Program of this lecture

Studied so far:

- **semantics**: behaviors of programs
- **properties**: safety, liveness, security...
- **approaches to verification**: typing, use of proof assistants, model checking

Today’s lecture: introduction to abstract interpretation

A **general framework for comparing semantics** introduced by Patrick Cousot and Radhia Cousot (1977)

- **abstraction**: use of a lattice of predicates
- **computing abstract over-approximations**, while preserving soundness
- **computing abstract over-approximations for loops**
Outline

1 Abstraction
   - Notion of abstraction
   - Abstraction and concretization functions
   - Galois connections

2 Abstract interpretation

3 Application of abstract interpretation

4 Conclusion
Abstraction example 1: signs

Abstraction: defined by a family of properties to use in proofs

Example:
- objects under study: sets of mathematical integers
- abstract elements: signs

Lattice of signs

\[
\begin{array}{c}
\bot \\
\downarrow \\
0 \\
\uparrow \\
\pm \\
\downarrow \\
\top \\
\end{array}
\]

- \(\bot\) denotes only \(\emptyset\)
- \(\pm\) denotes any set of positive integers
- \(0\) denotes any subset of \(\{0\}\)
- \(-\) denotes any set of negative integers
- \(\top\) denotes any set of integers

Note: the order in the abstract lattice corresponds to inclusion...
Abstraction example 1: signs

**Definition: abstraction relation**

- **Concrete elements**: elements of the original lattice \( c \in \mathcal{P}(\mathbb{Z}) \)
- **Abstract elements**: predicate \( a: \; \cdot \in \{\pm, 0, \ldots\} \)
- **Abstraction relation**: \( c \vdash_{S} a \) when \( a \) describes \( c \)

**Examples:**

- \( \{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\} \vdash_{S} \pm \)
- \( \{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\} \vdash_{S} \top \)

We use abstract elements to reason about operations:

- If \( c_0 \vdash_{S} \pm \) and \( c_1 \vdash_{S} \pm \), then \( \{x_0 + x_1 \mid x_i \in c_i\} \vdash_{S} \pm \)
- If \( c_0 \vdash_{S} \pm \) and \( c_1 \vdash_{S} \pm \), then \( \{x_0 \cdot x_1 \mid x_i \in c_i\} \vdash_{S} \pm \)
- If \( c_0 \vdash_{S} \pm \) and \( c_1 \vdash_{S} 0 \), then \( \{x_0 \cdot x_1 \mid x_i \in c_i\} \vdash_{S} 0 \)
- If \( c_0 \vdash_{S} \pm \) and \( c_1 \vdash_{S} \bot \), then \( \{x_0 \cdot x_1 \mid x_i \in c_i\} \vdash_{S} \bot \)
Abstraction example 1: signs

We can also consider the **union operation**:
- if \( c_0 \vdash_S \pm \) and \( c_1 \vdash_S \pm \), then \( c_0 \cup c_1 \vdash_S \pm \)
- if \( c_0 \vdash_S \pm \) and \( c_1 \vdash_S \perp \), then \( c_0 \cup c_1 \vdash_S \pm \)

But, what can we say about \( c_0 \cup c_1 \), when \( c_0 \vdash_S 0 \text{ and } c_1 \vdash_S \pm \) ?
- clearly, \( c_0 \cup c_1 \vdash_S \top \ldots \)
- but **no other relation holds**
- in the abstract, **we do not rule out negative values**

We can **extend the initial lattice**:
- \( \geq 0 \) denotes any set of positive or null integers
- \( \leq 0 \) denotes any set of negative or null integers
- \( \not= 0 \) denotes any set of non null integers
- if \( c_0 \vdash_S \pm \) and \( c_1 \vdash_S 0 \), then \( c_0 \cup c_1 \vdash_S \geq 0 \)
Abstraction example 2: constants

Definition: abstraction based on constants

- **Concrete elements:** $\mathcal{P}(\mathbb{Z})$
- **Abstract elements:** $\bot, \top, n$ where $n \in \mathbb{Z}$
  $\left( D^\#_C = \{\bot, \top\} \cup \{n \mid n \in \mathbb{Z}\} \right)$
- **Abstraction relation:** $c \vdash_C n \iff c \subseteq \{n\}$

We obtain a flat lattice:

Abstract reasoning:

- if $c_0 \vdash_C n_0$ and $c_1 \vdash_C n_1$, then $\{k_0 + k_1 \mid k_i \in c_i\} \vdash_C n_0 + n_1$
Abstraction example 3: Parikh vector

**Definition: Parikh vector abstraction**

- **concrete elements:** \( \mathcal{P}(\mathcal{A}^*) \) (sets of words over alphabet \( \mathcal{A} \))
- **abstract elements:** \( \{\bot, \top\} \cup (\mathcal{A} \to \mathbb{N}) \)
- **abstraction relation:** \( c \vdash \varphi : \mathcal{A} \to \mathbb{N} \) if and only if:

\[
\forall w \in c, \forall a \in \mathcal{A}, \ a \text{ appears } \varphi(a) \text{ times in } w
\]

**Abstract reasoning:**

- **concatenation:**

  if \( \varphi_0, \varphi_1 : \mathcal{A} \to \mathbb{N} \) and \( c_0, c_1 \) are such that \( c_i \vdash \varphi_i \),

  \[
  \{ w_0 \cdot w_1 \mid w_i \in c_i \} \vdash \varphi_0 + \varphi_1
  \]

