Traces Properties
Semantics and applications to verification

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Program of this lecture

Today’s lecture: we look back at program’s properties

- **families of properties:**
  what properties can be considered “similar” ? in what sense ?

- **proof techniques:**
  how can those kinds of properties be established ?

- **specification of properties:**
  are there languages to describe properties ?
In this lecture we look at trace properties.
A property is a set of traces, defining the admissible executions.

Safety properties:
- something (e.g., bad) will never happen
- proof by invariance

Liveness properties:
- something (e.g., good) will eventually happen
- proof by variance

Some interesting program properties do not fit this classification.
State properties

As usual, we consider $S = (S, \rightarrow, S_I)$

First approach: properties as sets of states

- A property $P$ is a set of states $P \subseteq S$
- $P$ is satisfied if and only if all reachable states belong to $P$, i.e., $[S]_R \subseteq P$ where $[S]_R = \{s_n \in S | \exists \langle s_0, \ldots, s_n \rangle \in [S]_R, s_0 \in S_I\}$

Examples:
- Absence of runtime errors:
  $$P = S \setminus \{\Omega\} \quad \text{where} \ \Omega \ \text{is the error state}$$
- Non termination (e.g., for an operating system):
  $$P = \{s \in S | \exists s' \in S, s \rightarrow s'\}$$
Second approach: properties as sets of traces

- A property $\mathcal{T}$ is a set of traces $\mathcal{T} \subseteq S^\infty$
- $\mathcal{T}$ is satisfied if and only if all traces belong to $\mathcal{T}$, i.e., $[S]^\infty \subseteq \mathcal{T}$

Examples:

- Obviously, **state properties** are trace properties
- **Functional properties**
  - e.g., “program $P$ takes one integer input $x$ and returns its absolute value”
- **Termination**: $\mathcal{T} = S^*$ (i.e., the system should have no infinite execution)
Monotonicity

**Property**

Let $\mathcal{P}_0, \mathcal{P}_1 \subseteq \mathcal{S}$ be two state properties, such that $\mathcal{P}_0 \subseteq \mathcal{P}_1$. Then $\mathcal{P}_0$ is stronger than $\mathcal{P}_1$, i.e. if program $S$ satisfies $\mathcal{P}_0$, then it also satisfies $\mathcal{P}_1$.

Let $\mathcal{T}_0, \mathcal{T}_1 \subseteq \mathcal{S}$ be two trace properties, such that $\mathcal{T}_0 \subseteq \mathcal{T}_1$. Then $\mathcal{T}_0$ is stronger than $\mathcal{T}_1$, i.e. if program $S$ satisfies $\mathcal{T}_0$, then it also satisfies $\mathcal{T}_1$.

**Proof:** straightforward application of the definition of state (resp., trace) properties
Safety properties

Informal definition: safety properties

A safety property is a property which specifies that some (bad) behavior will never occur.

- absence of runtime errors is a safety property ("bad thing": error)
- state properties is a safety property ("bad thing": reaching $\mathcal{S} \setminus \mathcal{P}$)
- non termination is a safety property ("bad thing": reaching a blocking state)
- “not reaching state $b$ after visiting state $a$” is a safety property (and not a state property)
- termination is not a safety property
Towards a formal definition

We intend to provide a **formal definition** of safety.

**How to refute a safety property?**

- we assume $S$ does **not** satisfy safety property $\mathcal{P}$
- thus, there exists a **counter-example trace**
  $\sigma = \langle s_0, \ldots, s_n, \ldots \rangle \in \mathcal{J}_S \setminus \mathcal{P}$;
  it may be finite or infinite...
- the intuitive definition says this trace **eventually exhibits some bad behavior**
- thus, there exists a rank $i \in \mathbb{N}$, such that the bad behavior has been observed before reaching $s_i$
- therefore, trace $\sigma' = \langle s_0, \ldots, s_i \rangle$ violates $\mathcal{P}$, i.e. $\sigma' \not\in \mathcal{P}$
- we remark $\sigma'$ is finite

**A safety property that does not hold can always be refuted with a finite counter-example**
Limit

Definition: upper closure operator (uco)

Function $\phi : S \rightarrow S$ is an **upper closure operator** iff:

- **monotone**
- **extensive**: $\forall x \in S, \ x \sqsubseteq \phi(x)$
- **idempotent**: $\forall x \in S, \ \phi(\phi(x)) = \phi(x)$

