Operational Semantics
Semantics and applications to verification

Xavier Rival

École Normale Supérieure

February 10, 2017
Program of this first lecture

Operational semantics

Mathematical description of the executions of a program

1. A model of programs: transition systems
   - definition, a small step semantics
   - a few common examples

2. Trace semantics: a kind of big step semantics
   - finite and infinite executions
   - fixpoint-based definitions
   - notion of compositional semantics
Outline

1. Transition systems and small step semantics
   - Definition and properties
   - Examples

2. Traces semantics

3. Summary
Transition systems: properties of the transition relation

A deterministic system is such that a state fully determines the next state:

$$\forall s_0, s_1, s'_1 \in S, \ (s_0 \rightarrow s_1 \land s_0 \rightarrow s'_1) \implies s_1 = s'_1$$

Otherwise, a transition system is non deterministic, i.e.:

$$\exists s_0, s_1, s'_1 \in S, \ s_0 \rightarrow s_1 \land s_0 \rightarrow s'_1 \land s_1 \neq s'_1$$

Notes:

- the transition relation $\rightarrow$ defines atomic execution steps; it is often called small-step semantics or structured operational semantics
- steps are discrete (not continuous)
  to describe both discrete and continuous behaviors, we would need to look at hybrid systems (beyond the scope of this lecture)
**Transition systems: initial and final states**

**Initial / final states:**
we often consider transition systems with a set of initial and final states:

- a set of **initial states** $S_I \subseteq S$ denotes states where the execution should start
- a set of **final states** $S_F \subseteq S$ denotes states where the execution should reach the end of the program

When needed, we add these to the definition of the transition systems $((S, \rightarrow, S_I, S_F))$.

**Blocking state** (not the same as final state):

- a state $s_0 \in S$ is **blocking** when it is the origin of no transition: $\forall s_1 \in S, \neg (s_0 \rightarrow s_1)$
- example: we often introduce an **error state** (usually noted $\Omega$ to denote the erroneous, blocking configuration)
Outline

1. Transition systems and small step semantics
   - Definition and properties
   - Examples

2. Traces semantics

3. Summary
We can clearly formalize the **word recognition** by a finite automaton using a transition system:

- We consider **automaton** $A = (Q, q_i, q_f, \rightarrow)$
- A “**state**” is defined by:
  - the **remaining of the word to recognize**
  - the **automaton state** that has been reached so far
  thus, $S = Q \times L^*$
- The **transition relation** $\rightarrow$ of the transition system is defined by:
  $$(q_0, aw) \rightarrow (q_1, w) \iff q_0 \xrightarrow{a} q_1$$
- The **initial** and **final states** are defined by:
  $$S_I = \{(q_i, w) \mid w \in L^*\} \quad S_F = \{(q_f, \epsilon)\}$$
Pure $\lambda$-calculus

A bare bones model of functional programing:

$\lambda$-terms

The set of $\lambda$-terms is defined by:

\[
t, u, \ldots \ ::= \ x \quad \text{variable} \\
| \quad \lambda x \cdot t \quad \text{abstraction} \\
| \quad t \ u \quad \text{application}
\]

$\beta$-reduction

- $(\lambda x \cdot t) \ u \rightarrow_\beta \ t[x \leftarrow u]$
- if $u \rightarrow_\beta v$ then $\lambda x \cdot u \rightarrow_\beta \lambda x \cdot v$
- if $u \rightarrow_\beta v$ then $u \ t \rightarrow_\beta v \ t$
- if $u \rightarrow_\beta v$ then $t \ u \rightarrow_\beta t \ v$

The $\lambda$-calculus defines a transition system:

- $\mathcal{S}$ is the set of $\lambda$-terms and $\rightarrow_\beta$ the transition relation
- $\rightarrow_\beta$ is non-deterministic; example ? though, ML fixes an execution order
- given a lambda term $t_0$, we may consider $(\mathcal{S}, \rightarrow_\beta, \mathcal{S}_I)$ where $\mathcal{S}_I = \{ t_0 \}$
- blocking states are terms with no redex $(\lambda x \cdot u) \ v$
A MIPS like assembly language: syntax

We now consider a (very simplified) assembly language

- machine integers: sequences of 32-bits (set: $B^{32}$)
- instructions are encoded over 32-bits (set: $I_{MIPS}$)
  and stored into the same space as data (i.e., $I_{MIPS} \subseteq B^{32}$)
- we assume a fixed set of addresses $A$