**Information preserved, information deleted:**

- **very precise** information about the number of occurrences
- **the order of letters is totally abstracted away (lost)**
Abstraction example 4: interval abstraction

**Definition: abstraction based on intervals**

- **concrete elements**: $\mathcal{P}(\mathbb{Z})$
- **abstract elements**: $\bot, \top, (a, b)$ where $a \in \{-\infty\} \cup \mathbb{Z}$, $b \in \mathbb{Z} \cup \{+\infty\}$ and $a \leq b$
- **abstraction relation**:
  \[
  \begin{align*}
  \emptyset & \vdash_I \bot \\
  S & \vdash_I (a, b) \iff \forall x \in S, \ a \leq x \leq b \\
  S & \vdash_I \top
  \end{align*}
  \]

**Operations**: TD
Abstraction example 5: non relational abstraction

**Definition: non relational abstraction**

- **Concrete elements:** $\mathcal{P}(X \to Y)$, inclusion ordering
- **Abstract elements:** $X \to \mathcal{P}(Y)$, pointwise inclusion ordering
- **Abstraction relation:** $c \vdash_{\mathcal{N}\mathcal{R}} a \iff \forall \phi \in c, \forall x \in X, \phi(x) \in a(x)$

**Information preserved, information deleted:**

- **Very precise** information about the image of the functions in $c$
- **Relations** such as (for given $x_0, x_1 \in X, y_0, y_1 \in Y$) the following are lost:

  $$\forall \phi \in c, \phi(x_0) = \phi(x_1)$$

  $$\forall \phi \in c, \forall x, x' \in X, \phi(x) \neq y_0 \lor \phi(x') \neq y_1$$
Notion of abstraction relation

Concrete order: so far, always inclusion
- the tighter the concrete set, the fewer behaviors
- smaller concrete sets correspond to more precise properties

Abstraction relation: \( c \vdash a \) when \( c \) satisfies \( a \)
- if \( c_0 \subseteq c_1 \) and \( c_1 \) satisfies \( a \), in all our examples, \( c_0 \) also satisfies \( a \)

Abstract order: in all our examples,
- it matches the abstraction relation as well:
  if \( a_0 \sqsubseteq a_1 \) and \( c \) satisfies \( a_0 \), then \( c \) also satisfies \( a_1 \)
- great advantage: we can reason about implication in the abstract, without looking back at the concrete properties

We will now formalize this in detail...
Outline

1. Abstraction
   - Notion of abstraction
   - Abstraction and concretization functions
   - Galois connections

2. Abstract interpretation

3. Application of abstract interpretation

4. Conclusion
Concretization function

We consider a **concrete lattice** \((C, \subseteq)\) and an **abstract lattice** \((A, \sqsubseteq)\).

So far, we used **abstraction relations**, that are consistent with orderings:

\[
\forall c_0, c_1 \in C, \forall a \in A, \ c_0 \subseteq c_1 \land c_1 \vdash a \implies c_0 \vdash a
\]

\[
\forall c \in C, \forall a_0, a_1 \in A, \ c \vdash a_0 \land a_0 \sqsubseteq a_1 \implies c \vdash a_1
\]

When we have a \(c\) (resp., \(a\)) and try to map it into a compatible \(a\) (resp. \(a\) \(c\)), the abstraction relation is not a convenient tool.

Hence, we shall use **adjoint functions** between \(C\) and \(A\).

- from concrete to abstract: **abstraction**
- from abstract to concrete: **concretization**
Concretization function

Our first adjoint function:

Definition: concretization function

Concretization function $\gamma : A \to C$ (if it exists) maps abstract $a$ into the weakest (i.e., most general) concrete $c$ that satisfies $a$ (i.e., $c \vdash a$).

Note: in common cases, there exists a $\gamma$.

- $c \vdash a$ if and only if $c \subseteq \gamma(a)$
Concretization function: a few examples

**Signs abstraction:**

\[ \gamma_{S} : \begin{array}{c|c}
\top & \mathbb{Z} \\
\pm & \mathbb{Z}^* \\
0 & \{0\} \\
\neg & \mathbb{Z}^* \\
\bot & \emptyset \\
\end{array} \]

**Constants abstraction:**

\[ \gamma_{C} : \begin{array}{c|c}
\top & \mathbb{Z} \\
\ bot \ & \emptyset \\
\end{array} \]

\[ \begin{array}{c|c}
\pm & \mathbb{Z}^* \\
0 & \{0\} \\
\neg & \emptyset \\
\bot & \emptyset \\
\end{array} \]

**Non relational abstraction:**

\[ \gamma_{NR} : (X \to P(Y)) \quad \mapsto \quad P(X \to Y) \]

\[ \begin{array}{c|c}
\Phi & \{\phi : X \to Y \mid \forall x \in X, \phi(x) \in \Phi(x)\} \\
\end{array} \]

**Parikh vector abstraction:** exercise!
Abstraction function

Our **second adjoint function:**

**Definition: abstraction function**

**Abstraction function** $\alpha : C \rightarrow A$ (if it exists) maps concrete $c$ into the most precise abstract $a$ that soundly describes $c$ (i.e., $c \vdash a$).

Note: in quite a few cases (including some in this course), there is no $\alpha$.