Definition: limit

The **limit operator** is defined by:

$$
\text{Lim} : \mathcal{P}(S^\infty) \rightarrow \mathcal{P}(S^\infty)
$$

$$
X \mapsto X \cup \{ \sigma \in S^\infty \mid \forall i \in \mathbb{N}, \ \sigma[i] \in X \}
$$

Operator $\text{Lim}$ is an **upper-closure operator**

**Proof**: exercise!
Prefix closure

We write $\sigma_{\mid i}$ for the prefix of length $i$ of trace $\sigma$:

$$
\langle s_0, \ldots, s_n \rangle_{\mid 0} = \epsilon \\
\langle s_0, \ldots, s_n \rangle_{\mid i+1} = \begin{cases} 
\langle s_0, \ldots, s_i \rangle & \text{if } i < n \\
\langle s_0, \ldots, s_n \rangle & \text{otherwise}
\end{cases}
$$

If $\sigma$ is finite, of length $n$, $|\sigma|_i = \min(n, i)$; if $\sigma$ is infinite, $|\sigma|_i = i$.

**Definition: prefix closure**

The prefix closure operator is defined by:

$$
\text{PCI} : \mathcal{P}(\Sigma^\infty) \rightarrow \mathcal{P}(\Sigma^*)
$$

$$
X \mapsto \{\sigma_{\mid i} \mid \sigma \in X, i \in \mathbb{N}\}
$$

**Properties:**

- PCI is monotone
- PCI is idempotent, i.e., $\text{PCI} \circ \text{PCI}(X) = \text{PCI}(X)$
Safety properties: formal definition

An upper closure operator

Operator \textbf{Safe} is defined by \textbf{Safe} = \textbf{Lim} \circ \textbf{PCI}.
It is an upper closure operator over \( \mathcal{P}(S^\infty) \)

Proof:

- \textbf{Safe} is monotone as \textbf{Lim} and \textbf{PCI} are
- \textbf{Safe} is extensive; indeed if \( X \subseteq S^\infty \) and \( \sigma \in X \), we can show that \( \sigma \in \textbf{Safe}(X) \):
  - if \( \sigma \) is a finite trace, it is one of its prefixes, so \( \sigma \in \textbf{PCI}(X) \subseteq \textbf{Lim}(\textbf{PCI}(X)) \)
  - if \( \sigma \) is an infinite trace, all its prefixes belong to \( \textbf{PCI}(X) \), so \( \sigma \in \textbf{Lim}(\textbf{PCI}(X)) \)
Proof (continued):

- **Safe** is idempotent:
  - as **Safe** is extensive and monotone **Safe** \(\subseteq** **Safe** \(\circ** **Safe**), so we simply need to show that **Safe** \(\circ** **Safe** \(\subseteq** **Safe**
  - let \(X \subseteq S^\infty\), \(\sigma \in **Safe**(**Safe**(**X**)); then:

\[
\sigma \in **Safe**(**Safe**(**X**)) \\
\Rightarrow \forall i, \sigma_{\left\lceil i \right\rceil} \in **PCI** \circ **Safe**(**X**) \quad \text{by def. of Lim} \\
\Rightarrow \forall i, \exists \sigma', j, \sigma_{\left\lceil i \right\rceil} = \sigma'_{\left\lceil j \right\rceil} \land \sigma' \in **Safe**(**X**) \quad \text{by def. of PCI} \\
\Rightarrow \forall i, \exists \sigma', j, \sigma_{\left\lceil i \right\rceil} = \sigma'_{\left\lceil j \right\rceil} \land \forall k, \sigma'_{\left\lceil k \right\rceil} \in **PCI**(**X**) \quad \text{by def. of Lim} \\
\Rightarrow \forall i, \exists \sigma', j, \sigma_{\left\lceil i \right\rceil} = \sigma'_{\left\lceil j \right\rceil} \land \sigma'_{\left\lceil i \right\rceil} \in **PCI**(**X**) \quad \text{with } i = j
\]