### Memory configurations

- **Program counter** $pc$
  - current instruction
- **General purpose registers**
  - $r_0 \ldots r_{31}$
- **Main memory** (RAM)
  - $mem : A \rightarrow B^{32}$
  - where $A \subseteq B^{32}$

### Instructions

- $i ::= (\in I_{MIPS})$
- $add r_d, r_s, r_{s'}$ addition
- $addi r_d, r_s, v$ add. $v \in B^{32}$
- $sub r_d, r_s, r_{s'}$ subtraction
- $b t$ branch
- $blt r_s, r_{s'}, t$ cond. branch
- $ld r_d, o, r_x$ relative load
- $st r_d, o, r_x$ relative store
- $v, t, o \in B^{32}, d, s, s', x \in [0, 31] $
A MIPS like assembly language: states

Definition: state

A state is a tuple \((\pi, \rho, \mu)\) which comprises:

- A **program counter** value \(\pi \in \mathbb{B}^{32}\)
- A function mapping each **general purpose register** to its value \(\rho : \{0, \ldots, 31\} \rightarrow \mathbb{B}^{32}\)
- A function mapping each **memory cell** to its value \(\mu : A \rightarrow \mathbb{B}^{32}\)

What would a **dangerous state** be?

- writing **over an instruction**
- reading or writing **outside the program’s memory**
- we cannot fully formalize these yet...

    as we need to formalize the behavior of each instruction first
A MIPS like assembly language: transition relation

We assume a state $s = (\pi, \rho, \mu)$ and that $\mu(\pi) = \ i$; then:

- if $i = \text{add } r_d, r_s, r_s'$, then:
  $$s \rightarrow (\pi + 4, \rho[d \leftarrow \rho(s) + \rho(s')], \mu)$$

- if $i = \text{addi } r_d, r_s, v$, then:
  $$s \rightarrow (\pi + 4, \rho[d \leftarrow \rho(s) + v], \mu)$$

- if $i = \text{sub } r_d, r_s, r_s'$, then:
  $$s \rightarrow (\pi + 4, \rho[d \leftarrow \rho(s) - \rho(s')], \mu)$$

- if $i = \text{bt } t$, then:
  $$s \rightarrow (t, \rho, \mu)$$
A MIPS like assembly language: transition relation

We assume a state $s = (\pi, \rho, \mu)$ and that $\mu(\pi) = i$; then:

- **if $i = \text{blt} \ r_s, r_s', t$, then:**
  $$s \rightarrow \begin{cases} (t, \rho, \mu) & \text{if } \rho(s) < \rho(s') \\ (\pi + 4, \rho, \mu) & \text{otherwise} \end{cases}$$

- **if $i = \text{ld} \ r_d, o, r_x$, then:**
  $$s \rightarrow \begin{cases} (\pi + 4, \rho[d \leftarrow \mu(\rho(x) + o)], \mu) & \text{if } \rho(x) + o \in A \\ \Omega & \text{otherwise} \end{cases}$$

- **if $i = \text{st} \ r_d, o, r_x$, then:**
  $$s \rightarrow \begin{cases} (\pi + 4, \rho, \mu[x \leftarrow \rho(d)]) & \text{if } \rho(x) + o \in A \\ \Omega & \text{otherwise} \end{cases}$$
A simple imperative language: syntax

We now look at a more classical imperative language (intuitively, a bare-bone subset of C):

- **variables** $X$: finite, predefined set of variables
- **labels** $L$: before and after each statement
- **values** $V$: $V_{\text{int}} \cup V_{\text{float}} \cup \ldots$

### Syntax

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$v \ (v \in V) \</td>
<td>x \ (x \in X) \</td>
</tr>
<tr>
<td>$c$</td>
<td>TRUE \</td>
<td>FALSE \</td>
</tr>
<tr>
<td>$i$</td>
<td>$x := e$; \</td>
<td>if($c$) $b$ else $b$ \</td>
</tr>
<tr>
<td>$b$</td>
<td>${i; \ldots; i; }$</td>
<td>block, program($P$)</td>
</tr>
</tbody>
</table>
A simple imperative language: states

A non-error state should fully describe the configuration at one instant of the program execution, including a memory state and a control state.