**Summary on adjoint functions:**

- $\alpha$ returns the **most precise abstract predicate** that holds true for its argument.
  this is called the **best abstraction**
- $\gamma$ returns the **most general concrete meaning** of its argument.
  hence, is called the **concretization**
Abstraction: a few examples

**Constants abstraction:**

\[ \alpha_C : (c \subseteq \mathbb{Z}) \mapsto \begin{cases} 
\bot & \text{if } c = \emptyset \\
n & \text{if } c = \{n\} \\
\top & \text{otherwise} 
\end{cases} \]

**Non relational abstraction:**

\[ \alpha_{NR} : (c \subseteq (X \rightarrow Y)) \mapsto (x \in X) \mapsto \{\phi(x) \mid \phi \in c\} \]

**Signs abstraction** and **Parikh vector abstraction:** exercises
Outline

1 Abstraction
   - Notion of abstraction
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Definition

So far, we have:
- abstraction $\alpha : C \to A$
- concretization $\gamma : A \to C$

How to tie them together?
They should agree on a same abstraction relation $\vdash$ !

Definition: Galois connection

A Galois connection is defined by a concrete lattice $(C, \subseteq)$, an abstract lattice $(A, \sqsubseteq)$, an abstraction function $\alpha : C \to A$ and a concretization function $\gamma : A \to C$ such that:

$$\forall c \in C, \forall a \in A, \alpha(c) \sqsubseteq a \iff c \subseteq \gamma(a) \quad (\iff c \vdash a)$$

Notation: 

$$(C, \subseteq) \xleftarrow{\gamma} (A, \sqsubseteq) \xrightarrow{\alpha}$$

Note: in practice, we never use $\vdash$; we use $\alpha, \gamma$ instead
Example: constants abstraction and Galois connection

**Constants lattice** $D^\#_C = \{ \bot, \top \} \uplus \{ n \mid n \in \mathbb{Z} \}$

\[
\begin{align*}
\alpha_C(c) & = \bot \quad \text{if } c = \emptyset & \gamma_C(\top) & \mapsto \mathbb{Z} \\
\alpha_C(c) & = n \quad \text{if } c = \{ n \} & \gamma_C(n) & \mapsto \{ n \} \\
\alpha_C(c) & = \top \quad \text{otherwise} & \gamma_C(\bot) & \mapsto \emptyset 
\end{align*}
\]

Thus:

- if $c = \emptyset$, $\forall a$, $c \subseteq \gamma_C(a)$, i.e., $c \subseteq \gamma_C(a) \iff \alpha_C(c) = \bot \subseteq a$
- if $c = \{ n \}$,
  \[
  \alpha_C(\{ n \}) = n \subseteq c \iff c = n \lor c = \top \iff c = \{ n \} \subseteq \gamma_C(a)
  \]
- if $c$ has at least two distinct elements $n_0, n_1$, $\alpha_C(c) = \top$ and $c \subseteq \gamma_C(a) \Rightarrow a = \top$, i.e., $c \subseteq \gamma_C(a) \iff \alpha_C(c) = \bot \subseteq a$

**Constant abstraction: Galois connection**

$c \subseteq \gamma_C(a) \iff \alpha_C(c) \subseteq a$, therefore, $(\mathcal{P}(\mathbb{Z}), \subseteq) \xrightarrow{\gamma_C} (D^\#_C, \subseteq)$
Example: non relational abstraction Galois connection

We have defined:

\[ \alpha_{NR} : (c \subseteq (X \to Y)) \mapsto (x \in X) \mapsto \{ f(x) \mid f \in c \} \]
\[ \gamma_{NR} : (\Phi \in (X \to \mathcal{P}(Y))) \mapsto \{ f : X \to Y \mid \forall x \in X, f(x) \in \Phi(x) \} \]

Let \( c \in \mathcal{P}(X \to Y) \) and \( \Phi \in (X \to \mathcal{P}(Y)) \); then:

\[ \alpha_{NR}(c) \subseteq \Phi \iff \forall x \in X, \alpha_{NR}(c)(x) \subseteq \Phi(x) \]
\[ \iff \forall x \in X, \{ f(x) \mid f \in c \} \subseteq \Phi(x) \]
\[ \iff \forall f \in c, \forall x \in X, f(x) \in \Phi(x) \]
\[ \iff \forall f \in c, f \in \gamma_{NR}(\Phi) \]
\[ \iff c \subseteq \gamma_{NR}(\Phi) \]

Non relational abstraction: Galois connection

\( c \subseteq \gamma_{NR}(a) \iff \alpha_{NR}(c) \subseteq a \), therefore,

\[ (\mathcal{P}(X \to Y), \subseteq) \xleftarrow{\gamma_{NR}} (X \to \mathcal{P}(Y), \subseteq) \xrightarrow{\alpha_{NR}} (\mathcal{P}(X \to Y), \subseteq) \]
Galois connections have **many useful properties**.

In the next few slides, we consider a Galois connection \((C, \subseteq) \xleftarrow{\gamma} (A, \subseteq)\) and establish a few interesting properties.