- if \(\sigma\) is finite, we let \(i = |\sigma|\), thus \(j\) has to be equal to \(n\) as well and \(\sigma = \sigma'_{\left\lceil i \right\rceil} \in **PCI**(**X**), thus \(\sigma \in **Lim**(**PCI**(**X**))
- if \(\sigma\) is infinite, \(|\sigma_{\left\lceil i \right\rceil}| = i\) and we may let \(i = k\) so

\[
\forall i, \sigma_{\left\lceil i \right\rceil} = \sigma'_{\left\lceil i \right\rceil} \in **PCI**(**X**) \\
\text{thus } \sigma \in **Lim**(**PCI**(**X**))
Safety properties: formal definition

**Safety: definition**
A trace property $\mathcal{T}$ is a **safety** property if and only if $\text{Safe}(\mathcal{T}) = \mathcal{T}$

**Theorem**
If $\mathcal{T}$ is a trace property, then $\text{Safe}(\mathcal{T)}$ is a safety property

**Proof:** straightforward, by idempotence of Safe
Example

We assume that:

- $S = \{a, b\}$
- $T$ states that *a should not be visited after state b is visited*;
  elements of $T$ are of the general form
  $$\langle a, a, a, \ldots, a, b, b, b, \ldots \rangle \text{ or } \langle a, a, a, \ldots, a, a, \ldots \rangle$$

Then:

- $\text{PCI}(T)$ elements are all finite traces which are of the above form (i.e.,
  made of $n$ occurrences of $a$ followed by $m$ occurrences of $b$, where
  $n, m$ are positive integers)
- $\text{Lim}(\text{PCI}(T))$ adds to this set the trace made made of infinitely many
  occurrences of $a$ and the infinite traces made of $n$ occurrences of $a$
  followed by infinitely many occurrences of $b$
- thus, $\text{Safe}(T) = \text{Lim}(\text{PCI}(T)) = T$

Therefore $T$ is indeed formally a safety property.
State properties are safety properties

**Theorem**

Any *state property* is also a *safety property*.

**Proof:** Let us consider *state property* \( P \).
It is equivalent to *trace property* \( T = P^\infty \):

\[
\text{Safe}(T) = \lim \text{PCI}(P^\infty) \\
= \lim (P^*) \\
= P^* \cup P^\omega \\
= P^\infty \\
= T
\]

Therefore \( T \) is indeed a safety property.
Intuition of the formal definition

Operator **Safe saturates** a set of traces $S$ with

- prefixes
- infinite traces all finite prefixes of which can be observed in $S$

Thus, if $\text{Safe}(S) = S$ and $\sigma$ is a trace, to establish that $\sigma$ is not in $S$, it is sufficient to discover a **finite prefix of $\sigma$** that cannot be observed in $S$.

Alternatively, if all finite prefixes of $\sigma$ belong to $S$ or can observed as a prefix of another trace in $S$, by definition of the limit operator, $\sigma$ **belongs to $S$** (even if it is infinite).

Thus, our definition **indeed captures properties that can be disproved with a counter-example**.
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Outline

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Traces Properties
Proof by invariance

- We consider transition system $S = (S, \rightarrow, S_I)$, and safety property $T$. Finite traces semantics is the least fixpoint of $F^*$. 
- We seek a way of verifying that $S$ satisfies $T$, i.e., that $\llbracket S \rrbracket^\infty \subseteq T$

Principle of invariance proofs

Let $\Pi$ be a set of finite traces; it is said to be an invariant if and only if:

- $\forall s \in S_I, \langle s \rangle \in \Pi$
- $F^*_\Pi \subseteq \Pi$

It is stronger than $T$ if and only if $\Pi \subseteq T$.

The “by invariance” proof method is based on finding an invariant that is stronger than $T$. 
The invariance proof method is sound: if we can find an invariant for $S$, that is stronger than $T$, then $S$ satisfies $T$.

Proof:
We assume that $I$ is an invariant of $S$ and that it is stronger than $T$, and we show that $S$ satisfies $T$:

- by induction over $n$, we can prove that $F^n_\ast(\{s \mid s \in S\}) \subseteq F^n_\ast(I) \subseteq I$
- therefore $[S]^\ast \subseteq I$
- thus, $\text{Safe}([S]^\ast) \subseteq \text{Safe}(I) \subseteq \text{Safe}(T)$ since $\text{Safe}$ is monotone
- we remark that $[S]^\propto = \text{Safe}([S]^\ast)$
- $T$ is a safety property so $\text{Safe}(T) = T$
- we conclude $[S]^\propto \subseteq T$, i.e., $S$ satisfies property $T$
Completeness

Theorem: completeness

The invariance proof method is complete: if $S$ satisfies $\mathcal{T}$, then we can find an invariant $I$ for $S$, that is stronger than $\mathcal{T}$.