The memory state defines the current contents of the memory:

$$m \in M = X \rightarrow V$$

The control state defines where the program currently is:
- analogous to the program counter
- can be defined by adding labels $$L = \{l_0, l_1, \ldots\}$$ between each pair of consecutive statements; then:

$$S = L \times M \cup \{\Omega\}$$

- or by the program remaining to be executed; then:

$$S = P \times M \cup \{\Omega\}$$
A simple imperative language: semantics of expressions

- The semantics \([e]\) of expression \(e\) should evaluate each expression into a value, given a memory state.
- **Evaluation errors** may occur: division by zero...
  - error value is also noted \(\Omega\)

Thus: \([e] : M \rightarrow V \uplus \{\Omega\}\)

**Definition**, by induction over the syntax:

\[
\begin{align*}
&[v](m) = v \\
&[x](m) = m(x) \\
&e_0 + e_1(m) = [e_0](m) \uplus [e_1](m) \\
&e_0 / e_1(m) = \begin{cases} 
\Omega & \text{if } [e_1](m) = 0 \\
[e_0](m) \uplus [e_1](m) & \text{otherwise}
\end{cases}
\end{align*}
\]

where \(\uplus\) is the machine implementation of operator \(\oplus\), and is \(\Omega\)-strict, i.e.,
\[\forall v \in V, \; v \oplus \Omega = \Omega \oplus v = \Omega.\]
A simple imperative language: semantics of conditions

- The semantics $[[c]]$ of condition $c$ should return a boolean value.
- It follows a similar definition to that of the semantics of expressions:
  $[[c]] : M \rightarrow \mathbb{V}_{bool} \cup \{\Omega\}$

Definition, by induction over the syntax:

$$
[[\text{TRUE}]](m) = \text{TRUE} \\
[[\text{FALSE}]](m) = \text{FALSE} \\
[[e_0 < e_1]](m) = \begin{cases} 
\text{TRUE} & \text{if } [[e_0]](m) < [[e_1]](m) \\
\text{FALSE} & \text{if } [[e_0]](m) \geq [[e_1]](m) \\
\Omega & \text{if } [[e_0]](m) = \Omega \text{ or } [[e_1]](m) = \Omega 
\end{cases} \\
[[e_0 = e_1]](m) = \begin{cases} 
\text{TRUE} & \text{if } [[e_0]](m) = [[e_1]](m) \\
\text{FALSE} & \text{if } [[e_0]](m) \neq [[e_1]](m) \\
\Omega & \text{if } [[e_0]](m) = \Omega \text{ or } [[e_1]](m) = \Omega 
\end{cases}
$$
A simple imperative language: transitions

Transitions describe local program execution steps, thus are defined by case analysis on the program statements

Case of assignment \( l_0 : x = e; l_1 \)

- if \( \llbracket e \rrbracket(m) \notin \Omega \), then \( (l_0, m) \rightarrow (l_1, m[x \leftarrow \llbracket e \rrbracket(m)]) \)
- if \( \llbracket e \rrbracket(m) = \Omega \), then \( (l_0, m) \rightarrow \Omega \)

Case of condition \( l_0 : \text{if}(c)\{ l_1 : b_t \} \text{else}\{ l_3 : b_f \}; l_4 \;} l_5 \)

- if \( \llbracket c \rrbracket(m) = \text{TRUE} \), then \( (l_0, m) \rightarrow (l_1, m) \)
- if \( \llbracket c \rrbracket(m) = \text{FALSE} \), then \( (l_0, m) \rightarrow (l_3, m) \)
- if \( \llbracket c \rrbracket(m) = \Omega \), then \( (l_0, m) \rightarrow \Omega \)
- \( (l_2, m) \rightarrow (l_5, m) \)
- \( (l_4, m) \rightarrow (l_5, m) \)
A simple imperative language: transitions

Case of loop \( l_0 : \text{while}(c)\{l_1 : b_t \ l_2\} \ l_3 \)

- if \( \llbracket c \rrbracket(m) = \text{TRUE} \), then \( \{ (l_0, m) \rightarrow (l_1, m), (l_2, m) \rightarrow (l_1, m) \} \)
- if \( \llbracket c \rrbracket(m) = \text{FALSE} \), then \( \{ (l_0, m) \rightarrow (l_3, m), (l_2, m) \rightarrow (l_3, m) \} \)
- if \( \llbracket c \rrbracket(m) = \Omega \), then \( \{ (l_0, m) \rightarrow \Omega, (l_2, m) \rightarrow \Omega \} \)

Case of \( \{ l_0 : i_0; l_1 : \ldots; l_{n-1} i_{n-1}; l_n \} \)

- the transition relation is defined by the individual instructions
Extending the language with non-determinism

The language we have considered so far is a bit limited:

- it is deterministic: at most one transition possible from any state
- it does not support the input of values

Changes if we model non deterministic inputs...