**Extensivity, contractivity**

- \(\alpha \circ \gamma\) is contractive: \(\forall a \in A, \alpha \circ \gamma(a) \subseteq a\)
- \(\gamma \circ \alpha\) is extensive: \(\forall c \in C, c \subseteq \gamma \circ \alpha(c)\)

**Proof:**

- let \(a \in A\); then, \(\gamma(a) \subseteq \gamma(a)\), thus \(\alpha(\gamma(a)) \subseteq a\)
- let \(c \in C\); then, \(\alpha(c) \subseteq \alpha(c)\), thus \(c \subseteq \gamma(\alpha(a))\)
Galois connection properties

Monotonicity of adjoints

- $\alpha$ is monotone
- $\gamma$ is monotone

Proof:

- **monotonicity of $\alpha$:** let $c_0, c_1 \in C$ such that $c_0 \subseteq c_1$; by extensivity of $\gamma \circ \alpha$, $c_1 \subseteq \gamma(\alpha(c_1))$, so by transitivity, $c_0 \subseteq \gamma(\alpha(c_1))$ by definition of the Galois connection, $\alpha(c_0) \subseteq \alpha(c_1)$

- **monotonicity of $\gamma$:** same principle

Note: many proofs can be derived by duality

If $(C, \subseteq) \xleftrightarrow{\gamma} (A, \sqsubseteq)$, then $(A, \sqsupseteq) \xleftrightarrow{\alpha} (C, \supseteq)$
Galois connection properties

**Iteration of adjoints**

- \( \alpha \circ \gamma \circ \alpha = \alpha \)
- \( \gamma \circ \alpha \circ \gamma = \gamma \)
- \( \alpha \circ \gamma \) (resp., \( \gamma \circ \alpha \)) is idempotent, hence a lower (resp., upper) closure operator

**Proof:**

- \( \alpha \circ \gamma \circ \alpha = \alpha \):  
  let \( c \in C \), then \( \gamma \circ \alpha(c) \subseteq \gamma \circ \alpha(c) \)  
  hence, by the Galois connection property, \( \alpha \circ \gamma \circ \alpha(c) \subseteq \alpha(c) \)  
  moreover, \( \gamma \circ \alpha \) is extensive and \( \alpha \) monotone, so \( \alpha(c) \subseteq \alpha \circ \gamma \circ \alpha(c) \)  
  thus, \( \alpha \circ \gamma \circ \alpha(c) = \alpha(c) \)

- the second point can be proved similarly (duality); the others follow
Galois connection properties

\(\alpha\) preserves least upper bounds

\[\forall c_0, c_1 \in C, \ \alpha(c_0 \cup c_1) = \alpha(c_0) \sqcup \alpha(c_1)\]

By duality:

\[\forall a_0, a_1 \in A, \ \gamma(c_0 \cap c_1) = \gamma(c_0) \cap \gamma(c_1)\]

Proof:
For all \(a \in A\):

\[\alpha(c_0 \cup c_1) \sqsubseteq a \iff c_0 \cup c_1 \subseteq \gamma(a)\]
\[\iff c_0 \subseteq \gamma(a) \land c_1 \subseteq \gamma(a)\]
\[\iff \alpha(c_0) \sqsubseteq a \land \alpha(c_1) \sqsubseteq a\]
\[\iff \alpha(c_0) \sqcup \alpha(c_1) \sqsubseteq a\]

Note: when \(C, A\) are complete lattices, this extends to any family of elements
Galois connection properties

**Uniqueness of adjoints**

- given $\gamma : C \to A$, there exists at most one $\alpha : A \to C$ such that $(C, \subseteq) \xleftarrow{\gamma} (A, \sqsubseteq)$, and, if it exists, $\alpha(c) = \cap \{a \in A \mid c \subseteq \gamma(a)\}$
- similarly, given $\alpha : A \to C$, there exists at most one $\gamma : C \to A$ such that $(C, \subseteq) \xrightarrow{\alpha} (A, \sqsubseteq)$, and it is defined dually

**Proof of the first point** (the other follows by duality):
we assume that there exist $\alpha$ so that we have a Galois connection and prove that, $\alpha(c) = \cap \{a \in A \mid c \subseteq \gamma(a)\}$ for a given $c \in C$.

- if $a \in A$ is such that $c \subseteq \gamma(a)$, then $\alpha(a) \sqsubseteq c$ thus, $\alpha(a)$ is a lower bound of $\{a \in A \mid c \subseteq \gamma(a)\}$.
- let $a_0 \in A$ be a lower bound of $\{a \in A \mid c \subseteq \gamma(a)\}$.
  since $\gamma \circ \alpha$ is extensive, $c \subseteq \gamma(\alpha(c))$ and $\alpha(c) \in \{a \in A \mid c \subseteq \gamma(a)\}$.
  hence, $a_0 \sqsubseteq \alpha(c)$

Thus, $\alpha(c)$ is the leaest upper bound of $\{a \in A \mid c \subseteq \gamma(a)\}$
Construction of adjoint functions

The adjoint uniqueness property is actually a very strong property:
- it allows to construct an abstraction from a concretization
- ... or to understand why no abstraction can be constructed :-) 

Turning an adjoint into a Galois connection (1)

Let \((C, \subseteq)\) and \((A, \sqsubseteq)\) be two lattices, such that any subset of \(A\) as a greatest lower bound and let \(\gamma : (A, \sqsubseteq) \to (C, \subseteq)\) be a monotone function.

Then, the function below defines a Galois connection:

\[
\alpha(c) = \sqcap \{ a \in A \mid c \subseteq \gamma(a) \}
\]

Example of abstraction with no \(\alpha\): when \(\sqcap\) is not defined on all families, e.g., lattice of convex polyhedra, abstracting sets of points in \(\mathbb{R}^2\).

