(X, \lambda = 1, r > 0)$ where undirected edges are drawn between Poisson points at distance at most $2r$ of each other. We focus on the restriction $G_n(r)$ of the network formed by the vertices that are inside a $\sqrt{n} \times \sqrt{n}$ box $B_n \subset \mathbb{R}^2$.

Theorem 2.9 (Penrose (1997)) *If* $\pi(2r_n)^2 = \log n + \alpha$ *then the number of isolated nodes inside* B_n *converges in distribution to a Poisson random variable of parameter* $\lambda = e^{-\alpha}$.

The following is a slight extension of Theorem 2.9 that can be proven following the same arguments as in the proof we have given for its discrete counterpart, Theorem 2.7.

Theorem 2.10 *Let* $\pi(2r_n)^2 = \log n + \alpha_n$ *and let* A_n *be the probability that there are no isolated nodes in* B_n. *We have that*

$$\lim_{n\to\infty} P(A_n) = e^{-e^{-\alpha}} \tag{19}$$

if and only if $\alpha_n \to \alpha$ *(where* $\alpha = \infty$ *is allowed).*

Above result shows that singleton nodes asymptotically disappear if and only if $\alpha_n \to \infty$. Additional results in [16] also lead to the following theorem, see also [12].

Theorem 2.11 *Let* $\pi(2r)^2 = \log n + \alpha_n$. *We have that* $G_n(r)$ *is connected w.h.p. if and only if* $\alpha_n \to \infty$.

It is interesting to note how the role of $\alpha_n \to \infty$ in Theorem 2.11 is replaced by $c_n \to 0$ in Corollary 2.8.

3 Critical node lifetimes

We now move from the static scenario examined in the previous section to a more dynamic one. The main idea in this section is that nodes in the network have limited lifetime and tend to become *inactive* over time. The application we have in mind is that of nodes equipped with batteries that progressively discharge, so that the probability that a single node becomes inactive increases over time. We wish to see how this new assumption can be incorporated with the scaling laws that we have derived in the previous section and the results of Penrose. Let us first give an informal picture: imagine to fix the system size n and let time t evolve. As nodes start progressively to fail, one might reasonably expects that there is a *critical time* t_n at which nodes with no active neighbours (i.e., *blind spots*) begin to appear in the network, and we are interested in finding the correct time scale at which this phenomenon can be observed.

To place above picture into a rigorous framework, we proceed in two steps. First, we extend the results given in the previous section, to derive scaling laws for the number of blind spots in the network at a given time t. Then we fix n and let t evolve, and depending on the way we scale the radii, we obtain the time scale t_n at which the number of blind spots converges to a non-trivial distribution. This can be effectively viewed as the critical time at which blind spots can be first observed, and holds for a given failure probability distribution that is related to the battery drainage of the nodes. We first illustrate the situation for the discrete and then treat the continuum model.

Let us denote by G_n the random $n \times n$ grid with edge probability p_n. For every n, all nodes have a common random lifetime distribution $q_n(t)$. That is, if we let T_i be the (random) failure time of node i, then for all $i \in G_n$, $P_n(T_i \leq t) = q_n(t)$. Node i is called *active* for $t < T_i$ and *inactive* for $t \geq T_i$. A blind spot is a node (either active or inactive) that is not connected to any active neighbours. For simplicity we consider a torus, but the results generalize to the square. Let $I_i(t)$ be the indicator random variable of the event that node i is a blind spot at time t. Now, $I_i(t) = 1$ if for all four neighbours of i, the neighbour is inactive or there is no connection. Accordingly, we have

$$\begin{aligned} P(I_i(t)=1) &= [q_n(t)+(1-q_n(t))(1-p_n)]^4 \\ &= [1-p_n+p_nq_n(t)]^4. \end{aligned} \tag{20}$$

We can now apply the Chen-Stein method to study the limiting distribution as $n \to \infty$ of the sum of the dependent random variables $I_i(t)$, $i = 1, \ldots, n^2$, as we did in Theorem 2.5 for the distribution of the isolated nodes. Note that blind spots in this case are increasing functions of independent random variables corresponding to the edges of the dual graph and the state of the neighbors at time t. Omitting the tedious computations and proceeding exactly as in Theorem 2.5, we have the following result for the asymptotic behaviour of blind spots for a given failure rate $q_n(t)$ and edge probability p_n, that is the analogous of Theorems 2.5 and 2.7 in this dynamic setting [2].