... with an input instruction:

- \( i ::= \ldots \mid x := \text{input}() \)
- \( l_0 : x := \text{input}(); l_1 \) generates transitions
  \[ \forall v \in V, \ (l_0, m) \rightarrow (l_1, m[x \leftarrow v]) \]
- one instruction induces non determinism

... with a random function:

- \( e ::= \ldots \mid \text{rand}() \)
- expressions have a non-deterministic semantics:
  \[ [e] : M \rightarrow \mathcal{P}(V \cup \{\Omega\}) \]
  \[ [\text{rand}()] (m) = V \]
  \[ [v] (m) = \{v\} \]
  \[ [c] : M \rightarrow \mathcal{P}(V_{\text{bool}} \cup \{\Omega\}) \]
- all instructions induce non determinism
Semantics of real world programming languages

C language:
- several norms: ANSI C’99, ANSI C’11, K&R...
- not fully specified:
  - undefined behavior
  - implementation dependent behavior: architecture (ABI) or implementation (compiler...)
  - unspecified parts: leave room for implementation of compilers and optimizations
- formalizations in HOL (C’99), in Coq (CompCert C compiler)

OCaml language:
- more formal...
- ... but still with some unspecified parts, e.g., execution order
Outline

1 Transition systems and small step semantics

2 Traces semantics
   - Definitions
   - Finite traces semantics
   - Fixpoint definition
   - Compositionality
   - Infinite traces semantics

3 Summary
Execution traces

- So far, we considered only states and atomic transitions
- We now consider program executions as a whole

**Definition: traces**

- A **finite trace** is a finite sequence of states $s_0, \ldots, s_n$, noted $\langle s_0, \ldots, s_n \rangle$
- An **infinite trace** is an infinite sequence of states $\langle s_0, \ldots \rangle$

Besides, we write:

- $S^*$ for the set of finite traces
- $S^\omega$ for the set of infinite traces
- $S^\infty = S^* \cup S^\omega$ for the set of finite or infinite traces
Operations on traces: concatenation

Definition: concatenation

The **concatenation operator** \( \cdot \) is defined by:

- \( h_{s_0}, \ldots, s_n \cdot h'_{s_0}, \ldots, s'_n \) = \( h_{s_0}, \ldots, s_n, s'_0, \ldots, s'_n \)
- \( h_{s_0}, \ldots, s_n \cdot h'_{s_0}, \ldots \) = \( h_{s_0}, \ldots, s_n, s'_0, \ldots \)
- \( h_{s_0}, \ldots, s_n, \ldots \cdot \sigma' \) = \( h_{s_0}, \ldots, s_n, \ldots \)

We also define:

- the **empty trace** \( \epsilon \), neutral element for \( \cdot \).
- the **length** operator \(|.|\):

\[
\begin{cases}
|\epsilon| &= 0 \\
|h_{s_0}, \ldots, s_n|i| &= n + 1 \\
|h_{s_0}, \ldots|i| &= \omega
\end{cases}
\]
Comparing traces: the prefix order relation

Definition: prefix order relation

Relation $\prec$ is defined by:

\[ h_{s_0}, \ldots, s_n \prec h'_{s_0}, \ldots, s'_n \iff \begin{cases} n \leq n' \\
\forall i \in \mathbb{N}, \ s_i = s'_i \end{cases} \]

\[ h_{s_0}, \ldots i \prec h'_{s_0}, \ldots i \iff \forall i \in \mathbb{N}, \ s_i = s'_i \]

\[ h_{s_0}, \ldots, s_n i \prec h'_{s_0}, \ldots i \iff \forall i \in [0, n], \ s_i = s'_i \]

Proof: straightforward application of the definition of order relations
Outline

1. Transition systems and small step semantics

2. Traces semantics
   - Definitions
   - Finite traces semantics
   - Fixpoint definition
   - Compositionality
   - Infinite traces semantics

3. Summary
Semantics of finite traces

We consider a transition system $S = (S, \rightarrow)$

**Definition**

The *finite traces semantics* $\llbracket S \rrbracket^*$ is defined by:

$$\llbracket S \rrbracket^* = \{ s_0, \ldots, s_n \in S^* \mid \forall i, s_i \rightarrow s_{i+1} \}$$

**Example:**

- contrived transition system $S = (\{a, b, c, d\}, \{(a, b), (b, a), (b, c)\})$
- finite traces semantics:
  $$\llbracket S \rrbracket^* = \{ \epsilon, a, b, a, b, a, b, c, d \}$$
Interesting subsets of the finite trace semantics

We consider a transition system \( S = (\mathcal{S}, \rightarrow, \mathcal{S}_I, \mathcal{S}_F) \):