Proof:
We assume that $[S]^{\infty}$ satisfies $\mathcal{T}$, and show that we can exhibit an invariant.
Then, $I = [S]^{\infty}$ is an invariant of $S$ by definition of $[.]^{\infty}$, and it is stronger than $\mathcal{T}$.

Caveat:
- $[S]^{\infty}$ is most likely not a very easy to express invariant
- it is just a convenient completeness argument
- so, completeness does not mean the proof is easy!
Safety properties
Proof method

Example

We consider the proof that the program below computes the sum of the elements of an array, i.e., when the exit is reached, $s = \sum_{k=0}^{n-1} t[k]$:

\[
\begin{align*}
i, s & \text{ integer variables} \\
t & \text{integer array of length } n \\
l_0 &: (\text{true}) \\
& \quad s = 0; \\
l_1 &: (s = 0) \\
& \quad i = 0; \\
l_2 &: (i = 0 \land s = 0) \\
& \quad \text{while}(i < n) \{ \\
l_3 &: (0 \leq i < n \land s = \sum_{k=0}^{i-1} t[k]) \\
& \quad s = s + t[i]; \\
l_4 &: (0 \leq i < n \land s = \sum_{k=0}^{i} t[k]) \\
& \quad i = i + 1; \\
l_5 &: (1 \leq i \leq n \land s = \sum_{k=0}^{i-1} t[k]) \\
& \} \\
l_6 &: (i = n \land s = \sum_{k=0}^{n-1} t[k])
\end{align*}
\]

Principle of the proof:

- for each program point $l$, we have a local invariant $I_l$ (denoted by a logical formula instead of a set of states in the figure)
- the global invariant $I$ is defined by:
  \[
  I = \{\langle (l_0, m_0), \ldots, (l_n, m_n) \mid \forall n, m_n \in I_{l_n}\}
  \]
Outline
Liveness properties

Informal definition: liveness properties

A liveness property is a property which specifies that some (good) behavior will eventually occur.

- **termination** is a liveness property
  “good behavior”: reaching a blocking state (no more transition available)

- “state a will eventually be reached by all execution” is a liveness property
  “good behavior”: reaching state a

- The absence of runtime errors is *not* a liveness property
Intuition towards a formal definition

We intend to provide a formal definition of liveness.

How to refute a liveness property?
- we consider liveness property $\mathcal{T}$ (think $\mathcal{T}$ is termination)
- we assume $\mathcal{S}$ does not satisfy liveness property $\mathcal{T}$
- thus, there exists a counter-example trace $\sigma \in [\mathcal{S}] \setminus \mathcal{T}$;
- let us assume $\sigma$ is actually finite...

the definition of liveness says some (good) behavior should eventually occur:
  - how do we know that $\sigma$ cannot be extended into a trace $\sigma \cdot \sigma'$ that will satisfy this behavior?
  - maybe that after a few more computation steps, $\sigma$ will reach a blocking state...
Intuition towards a formal definition

To refutate a liveness property, we need to look at infinite traces.

Example: if we run a program, and do not see it return...
- should we do Ctrl+C and conclude it does not terminate?
- should we just wait a few more seconds minutes, hours, years?

Towards a formal definition: we expect any finite trace be the prefix of a trace in $T$
as finite executions cannot be used to disprove $T$

Formal definition (incomplete)

$$PCI(T) = S^*$$
Definition

Formal definition

Operator \textbf{Live} is defined by \( \text{Live}(\mathcal{T}) = \mathcal{T} \cup (\mathcal{S}^\infty \setminus \text{Safe}(\mathcal{T})) \). Given property \( \mathcal{T} \), the following three statements are equivalent:

(i) \( \text{Live}(\mathcal{T}) = \mathcal{T} \)

(ii) \( \text{PCI}(\mathcal{T}) = \mathcal{S}^* \)

(iii) \( \text{Lim} \circ \text{PCI}(\mathcal{T}) = \mathcal{S}^\infty \)

When they are satisfied, \( \mathcal{T} \) is said to be a \textit{liveness property}

Example: \textit{termination}

- the property is \( \mathcal{T} = \mathcal{S}^* \) (i.e., there should be no infinite execution)
- clearly, it satisfies (ii): \( \text{PCI}(\mathcal{T}) = \mathcal{S}^* \) thus termination indeed satisfies this definition
Proof of equivalence