Exercise: state the dual property and apply the same principle to the concretization
A characterization of Galois connections

Let \((C, \subseteq)\) and \((A, \sqsubseteq)\) be two lattices, and \(\alpha : C \to A\) and \(\gamma : A \to C\) be two monotone functions, such that:

- \(\alpha \circ \gamma\) is contractive
- \(\gamma \circ \alpha\) is extensive

Then, we have a Galois connection

\[(C, \subseteq) \leftrightarrow (A, \sqsubseteq)\]

Proof:

- let \(c \in C\) and \(a \in A\) such that \(\alpha(c) \sqsubseteq a\).
  then: \(\gamma(\alpha(c)) \subseteq \gamma(a)\) (as \(\gamma\) is monotone)
  \(c \subseteq \gamma(\alpha(c))\) (as \(\gamma \circ \alpha\) is extensive)
  thus, \(c \subseteq \gamma(a)\), by transitivity

the other implication can be proved by duality
Outline

1. Abstraction

2. Abstract interpretation
   - Abstract computation
   - Fixpoint transfer

3. Application of abstract interpretation

4. Conclusion
Constructing a static analysis

We have set up a notion of \textit{abstraction}:

- it describes \textit{sound} approximation of \textit{concrete properties} with abstract predicates
- there are several ways to formalize it (abstraction, concretization...)
- we now wish to \textit{compute sound abstract predicates}

In the following, we assume

- a \textit{Galois connection} \[(C, \subseteq) \xrightarrow{\gamma} (A, \sqsubseteq)\]
- a \textit{concrete semantics} \[[.]]\, with a \textit{constructive definition} i.e., \[[P]]\ is defined by constructive equations (\([P] = f(\ldots))\), least fixpoint formula (\([[P]] = \text{lfp}_{\emptyset} f\))...
Abstract transformer

A fixed concrete element $c_0$ can be abstracted by $\alpha(c_0)$.

We now consider a monotone concrete function $f : C \to C$

- given $c \in C$, $\alpha \circ f(c)$ abstracts the image of $c$ by $f$
- if $c \in C$ is abstracted by $a \in A$, then $f(c)$ is abstracted by $\alpha \circ f \circ \gamma(a)$:
  
  $c \subseteq \gamma(a)$ by assumption
  $f(c) \subseteq f(\gamma(a))$ by monotonicity of $f$
  $\alpha(f(c)) \subseteq \alpha(f(\gamma(a)))$ by monotonicity of $\alpha$

Definition: best and sound abstract transformers

- the best abstract transformer approximating $f$ is $f^\# = \alpha \circ f \circ \gamma$
- a sound abstract transformer approximating $f$ is any operator $f^\# : A \to A$, such that $\alpha \circ f \circ \gamma \subseteq f^\#$ (or equivalently, $f \circ \gamma \subseteq \gamma \circ f^\#$)
Example: lattice of signs

- $f : D_C^# \rightarrow D_C^#$, $c \mapsto \{-n \mid n \in c\}$
- $f^# = \alpha \circ f \circ \gamma$

Lattice of signs:

Abstract negation operator:

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- here, the best abstract transformer is very easy to compute
- no need to use an approximate one
We can generalize this to \textit{n-ary operators}, such as \textit{boolean operators} and \textit{arithmetic operators}.

\textbf{Definition: sound and exact abstract operators}

Let $g : C^n \rightarrow C$ be a monotone \textit{n-ary operator}. Then:

- \textbf{the best abstract operator} approximating $g$ is defined by:
  
  $$g^\# : A^n \rightarrow A \quad (a_0, \ldots, a_{n-1}) \mapsto \alpha \circ g(\gamma(a_0), \ldots, \gamma(a_{n-1}))$$

- \textbf{a sound abstract transformer} approximating $g$ is any operator $g^\# : A^n \rightarrow A$, such that
  
  $$\forall (a_0, \ldots, a_{n-1}) \in A^n, \quad \alpha \circ g(\gamma(a_0), \ldots, \gamma(a_{n-1})) \sqsubseteq g^\#(a_0, \ldots, a_{n-1})$$
Example: lattice of signs arithmetic operators

**Application:**

- $\oplus: C^2 \to C, (c_0, c_1) \mapsto \{n_0 + n_1 \mid n_i \in c_i\}$
- $\otimes: C^2 \to C, (c_0, c_1) \mapsto \{n_0 \cdot n_1 \mid n_i \in c_i\}$

**Best abstract operators:**

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</tr>
<tr>
<td>$-$</td>
<td>$\perp$</td>
<td>$\pm$</td>
<td>0</td>
<td>$\perp$</td>
<td>$\top$</td>
</tr>
<tr>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>$\perp$</td>
<td>$\top$</td>
</tr>
<tr>
<td>$\pm$</td>
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<td>0</td>
<td>$\perp$</td>
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<td>$\top$</td>
<td>$\perp$</td>
<td>$\top$</td>
<td>0</td>
<td>$\top$</td>
<td>$\top$</td>
</tr>
</tbody>
</table>

**Example of loss in precision:**

- $\{8\} \in \gamma_S(\pm)$ and $\{-2\} \in \gamma_S(-)$
- $\oplus^*(\pm, -) = \top$ is a lot worse than $\alpha_S(\oplus(\{8\}, \{-2\})) = \pm$
Example: lattice of signs set operators