- the **initial traces**, i.e., starting from an initial state:
  \[
  \{s_0, \ldots, s_n | s_0 \in \mathcal{S}_I\}
  \]

- the **traces reaching a blocking state**:
  \[
  \{\sigma \in \mathcal{J}[S]^* | \forall \sigma' \in \mathcal{J}[S]^*, \sigma < \sigma' \implies \sigma = \sigma'\}
  \]

- the **traces ending in a final state**:
  \[
  \{s_0, \ldots, s_n | s_n \in \mathcal{S}_F\}
  \]

- the **maximal traces** are both initial and final

**Example** (same transition system, with \( \mathcal{S}_I = \{a\} \) and \( \mathcal{S}_F = \{c\} \)):

- traces from an initial state ending in a final state are all of the form: \( ha, b, \ldots, a, b, a, b, ci \)
Example: finite automaton

We consider the example of the previous course:

\[ L = \{a, b\} \quad Q = \{q_0, q_1, q_2\} \]

\[ q_i = q_0 \quad q_f = q_2 \]

\[ q_0 \xrightarrow{a} q_1 \quad q_1 \xrightarrow{b} q_2 \quad q_2 \xrightarrow{a} q_1 \]

Then, we have the following traces:

\[ \tau_0 = h(q_0, ab), (q_1, b), (q_2, \epsilon)i \]
\[ \tau_1 = h(q_0, abab), (q_1, bab), (q_2, ab), (q_1, b), (q_2, \epsilon)i \]
\[ \tau_2 = h(q_0, ababab), (q_1, babab), (q_2, abab), (q_1, bab)i \]
\[ \tau_3 = h(q_0, abaaa), (q_1, baaa), (q_2, aaa), (q_1, aa)i \]

Then:

- \( \tau_0, \tau_1 \) are initial traces, reaching a final state
- \( \tau_2 \) is an initial trace, and is not maximal
- \( \tau_3 \) reaches a blocking state, but not a final state
Example: $\lambda$-term

We consider $\lambda$-term $\lambda y \cdot (((\lambda x \cdot y)((\lambda x \cdot x x)(\lambda x \cdot x x)))$, and show two traces generated from it (at each step the reduced lambda is shown in red):

$$\tau_0 = h \quad \lambda y \cdot (((\lambda x \cdot y)((\lambda x \cdot x x)(\lambda x \cdot x x))))$$
$$\tau_0 = \lambda y \cdot y \quad i$$

$$\tau_1 = h \quad \lambda y \cdot (((\lambda x \cdot y)((\lambda x \cdot x x)(\lambda x \cdot x x))))$$
$$\lambda y \cdot (((\lambda x \cdot y)((\lambda x \cdot x x)(\lambda x \cdot x x))))$$
$$\lambda y \cdot (((\lambda x \cdot y)((\lambda x \cdot x x)(\lambda x \cdot x x))))$$
$$\lambda y \cdot (((\lambda x \cdot y)((\lambda x \cdot x x)(\lambda x \cdot x x)))) \quad i$$

Then:

- $\tau_0$ is a maximal trace; it reaches a blocking state (no more reduction can be done)
- $\tau_1$ can be extended for arbitrarily many steps; the second part of the course will study infinite traces
Example: imperative program

Similarly, we can write the traces of a simple imperative program:

\[
\begin{align*}
  l_0 : &amp; \quad x := 1; \\
  l_1 : &amp; \quad y := 0; \\
  l_2 : &amp; \quad \textbf{while}(x < 4) \{ \\
  l_3 : &amp; \quad y := y + x; \\
  l_4 : &amp; \quad x := x + 1; \\
  l_5 : &amp; \quad \} \\
  l_6 : &amp; \quad (\text{final program point})
\end{align*}
\]

\[
\tau = h \ (l_0, (x = 6, y = 8)), (l_1, (x = 1, y = 8)), \\
    (l_2, (x = 1, y = 0)), (l_3, (x = 1, y = 0)), \\
    (l_4, (x = 1, y = 1)), (l_5, (x = 2, y = 1)), \\
    (l_3, (x = 2, y = 1)), (l_4, (x = 2, y = 3)), \\
    (l_5, (x = 3, y = 3)), (l_3, (x = 3, y = 3)), \\
    (l_4, (x = 3, y = 6)), (l_5, (x = 4, y = 6)), \\
    (l_6, (x = 4, y = 6))
\]

- very **precise** description of what the program does...
- ... but **quite cumbersome**
Outline

1 Transition systems and small step semantics

2 Traces semantics
   - Definitions
   - Finite traces semantics
   - Fixpoint definition
   - Compositionality
   - Infinite traces semantics