Theorem 3.1 *Let λ be a positive constant and let t_n be such that*

$$n^2[1-p_n+p_nq_n(t_n)]^4 = \lambda. \tag{21}$$

Then the number of blind spots in G_n converges in distribution to a Poisson random variable of parameter λ. Furthermore, letting A_n be the event that at time t_n there are no blind spots in G_n, if

$$p_n - p_nq_n(t_n) = 1 - \frac{c_n}{\sqrt{n}}, \tag{22}$$

then

$$\lim_{n\to\infty} P(A_n) = e^{-\lambda} \tag{23}$$

if and only if $c_n \to \lambda^{\frac{1}{4}}$.

We explicitly note that if $q_n(t) = 0$, then (22) reduces to the scaling of p_n given in Theorem 2.7.

We can now use Theorem 3.1 to derive the critical threshold time t_n at which blind spots begin to appear in the random network. To do this, we must fix n and let t evolve to infinity. Clearly, the critical time scale must depend on the given failure rate $q_n(t)$. Accordingly, we let

$$q_n(t) = 1 - e^{-t/\tau_n}, \tag{24}$$

which captures the property that a given node is more likely to fail as time increases. In (24), τ_n can be interpreted as the time constant of the battery drainage of a given node in a network of size n^2 . It is clear that other expressions of the failure rate, different from (24), can also be assumed. By substituting (24)

[2] Considering a square instead than a torus, one needs to trivially modify eq. (20) and patiently go through even more tedious, but essentially similar computations.

into (22) we have that the critical time at which blind spots begin to appear in the random network, is given by

$$\begin{aligned} t_n &= -\tau_n \log\left(\frac{1}{p_n} - \frac{c_n}{p_n\sqrt{n}}\right) \\ &= -\tau_n \log\left(\frac{1-\frac{c_n}{\sqrt{n}}}{p_n}\right). \end{aligned} \quad (25)$$

Some observations are now in order. Note that if $p_n = 1 - \frac{c_n}{\sqrt{n}}$ then $t_n = 0$, which is coherent with Theorem 2.7, stating that in this case there are isolated nodes even if all the nodes are active all the time, whenever $c_n \to c$. On the other hand, it is clear from (25) that if p_n approaches one at a faster rate than $1 - \frac{c_n}{\sqrt{n}}$, then the critical time scale required to observe blind spots increases. In practice, what happens is that a rate higher than what is required to avoid blind spots when all the nodes in the grid are active, provides some 'slack' that contrasts the effect of nodes actually becoming inactive over time.

We now consider the continuous case. We assume that any Poisson point can become inactive before time t with probability $q_n(t)$. We also let the probability of a node being active at time t be $s_n(t) = 1 - q_n(t)$. The type of critical time – analogous to the discrete setting above – now strongly depends on the type of scaling that we use for the radii.

We first derive scaling laws for the number of *active* blind spots in the network.

Theorem 3.2 *If at the times t_n we have*

$$\pi(2r_n)^2 = \frac{\log ns_n(t_n) + \alpha_n}{s_n(t_n)},$$

where $\alpha_n \to \alpha$ and $ns_n(t_n) \to \infty$, then the number of active blind spots in B_n at time t_n converges in distribution (as $n \to \infty$) to a Poisson random variable of parameter $e^{-\alpha}$.

Furthermore, if at the times t_n we have

$$\pi(2r_n)^2 = \frac{\log n + \alpha_n}{s_n(t_n)},$$

where $\alpha_n \to \alpha$ and $s_n(t_n) \to s$, then the number of active blind spots in B_n at time t_n converges in distribution to a Poisson random variable of parameter $se^{-\alpha}$.

Proof of Theorem 3.2 By Theorem 2.9 we have that for a continuum percolation model of unit density inside the box B_n, if $\pi(2r)^2 = \log n + \alpha_n$ with $\alpha_n \to \alpha$, then the number of isolated nodes converges to a Poisson distribution of parameter $e^{-\alpha}$. Writing s_n for $s_n(t_n)$, the same clearly holds for the continuum model in the box B_{ns_n} with density 1, and a radius r_n satisfying $\pi(2r_n)^2 = \log ns_n + \alpha_n$, as long as $ns_n \to \infty$. Indeed, under this last condition this just constitutes a subsequence of our orignal sequence of models. When we now scale the latter sequence of models back to the original size B_n, that is, we multiply all lenghts by $s_n^{-1/2}$, we obtain a sequence of models with density s_n and radii r_n given by

$$\pi(2r_n)^2 = \frac{\log ns_n + \alpha_n}{s_n}. \tag{26}$$

It immediately follows that in this model the number of isolated nodes converges to a Poisson distribution of parameter $e^{-\alpha}$, and the first claim follows.