Best abstract operators approximating $\cup$ and $\cap$:

$$\begin{array}{|c|c|c|c|c|c|}
\hline
\cup# & \bot & \neg & 0 & \pm & \top \\
\hline
\bot & \bot & \neg & 0 & \pm & \top \\
\neg & \neg & \neg & \top & \top & \top \\
0 & 0 & \top & 0 & \top & \top \\
\pm & \pm & \top & \top & \pm & \top \\
\top & \top & \top & \top & \top & \top \\
\hline
\end{array}$$

$$\begin{array}{|c|c|c|c|c|c|}
\hline
\cap# & \bot & \neg & 0 & \pm & \top \\
\hline
\bot & \bot & \bot & \bot & \bot & \bot \\
\neg & \neg & \neg & \bot & \bot & \bot \\
0 & 0 & \bot & 0 & \bot & \bot \\
\pm & \bot & \bot & \bot & \pm & \bot \\
\top & \bot & \bot & \bot & \top & \top \\
\hline
\end{array}$$

Example of loss in precision:

$$\gamma(\neg) \cup \gamma(\pm) = \{ n \in \mathbb{Z} | n \neq 0 \} \subset \gamma(\top)$$
Outline

1 Abstraction

2 Abstract interpretation
   • Abstract computation
   • Fixpoint transfer

3 Application of abstract interpretation

4 Conclusion
Fixpoint transfer

What about loops? Semantic functions defined by fixpoints?

Theorem: exact fixpoint transfer

We consider a Galois connection \((C, \subseteq) \leq \frac{\gamma}{\alpha} (A, \subseteq)\), two functions \(f : C \rightarrow C\) and \(f^\# : A \rightarrow A\) and two elements \(c_0 \in C\), \(a_0 \in A\) such that:

- \(f\) is continuous
- \(f^\#\) is monotone
- \(\alpha \circ f = f^\# \circ \alpha\)
- \(\alpha(c_0) = a_0\)

Then:

- Both \(f\) and \(f^\#\) have a least-fixpoint (Tarski’s fixpoint theorem)
- \(\alpha(\text{lfp}_{c_0} f) = \text{lfp}_{a_0} f^\#\)
Fixpoint transfer: proof

- \( \alpha(\text{lfp}_{c_0} f) \) is a fixpoint of \( f^\# \) since:
  \[
  f^\#(\alpha(\text{lfp}_{c_0} f)) = \alpha(f(\text{lfp}_{c_0} f)) = \alpha(\text{lfp}_{c_0} f)
  \]
  since \( \alpha \circ f = f^\# \circ \alpha \)
  by definition of the fixpoints

- To show that \( \alpha(\text{lfp}_{c_0} f) \) is the least-fixpoint of \( f^\# \),
  we assume that \( X \) is another fixpoint of \( f^\# \) greater than \( a_0 \) and we show that
  \( \alpha(\text{lfp}_{c_0} f) \subseteq X \), i.e., that \( \text{lfp}_{c_0} f \subseteq \gamma(X) \).
  As \( \text{lfp}_{c_0} f = \bigcup_{n \in \mathbb{N}} f^n_{0}(c_0) \), it amounts to proving that
  \( \forall n \in \mathbb{N}, f^n_{0}(c_0) \subseteq \gamma(X) \).
  By induction over \( n \):
  \begin{itemize}
  \item \( f^0_{0}(c_0) = c_0 \), thus \( \alpha(f^0_{0}(c_0)) = a_0 \subseteq X \); thus, \( f^0_{0}(c_0) \subseteq \gamma(X) \).
  \item let us assume that \( f^n_{0}(c_0) \subseteq \gamma(X) \), and let us show that
    \( f^{n+1}_{0}(c_0) \subseteq \gamma(X) \), i.e. that \( \alpha(f^{n+1}_{0}(c_0)) \subseteq X \):
    \[
    \alpha(f^{n+1}_{0}(c_0)) = \alpha \circ f(f^n_{0}(c_0)) = f^\# \circ \alpha(f^n_{0}(c_0)) \subseteq f^\#(X) = X
    \]
    as \( \alpha(f^n_{0}(c_0)) \subseteq X \) and \( f^\# \) is monotone.
Constructive analysis of loops

How to get a constructive version of fixpoint transfer?

Theorem: fixpoint abstraction

Under the assumptions of the previous theorem, and with the following additional hypothesis:

- lattice \( A \) is of finite height

We compute the sequence \((a_n)_{n \in \mathbb{N}}\) defined by \(a_{n+1} = a_n \sqcup f^\#(a_n)\).

Then, \((a_n)_{n \in \mathbb{N}}\) converges and its limit \(a_\infty\) is such that \(\alpha(\text{lfp}_{c_0} f) = a_\infty\).

Proof: exercise.

Note:

- the assumptions we have made are too restrictive in practice
- more general fixpoint abstraction methods in the next lectures
Outline

1. Abstraction
2. Abstract interpretation
3. Application of abstract interpretation
4. Conclusion
Comparing existing semantics

1. A **concrete semantics** \([P]\) is given: e.g., big steps operational semantics
2. An **abstract semantics** \([P]^{\#}\) is given: e.g., denotational semantics
3. **Search for an abstraction relation between them**
   e.g., \([P]^{\#} = \alpha([P])\), or \([P] \subseteq \gamma([P]^{\#})\)

Examples:
- finite traces semantics as an abstraction of bi-finitary trace semantics
- denotational semantics as an abstraction of trace semantics
- types as an abstraction of denotational semantics
- ...

Payoff:
- better understanding of ties across semantics
- chance to generalize existing definitions
Derivation of a static analysis

1. Start from a **concrete semantics** $[P]$
2. **Choose an abstraction** defined by a Galois connection or a concretization function (usually)
3. Derive an abstract semantics $[P]^\#$ such that $[P] \subseteq \gamma([P]^\#)$

**Examples:**
- derivation of an analysis with a numerical lattice (constants, intervals...)
- construction of an analysis for a complex programming language

**Payoff:**
- the derivation of the abstract semantics is quite systematic
- this process offers good opportunities for a modular analysis design

There are many ways to apply abstract interpretation.
We now apply this to a very simple language, and derive a static analysis step by step, from a concrete semantics and an abstraction.