3 Summary
Towards a fixpoint definition

We consider again our contrived transition system

\[ S = (\{a, b, c, d\}, \{(a, b), (b, a), (b, c)\}) \]

Traces by length:

<table>
<thead>
<tr>
<th>(i)</th>
<th>traces of length (i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(\epsilon)</td>
</tr>
<tr>
<td>1</td>
<td>(a, b, c, d)</td>
</tr>
<tr>
<td>2</td>
<td>(a, b, a, b, c, d)</td>
</tr>
<tr>
<td>3</td>
<td>(a, b, a, b, a, b, c, d)</td>
</tr>
<tr>
<td>4</td>
<td>(a, b, a, b, a, b, a, b, c, d)</td>
</tr>
</tbody>
</table>

Like the automaton in lecture 1, this suggests a least fixpoint definition: traces of length \(i + 1\) can be derived from the traces of length \(i\), by adding a transition.
We define a **semantic function**, that computes the traces of length \(i + 1\) from the traces of length \(i\) (where \(i \geq 1\)), and adds the traces of length 1:

**Finite traces semantics as a fixpoint**

Let \(I = \{\epsilon\} \cup \{s_i \mid s \in S\}\). Let \(F_*\) be the function defined by:

\[
F_* : P(S^*) \rightarrow P(S^*)
\]

\[
X \rightarrow I \cup \{s_0, \ldots, s_n, s_{n+1}i \mid s_0, \ldots, s_n i \in X \land s_n \rightarrow s_{n+1}\}
\]

Then, \(F_*\) is **continuous** and thus has a least-fixpoint and:

\[
\text{lfp} F_* = [S]^* = \bigcup_{n \in \mathbb{N}} F_*^n(\emptyset)
\]
Fixpoint definition: proof (1), fixpoint existence

First, we prove that \( F_* \) is **continuous**.

Let \( X \subseteq P(\mathbb{S}^*) \) such that \( X \not= \emptyset \) and \( A = \bigcup_{U \in \mathcal{X}} U \). Then:

\[
F_*(\bigcup_{X \in \mathcal{X}} X) = \bigcup \{s_0, \ldots, s_n, s_{n+1} | (s_0, \ldots, s_n) \in \bigcup_{U \in \mathcal{X}} U \land s_n \rightarrow s_{n+1}\}
\]

\[
= \bigcup \{s_0, \ldots, s_n, s_{n+1} | \exists U \in X, (s_0, \ldots, s_n) \in U \land s_n \rightarrow s_{n+1}\}
\]

\[
= \bigcup_{U \in \mathcal{X}} \{s_0, \ldots, s_n, s_{n+1} | s_0, \ldots, s_n \in U \land s_n \rightarrow s_{n+1}\}
\]

\[
= \bigcup_{U \in \mathcal{X}} F_*(U)
\]

In particular, this is true for any increasing chain \( X \) (here, we considered any non empty family), hence \( F_* \) is continuous.

As \((P(\mathbb{S}^*), \subseteq)\) is a CPO, the continuity of \( F_* \) entails the **existence of a least-fixpoint** (Kleene theorem); moreover, it implies that:

\[
\text{lfp } F_* = \bigcup_{n \in \mathbb{N}} F^n_*(\emptyset)
\]
Fixpoint definition: proof (2), fixpoint equality

We now show that $[S]^* \text{ is equal to lfp } F_*$, by showing the property below, by induction over $n$:

$$\forall k < n, h_0, \ldots, s_k i \in F_*^n(\emptyset) \iff h_0, \ldots, s_k i \in [S]^*$$

- at rank 0, only trace $\epsilon$ needs to be considered, and its case is trivial
- at rank $n + 1$, we need to consider both traces of length 1 (the case of which is trivial) and traces of length $n + 1$ for some integer $n \geq 1$:

\[
\begin{align*}
    h_0, \ldots, s_k, s_{k+1} i & \in [S]^* \\
    \iff & h_0, \ldots, s_k i \in [S]^* \land s_k \rightarrow s_{k+1} \\
    \iff & h_0, \ldots, s_k i \in F_*^n(\emptyset) \land s_k \rightarrow s_{k+1} \quad (k < n \text{ since } k + 1 < n + 1) \\
    \iff & h_0, \ldots, s_k, s_{k+1} i \in F_*^{n+1}(\emptyset)
\end{align*}
\]
Trace semantics fixpoint form: example

Example, with the same simple transition system $S = (S, \rightarrow)$:

- $S = \{a, b, c, d\}$
- $\rightarrow$ is defined by $a \rightarrow b$, $b \rightarrow a$ and $b \rightarrow c$

Then, the first iterates are:

\[
\begin{align*}
F^0_\ast(\emptyset) &= \emptyset \\
F^1_\ast(\emptyset) &= \{\epsilon, ha, hbi, hci, hdi\} \\
F^2_\ast(\emptyset) &= F^1_\ast(\emptyset) \cup \{hb, ai, ha, bi, hb, ci\} \\
F^3_\ast(\emptyset) &= F^2_\ast(\emptyset) \cup \{ha, b, ai, hb, a, bi, ha, b, ci\} \\
F^4_\ast(\emptyset) &= F^3_\ast(\emptyset) \cup \{hb, a, b, ai, ha, b, a, bi, hb, a, b, ci\} \\
F^5_\ast(\emptyset) &= F^4_\ast(\emptyset) \cup \{ha, b, a, b, ai, hb, a, b, a, bi, ha, b, a, b, ci\} \\
F^6_\ast(\emptyset) &= \ldots
\end{align*}
\]

The traces of $[S]^* \ast$ of length $n + 1$ appear in $F^n_\ast(\emptyset)$
Outline

1 Transition systems and small step semantics

2 Traces semantics
   - Definitions
   - Finite traces semantics
   - Fixpoint definition
   - Compositionality
   - Infinite traces semantics

3 Summary
The traces semantics definition we have seen is **global**: 
- the **whole system** defines a **transition relation**
- we **iterate** this relation until we get a fixpoint

Though, a **modular** definition would be nicer, to allow reasoning on program fragments, or derive properties of a program from properties of its pieces...

**Can we derive a more modular expression of the semantics?**
Notion of compositional semantics

Observation: programs often have an inductive structure

- $\lambda$-terms are defined by induction over the syntax
- Imperative programs are defined by induction over the syntax
- There are exceptions: our MIPS language does not naturally look that way

Definition: compositional semantics

A semantics $\llbracket . \rrbracket$ is said to be **compositional** when the semantics of a program can be defined as a function of the semantics of its parts, i.e.,

When program $\pi$ writes down as $C[\pi_0, \ldots, \pi_k]$ where $\pi_0, \ldots, \pi_k$ are its components, there exists a function $F_C$ such that $\llbracket \pi \rrbracket = F_C(\llbracket \pi_0 \rrbracket, \ldots, \llbracket \pi_k \rrbracket)$, where $F_C$ depends only on syntactic construction $F_C$. 
Case of a simplified imperative language

Case of a sequence of two instructions \( b \equiv l_0 : i_0; l_1 : i_1; l_2: \)

\[
[b]^* = [i_0]^* \cup [i_1]^* \cup \{h_{s_0}, \ldots, s_m i | \exists n \in [0, m], h_{s_0}, \ldots, s_n i \in [i_0]^* \land h_{s_n}, \ldots, s_m i \in [i_1]^* \}
\]

This amounts to **concatenating** traces of \([i_0]^*\) and \([i_1]^*\) that share a state in common (necessarily at point \(l_1\)).

Cases of a condition, a loop: similar

- by concatenation of traces around junction points
- by doing a least-fixpoint computation over loops

We can provide a compositional semantics for our simplified imperative language
Case of $\lambda$-calculus

Case of a $\lambda$-term $t = (\lambda x \cdot u) \, v$:

- executions may start with a reduction in $u$
- executions may start with a reduction in $v$
- executions may start with the reduction of the head redex
- an execution may mix reductions steps in $u$ and $v$ in an arbitrary order

No nice compositional trace semantics of $\lambda$-calculus...
Outline

1. Transition systems and small step semantics

2. Traces semantics
   - Definitions
   - Finite traces semantics
   - Fixpoint definition
   - Compositionality
   - Infinite traces semantics

3. Summary
Non termination

Can the finite traces semantics express non termination?

Consider the case of our contrived system:

\[ S = \{a, b, c, d\} \quad (\rightarrow) = \{(a, b), (b, a), (b, c)\} \]

**System behaviors:**

- this system clearly has non-terminating behaviors:
  it can loop from \( a \) to \( b \) and back forever
- the finite traces semantics does show the existence of this cycle as there exists an infinite chain of finite traces for the prefix order \( \prec \):
  \[ h_a, b, a, b, a, b, a, b, a, b, a, b, a, \ldots \in [S]^* \]
- though, the existence of this chain is not very obvious

Thus, we now define a semantics made of infinite traces
Semantics of infinite traces