To prove the second claim, we simply write

$$\frac{\log n + \alpha_n}{s_n} = \frac{\log ns_n + \alpha_n - \log s_n}{s_n},$$

and then the claim follows from the previous result, with α_n replaced by $\alpha_n - \log s_n$. □

To see how these results can be used, consider the situation in which $s_n(t) = e^{-t/\tau_n}$. Since $ns_n(t_n) = ne^{-t_n/\tau_n}$, it follows that $ns_n(t_n) \to \infty$ if and only if $\log n - t_n/\tau_n \to \infty$. Hence, when we use the first scaling in Theorem 3.2, along any sequence t_n that satisfies this last condition, the number of active blind spots converges in distribution to a Poisson law with parameter $e^{-\alpha}$.

On the other hand, if for some s, $t_n = -\tau_n \log s + o(\tau_n)$, then $s_n(t_n) \to s$ and when we use the second scaling in Theorem 3.2, it follows that along the sequence t_n, the number of active blind spots converges to a Poisson law with parameter $se^{-\alpha}$. Note that different scalings of the radii lead to different critical times.

For blind spots which are not necessarily active, we have the following result.

Theorem 3.3 *If at the times t_n we have*

$$\pi(2r_n)^2 = \frac{\log n + \alpha_n}{s_n(t_n)},$$

where $\alpha_n \to \alpha$, then the number of blind spots in in B_n converges in distribution to a Poisson random variable of parameter $\lambda = e^{-\alpha}$.

Before we prove this result, a few remarks are appropriate. First of all, this result cannot be obtained from simple scaling the original results of Penrose, as we did in the proof of Theorem 3.2. Secondly, the role of s_n is quite different here. In particular, we need no longer require anything about the asymptotic behaviour of $s_n(t_n)$. The previous requirement $ns_n(t_n) \to \infty$ was to make sure that enough *active* blind spots would be there, compared to the volume of the box, to ensure Poisson convergence. But since we allow all blind spots here, this is not needed anymore. Due to the particular scaling of the radii in this case, there is no notion of a critical time. Loosely speaking, in this case the choice of the critical radius 'scales away' the notion of a critical time and ensures Poisson convergence for any choice of the failure probability. On the other hand, in Theorem 3.2, for a given choice of the radius, the critical sequence t_n must satisfy additional conditions due to the thinning of the system to ensure Poisson convergence.

Proof of Theorem 3.3 The proof is based on the application of the Chen-Stein bound (4) and is a slight modification of the proof of Proposition 2.9 in [16]. The required modification is to change the event of not having Poisson points in a region of radius $2r_n$, to the event of not having any *active* Poisson point in such a region and to go through the computations. Precise details where the proof in [16] is modified are given next.

The proof is based on a suitable discretisation of the space, followed by the evaluation of the limiting behaviour of the event that a node is isolated. Let us describe the discretisation first. We start working on a torus and we partition B_n into m^2 subsquares (denoted by $u_i, i = 1, \ldots, u_{m^2}$) of side length $\sqrt{n}/m$, and centered in $c_1, \ldots, c_{m^2}$ respectively. Let A_i^{mn} be the event that u_i contains exactly one Poisson point. For any fixed n we have

$$\lim_{m\to\infty} \frac{P(A_i^{mn})}{\frac{n}{m^2}} = 1. \tag{27}$$

Note that events A_i^{mn} are independent of each other for all i, and that the limit above does not depend on i. We now turn to consider node isolation events, writing $s_n = s_n(t_n)$. Let D_n be a disc of radius $2r_n$ such that

$$\pi(2r_n)^2 = \frac{\log n + \alpha_n}{s_n},$$

centered at c_i. We let B_i^{nm} be the event that the region of all subsquares intersecting $\{D_n \setminus u_i\}$ does not contain any active Poisson point. For any fixed n, we have

$$\lim_{m\to\infty} \frac{P(B_i^{nm})}{e^{-\pi(2r_n)^2 s_n}} = 1. \tag{28}$$

Note that in (28) the limit does not depend on i, because of the torus assumption. Note also that events B_i^{nm} are certainly independent of each other for boxes u_i centered at points c_i further than $5r_n$ away from each other, because in this case the corresponding discs D_n intersect disjoint subsquares.