- we assume a fixed set of \( n \) integer variables \( x_0, \ldots, x_{n-1} \)
- we consider the language defined by the grammar below:

\[
P ::= x_i = n \quad \text{where } n \in \mathbb{Z} \\
    | x_i = x_j + x_k \\
    | x_i = x_j - x_k \\
    | x_i = x_j \cdot x_k \\
    | \text{input}(x_i) \quad \text{reading of a positive input} \\
    | \text{if}(x_i > 0) \ P \ \text{else} \ P \\
    | \text{while}(x_i > 0) \ P
\]

- a state is a vector \( \sigma = (\sigma_0, \ldots, \sigma_{n-1}) \in \mathbb{Z}^n \)
- a single initial state \( \sigma_{\text{init}} = (0, \ldots, 0) \)
Concrete semantics

We let \( [P] : \mathcal{P}(\mathbb{Z}^n) \rightarrow \mathcal{P}(\mathbb{Z}^n) \) be defined by:

\[
\begin{align*}
[x_i = n](S) &= \{ \sigma[i \leftarrow n] \mid \sigma \in S \} \\
[x_i = x_j + x_k](S) &= \{ \sigma[i \leftarrow \sigma_j + \sigma_k] \mid \sigma \in S \} \\
[x_i = x_j - x_k](S) &= \{ \sigma[i \leftarrow \sigma_j - \sigma_k] \mid \sigma \in S \} \\
[x_i = x_j \cdot x_k](S) &= \{ \sigma[i \leftarrow \sigma_j \cdot \sigma_k] \mid \sigma \in S \} \\
\text{input}(x_i)(S) &= \{ \sigma[i \leftarrow n] \mid \sigma \in S \land n > 0 \} \\
\text{if}(x_i > 0) \, P_0 \text{ else } P_1](S) &= [P_0]([\sigma \in S \mid \sigma_i > 0]) \\
&\quad \cup [P_1]([\sigma \in S \mid \sigma_i \leq 0]) \\
\text{while}(x_i > 0) \, P](S) &= \{ \sigma \in \text{lfp}_S f \mid \sigma_i \leq 0 \} \\
&\quad \text{where } f : S' \mapsto [P](\{ \sigma \in S' \mid \sigma_i > 0 \})
\end{align*}
\]

- given a complete program \( P \), the reachable states are defined by \( [P](\{ \sigma_{\text{init}} \}) \)
Abstraction

We compose two abstractions:

- **non relational abstraction**: the values a variable may take is abstracted separately from the other variables.
- **sign abstraction**: the set of values observed for each variable is abstracted into the lattice of signs.

Abstraction

- **concrete domain**: \((\mathcal{P}(\mathbb{Z}^n), \subseteq)\)
- **abstract domain**: \((D^\#, \sqsubseteq)\), where \(D^\# = (D^S_\#)^n\) and \(\sqsubseteq\) is the pointwise ordering.
- **Galois connection** \((\mathcal{P}(\mathbb{Z}), \subseteq) \xrightarrow{\gamma} (D^\#, \sqsubseteq)\), defined by

\[
\begin{align*}
\alpha & : S & \mapsto (\alpha_S(\{\sigma_0 | \sigma \in S\}), \ldots, \alpha_S(\{\sigma_{n-1} | \sigma \in S\})) \\
\gamma & : S^\# & \mapsto \{ \sigma \in \mathbb{Z}^n | \forall i, \sigma_i \in \gamma_c(S^\#_i) \}
\end{align*}
\]
Computation of the abstract semantics

We search for an abstract semantics $\llbracket P \rrbracket^\sharp : D^\sharp \rightarrow D^\sharp$ such that:

$$\alpha \circ \llbracket P \rrbracket = \llbracket P \rrbracket^\sharp \circ \alpha$$

We observe that:

$$\alpha(S) = (\alpha_S(\{\sigma_0 \mid \sigma \in S\}), \ldots, \alpha_S(\{\sigma_{n-1} \mid \sigma \in S\}))$$

$$\alpha \circ \llbracket P \rrbracket(S) = (\alpha_S(\{\sigma_0 \mid \sigma \in \llbracket P \rrbracket(S)\}), \ldots, \alpha_S(\{\sigma_{n-1} \mid \sigma \in \llbracket P \rrbracket(S)\}))$$

We start with $x_i = n$:

$$\alpha \circ \llbracket x_i = n \rrbracket(S)$$

$$= (\alpha_S(\{\sigma_0 \mid \sigma \in \llbracket P \rrbracket(\{\sigma[i \leftarrow n] \mid \sigma \in S\})\}), \ldots,$$

$$\alpha_S(\{\sigma_{n-1} \mid \sigma \in \llbracket P \rrbracket(\{\sigma[i \leftarrow n] \mid \sigma \in S\})\}))$$

$$= (\alpha_S(\{\sigma_0 \mid \sigma \in S\}), \ldots, \alpha_S(\{\sigma_{n-1} \mid \sigma \in S\}))[i \leftarrow \alpha_S(n)]$$

$$= \alpha(S)[i \leftarrow \alpha_S(n)]$$

$$= \llbracket x_i = n \rrbracket^\sharp(\alpha(S))$$

where $\llbracket x_i = n \rrbracket^\sharp(S^\sharp) = S^\sharp[i \leftarrow \alpha_S(n)]$
Other assignments are treated in a similar manner:

\[
\begin{align*}
\llbracket x_i = x_j + x_k \rrbracket#(S#) &= S#[i \leftarrow x_j \oplus S#_k] \\
\llbracket x_i = x_j - x_k \rrbracket(S) &= S#[i \leftarrow x_j \ominus S#_k] \\
\llbracket x_i = x_j \cdot x_k \rrbracket(S) &= S#[i \leftarrow x_j \odot S#_k] \\
\llbracket \text{input}(x_i) \rrbracket(S) &= S#[i \leftarrow \pm]
\end{align*}
\]