We consider a transition system $S = (S, \rightarrow)$

**Definition**

The **infinite traces semantics** $[S]^{\omega}$ is defined by:

$$[S]^{\omega} = \{s_0, \ldots | \forall i, s_i \rightarrow s_{i+1}\}$$

**Infinite traces starting from an initial state** (considering $S = (S, \rightarrow, S_I, S_F)$):

$$\{s_0, \ldots | s_0 \in S_I\}$$

**Example:**

- contrived transition system defined by
  
  $S = \{a, b, c, d\}$  \hspace{1cm} $(\rightarrow) = \{(a, b), (b, a), (b, c)\}$

- the infinite traces semantics contains **exactly two** traces
  
  $$[S]^{\omega} = \{ha, b, \ldots, a, b, a, b, \ldots i, hb, a, \ldots, b, a, b, a, \ldots i\}$$
Fixpoint form

Can we also provide a fixpoint form for $[S]^\omega$?

Intuitively, $s_0, s_1, \ldots \in [S]^\omega$ if and only if $\forall n, s_n \rightarrow s_{n+1}$, i.e.,

$$\forall n \in \mathbb{N}, \forall k \leq n, s_k \rightarrow s_{k+1}$$

Let $F_\omega$ be defined by:

$$F_\omega : \mathcal{P}(S^\omega) \rightarrow \mathcal{P}(S^\omega)$$

$$X \mapsto \{s_0, s_1, \ldots, s_n, \ldots \mid s_0 \rightarrow s_1 \}$$

Then, we can show by induction that:

$$\sigma \in [S]^\omega \iff \forall n \in \mathbb{N}, \sigma \in F_\omega^n(S^\omega) \iff \bigcap_{n \in \mathbb{N}} F_\omega^n(S^\omega)$$
Fixpoint form of the semantics of infinite traces

Infinite traces semantics as a fixpoint

Let $F_\omega$ be the function defined by:

$$F_\omega : \mathcal{P}(\mathcal{S}_\omega) \rightarrow \mathcal{P}(\mathcal{S}_\omega)$$

$$X \mapsto \{s_0, s_1, \ldots, s_n, \ldots \mid s_1, \ldots, s_n, \ldots \in X \land s_0 \rightarrow s_1\}$$

Then, $F_\omega$ is $\cap$-continuous and thus has a greatest-fixpoint; moreover:

$$\text{gfp } F_\omega = \mathcal{J} \mathcal{S} \mathcal{K}_\omega = \bigcap_{n \in \mathbb{N}} F^n_\omega(\mathcal{S}_\omega)$$

Proof sketch:

- the $\cap$-continuity proof is similar as for the $\cup$-continuity of $F_*$
- by the dual version of Kleene’s theorem, $\text{gfp } F_\omega$ exists and is equal to $\bigcap_{n \in \mathbb{N}} F^n_\omega(\mathcal{S}_\omega)$, i.e. to $\mathcal{J} \mathcal{S} \mathcal{K}_\omega$ (similar induction proof)
Fixpoint form of the infinite traces semantics: iterates

Example, with the same simple transition system:

- \( S = \{a, b, c, d\} \)
- \( \rightarrow \) is defined by \( a \rightarrow b \), \( b \rightarrow a \) and \( b \rightarrow c \)

Then, the first iterates are:

\[
\begin{align*}
F_0^\omega(S^\omega) &= S^\omega \\
F_1^\omega(S^\omega) &= a \cdot S^\omega \cup b \cdot S^\omega \cup b \cdot S^\omega \\
F_2^\omega(S^\omega) &= b \cdot a \cdot S^\omega \cup a \cdot b \cdot S^\omega \cup a \cdot b \cdot S^\omega \\
F_3^\omega(S^\omega) &= a \cdot b \cdot a \cdot S^\omega \cup b \cdot a \cdot b \cdot S^\omega \cup b \cdot a \cdot b \cdot S^\omega \\
F_4^\omega(S^\omega) &= \ldots
\end{align*}
\]

Intuition

- at iterate \( n \), prefixes of length \( n + 1 \) match the traces in the infinite semantics
- only \( a, b, \ldots, a, b, a, b, \ldots \) and \( b, a, \ldots, b, a, b, a, \ldots \) belong to all iterates
Outline

1. Transition systems and small step semantics
2. Traces semantics
3. Summary
Summary

We have discussed today:

- **small-step / structural operational semantics:** individual program steps

- **big-step / natural semantics:** program executions as sequences of transitions

- their fixpoint definitions and properties will play a great role to design verification techniques

Next lectures:

- another family of semantics, more compact and compositional

- semantic program and proof methods