We now define the following random variables

$$I_i = \begin{cases} 1 & \text{if } A_i^{mn} \text{ and } B_i^{nm} \text{ occur,} \\ 0 & \text{otherwise,} \end{cases} \tag{29}$$

$$W_n^m = \sum_{i=0}^{m^2} I_i, \quad W_n = \lim_{m\to\infty} W_n^m. \tag{30}$$

We want to use the Chen-Stein bound in Theorem 2.4. Accordingly, we define a neighbourhood of dependence $\mathcal{N}_i$ for each $i \leq m^2$ as

$$\mathcal{N}_i = \{j : |s_i - s_j| \leq 5r_n\}, \tag{31}$$

Note that the I_i is independent from I_j for all indices outside the neighbourhood of independence of i. We also define

$$\begin{aligned}
b_1 &\equiv \sum_{i=1}^{m^2} \sum_{j \in \mathcal{N}_i} E(I_i)E(I_j) \\
b_2 &\equiv \sum_{i=1}^{m^2} \sum_{j \in \mathcal{N}_i, j \neq i} E(I_i I_j). \qquad (32)
\end{aligned}$$

By Theorem 2.4 we have that

$$d_{TV}[W, Po(\lambda)] \leq 2(b_1 + b_2), \qquad (33)$$

where $\lambda := E(W_n^m)$ can be computed as follows. Writing $a \sim_m b$ if $a/b \to 1$ as $m \to \infty$, by monotone convergence of (30) and using (27), (28) we have

$$\begin{aligned}
E(W_n^m) &\sim_m ne^{-\pi(2r_n^2)s_n} \\
&= e^{\log n - \frac{(\log n + \alpha_n)}{s_n} s_n} \\
&= e^{-\alpha_n}. \qquad (34)
\end{aligned}$$

We then also have that

$$\lim_{n\to\infty} \lim_{m\to\infty} E(W_n^m) = \lim_{n\to\infty} e^{-\alpha_n} = e^{-\alpha}. \qquad (35)$$

The remaining part of the proof that computes the right-hand side of (33) and shows that it tends to zero, as well as the adaptation from a torus to a square, is identical to [16]. □

We wish to compare the statements of Theorems 3.2 and 3.3 with the corresponding results in [18]. These authors consider a model of density n inside the box of unit area where each node can be active with probability s, uniformly in n. In this case they show that if $\pi(2r)^2 = \frac{\log n + \alpha}{sn}$, then the number of active nodes without active neighbours, converges to a Poisson random variable of parameter $se^{-\alpha}$; and the number of nodes (active or inactive) without active neighbours converges to a Poisson random variable of parameter $e^{-\alpha}$. We can recover these results by simply scaling all distance lengths in our proofs by $\sqrt{n}$, considering a density of nodes $\lambda = n$, and letting $s_n(t) = s$ for all t. Our proofs are much simpler though, and our results much more general.

As to the results in [14], we note that these contain unspecified order terms that do not allow to recover results in [18]. Our theorems are tighter, and also our proofs much simpler.

Finally, results reported in the discrete setting are new and reveal interestingly differences with the continuum case for both connectivity [16] and critical times.

4 Conclusion

Ad-hoc and sensor networks are often modelled as random connection graphs. In these systems nodes are subject to battery drainage, and can become inactive over time. By applying the Chen-Stein method of Poisson approximation, we have shown how to obtain a critical time scale at which the number of blind spots has a non-trivial limiting distribution in different models of random networks. This can be effectively viewed as the critical time at which blind spots begin to arise in the network. Some of our results reduce to previous ones of [18] in the special case when the failure probability does not depend on n and there is no time evolution. Our results are stronger than the ones of [14], where some order terms have not been completely characterized, and naturally follow from Poisson approximation and scaling relations applied to the work in [16]. Finally, the paper serves as an illustration of the Chen-Stein method applied to random communication networks, which allows to obtain derivations in an elegant and compact way.

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