Proofs are left as exercises.
We now consider the case of tests:

\[
\alpha \circ \llbracket \text{if}(x_i > 0) \ P_0 \ \text{else} \ P_1 \rrbracket(S)
\]
\[
= \alpha(\llbracket P_0 \rrbracket(\{\sigma \in S \mid \sigma_i > 0\})) \cup \llbracket P_1 \rrbracket(\{\sigma \in S \mid \sigma_i \leq 0\}))
\]
\[
= \alpha(\llbracket P_0 \rrbracket(\{\sigma \in S \mid \sigma_i > 0\})) \sqcup \alpha(\llbracket P_1 \rrbracket(\{\sigma \in S \mid \sigma_i \leq 0\}))
\]

as \( \alpha \) preserves least upper bounds.

\[
= \llbracket P_0 \rrbracket^\#(\alpha(\{\sigma \in S \mid \sigma_i > 0\})) \cup \llbracket P_1 \rrbracket^\#(\alpha(\{\sigma \in S \mid \sigma_i \leq 0\}))
\]
\[
= \llbracket P_0 \rrbracket^\#(\alpha(S) \cap T[i \leftarrow \pm]) \cup \llbracket P_1 \rrbracket^\#(\alpha(S))
\]

where

\[
\llbracket \text{if}(x_i > 0) \ P_0 \ \text{else} \ P_1 \rrbracket^\#(S^\#) = \llbracket P_0 \rrbracket^\#(S^\# \cap T[i \leftarrow \pm]) \cup \llbracket P_1 \rrbracket^\#(S^\#)
\]

In the case of loops:

\[
\llbracket \text{while}(x_i > 0) \ P \rrbracket^\#(S^\#) = \text{lfp}_{S^\#} f^\#
\]

where

\[
f^\# : S^\# \mapsto S^\# \sqcup \llbracket P \rrbracket^\#(S^\# \cap T[i \leftarrow \pm])
\]

Proof: exercise
Abstract semantics

Abstract semantics and soundness

We have derived the following definition of $[[P]]^\#$:

\[
\begin{align*}
[x_i = n]^\#(S^\#) &= S^\#[i \leftarrow \alpha_S(n)] \\
[x_i = x_j + x_k]^\#(S^\#) &= S^\#[i \leftarrow S_j^\# \oplus^\# S_k^\#] \\
[x_i = x_j - x_k](S) &= S^\#[i \leftarrow S_j^\# \ominus^\# S_k^\#] \\
[x_i = x_j \cdot x_k](S) &= S^\#[i \leftarrow S_j^\# \otimes^\# S_k^\#] \\
[\text{input}(x_i)](S) &= S^\#[i \leftarrow \top] \\
[\text{if}(x_i > 0) \ P_0 \ \text{else} \ P_1]^\#(S^\#) &= \[P_0]^\#(S^\# \sqcap \top[i \leftarrow \top]) \cup \[P_1]^\#(S^\#) \\
[\text{while}(x_i > 0) \ P]^\#(S^\#) &= \text{lfp}_{S^\#} f^\# \text{ where } \\
f^\#: S^\# \mapsto S^\# \sqcup [[P]^\#(S^\# \sqcap \top[i \leftarrow \top])] \\
\end{align*}
\]

Furthermore, for all program $P$: $\alpha \circ [[P]] = [[P]]^\# \circ \alpha$

An over-approximation of the final states is computed by $[[P]]^\#(\top)$. 
Example

Factorial function:

\[
\text{input}(x_0); \\
x_1 = 1; \\
x_2 = 1; \\
\text{while}(x_0 > 0)\{ \\
  x_1 = x_0 \cdot x_1; \\
  x_0 = x_0 - x_2; 
\}
\]

Abstract state before the loop: \((\pm, \pm, \pm)\)

Iterates on the loop:

<table>
<thead>
<tr>
<th>iterate</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_0)</td>
<td>(\pm)</td>
<td>(\top)</td>
<td>(\top)</td>
</tr>
<tr>
<td>(x_1)</td>
<td>(\pm)</td>
<td>(\pm)</td>
<td>(\pm)</td>
</tr>
<tr>
<td>(x_2)</td>
<td>(\pm)</td>
<td>(\pm)</td>
<td>(\pm)</td>
</tr>
</tbody>
</table>

Abstract state after the loop: \((\top, \pm, \pm)\)
Outline

1. Abstraction
2. Abstract interpretation
3. Application of abstract interpretation
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Summary

This lecture:
- abstraction and its formalization
- computation of an abstract semantics in a very simplified case

Next lectures:
- construction of a few non trivial abstractions
- more general ways to compute sound abstract properties

The project will also allow to practice these notions