Proof of equivalence:

- **(i) implies (ii):**
  
  we assume that \( \text{Live}(\mathcal{T}) = \mathcal{T} \), i.e., \( \mathcal{T} \cup (\mathcal{S}^\infty \setminus \text{Safe}(\mathcal{T})) = \mathcal{T} \)
  
  therefore, \( \mathcal{S}^\infty \setminus \text{Safe}(\mathcal{T}) \subseteq \mathcal{T} \);
  
  let \( \sigma \in \mathcal{S}^* \), and let us show that \( \sigma \in \text{PCI}(\mathcal{T}) \); clearly, \( \sigma \in \mathcal{S}^\infty \), thus:
  
  ▶ either \( \sigma \in \text{Safe}(\mathcal{T}) = \text{Lim}(\text{PCI}(\mathcal{T})) \), so all its prefixes are in \( \text{PCI}(\mathcal{T}) \) and \( \sigma \in \text{PCI}(\mathcal{T}) \)
  
  ▶ or \( \sigma \in \mathcal{T} \), which implies that \( \sigma \in \text{PCI}(\mathcal{T}) \)

- **(ii) implies (iii):**
  
  if \( \text{PCI}(\mathcal{T}) = \mathcal{S}^* \), then \( \text{Lim} \circ \text{PCI}(\mathcal{T}) = \mathcal{S}^\infty \)

- **(iii) implies (i):**
  
  if \( \text{Lim} \circ \text{PCI}(\mathcal{T}) = \mathcal{S}^\infty \), then
  
  \[ \text{Live}(\mathcal{T}) = \mathcal{T} \cup (\mathcal{S}^\infty \setminus (\mathcal{T} \cup \text{Lim} \circ \text{PCI}(\mathcal{T}))) = \mathcal{T} \cup (\mathcal{S}^\infty \setminus \mathcal{S}^\infty) = \mathcal{T} \]
Example

We assume that:

- $S = \{a, b, c\}$
- $T$ states that $b$ should eventually be visited, after $a$ has been visited; elements of $T$ can be described by

$$T = S^* \cdot a \cdot S^* \cdot b \cdot S^\infty$$

Then $T$ is a liveness property:

- let $\sigma \in S^*$; then $\sigma \cdot a \cdot b \in T$, so $\sigma \in \text{PCI}(T)$
- thus, $\text{PCI}(T) = S^*$
A property of Live

**Theorem**
If $\mathcal{T}$ is a trace property, then $\text{Live}(\mathcal{T})$ is a liveness property (i.e., operator Live is idempotent).

**Proof:** we show that $\text{PCI} \circ \text{Live}(\mathcal{T}) = S^*$, by considering $\sigma \in S^*$ and proving that $\sigma \in \text{PCI} \circ \text{Live}(\mathcal{T})$; we first note that:

\[
\text{PCI} \circ \text{Live}(\mathcal{T}) = \text{PCI}(\mathcal{T}) \cup \text{PCI}(S^* \setminus \text{Safe}(\mathcal{T}))
= \text{PCI}(\mathcal{T}) \cup \text{PCI}(S^* \setminus \text{Lim} \circ \text{PCI}(\mathcal{T}))
\]

- if $\sigma \in \text{PCI}(\mathcal{T})$, this is obvious.
- if $\sigma \notin \text{PCI}(\mathcal{T})$, then:
  - $\sigma \notin \text{Lim} \circ \text{PCI}(\mathcal{T})$ by definition of the limit
  - thus, $\sigma \in S^* \setminus \text{Lim} \circ \text{PCI}(\mathcal{T})$
  - $\sigma \in \text{PCI}(S^* \setminus \text{Lim} \circ \text{PCI}(\mathcal{T}))$ as PCI is extensive, which proves the above result
Outline
Termination proof with ranking function

- We consider only termination
- We consider transition system \( S = (S, \rightarrow, S_I) \), and liveness property \( T \)
- We seek a way of verifying that \( S \) satisfies termination, i.e., that \([S]^{\infty} \subseteq S^*\)

**Definition: ranking function**

A **ranking function** is a function \( \phi : S \rightarrow E \) where:

- \((E, \sqsubseteq)\) is a well-founded ordering
- \(\forall s_0, s_1 \in S, s_0 \rightarrow s_1 \implies \phi(s_1) \sqsubseteq \phi(s_0)\)

**Theorem**

If \( S \) has a ranking function \( \phi \), it satisfies termination.
Example

We consider the termination of the array sum program:

\begin{align*}
& i, s \text{ integer variables} \\
& t \text{ integer array of length } n \\
& l_0 : \quad s = 0; \\
& l_1 : \quad i = 0; \\
& l_2 : \quad \textbf{while}(i < n)\{ \\
& \quad l_3 : \quad s = s + t[i]; \\
& \quad l_4 : \quad i = i + 1; \\
& \quad l_5 : \quad \} \\
& l_6 : \quad \ldots
\end{align*}

Ranking function:

\[
\phi : \mathbb{S} \longrightarrow \mathbb{N}
\]

\[
\begin{align*}
& (l_0, m) \mapsto 3 \cdot n + 6 \\
& (l_1, m) \mapsto 3 \cdot n + 5 \\
& (l_2, m) \mapsto 3 \cdot n + 4 \\
& (l_3, m) \mapsto 3 \cdot (n - m(i)) + 3 \\
& (l_4, m) \mapsto 3 \cdot (n - m(i)) + 2 \\
& (l_5, m) \mapsto 3 \cdot (n - m(i)) + 1 \\
& (l_6, m) \mapsto 0
\end{align*}
\]
Proof by variance

- We consider transition system $S = (S, \rightarrow, S_T)$, and liveness property $T$; infinite traces semantics is the least fixpoint of $F_\omega$.
- We seek a way of verifying that $S$ satisfies $T$, i.e., that $[S]^\infty \subseteq T$.

Principle of variance proofs

Let $(I_n)_{n \in \mathbb{N}}, I_\omega$ be elements of $S^\infty$; these are said to form a variance proof of $T$ if and only if:

- $S^\infty \subseteq I_0$
- for all $k \in \{1, 2, \ldots, \omega\}$, $\forall s \in S$, $\langle s \rangle \in I_k$
- for all $k \in \{1, 2, \ldots, \omega\}$, there exists $l < k$ such that $F_\omega(I_l) \subseteq I_k$
- $I_\omega \subseteq T$

Proofs of soundness and completeness: exercise
### The decomposition theorem

**Theorem**

Let $\mathcal{T} \subseteq S^\infty$; it can be decomposed into the **conjunction** of safety property $\text{Safe}(\mathcal{T})$ and liveness property $\text{Live}(\mathcal{T})$:

$$\mathcal{T} = \text{Safe}(\mathcal{T}) \cap \text{Live}(\mathcal{T})$$

- **Reading:**
  
  *Recognizing Safety and Liveness.*
  
  Bowen Alpern and Fred B. Schneider.
  

- **Consequence of this result:**
  
  the proof of any trace property can be decomposed into
  
  - a proof of safety
  - a proof of liveness
Proof

- **safety part:**
  Safe is idempotent, so Safe($\mathcal{T}$) is a safety property.

- **liveness part:**
  Live is idempotent, so Live($\mathcal{T}$) is a liveness property.

- **decomposition:**
  \[
  \text{Safe}(\mathcal{T}) \cap \text{Live}(\mathcal{T}) = (\mathcal{S}^\infty \setminus \text{Safe}(\mathcal{T}) \cup \mathcal{T}) \cap \text{Safe}(\mathcal{T}) \\
  = (\mathcal{S}^\infty \setminus \text{Safe}(\mathcal{T}) \cap \text{Safe}(\mathcal{T})) \cup (\mathcal{T} \cap \text{Safe}(\mathcal{T})) \\
  = \mathcal{T}
  \]
Decomposition of trace properties

Example: verification of total correctness

\begin{enumerate}
    \item i, s integer variables
    \item t integer array of length \( n \)
\end{enumerate}

\begin{itemize}
    \item \( l_0: \) \( s = 0; \)
    \item \( l_1: \) \( i = 0; \)
    \item \( l_2: \) \textbf{while}(i < n){
    \item \( l_3: \) \( s = s + t[i]; \)
    \item \( l_4: \) \( i = i + 1; \)
    \item \( l_5: \) }
    \item \( l_6: \) ...
\end{itemize}

Property to prove:
\begin{enumerate}
    \item total correctness
\end{enumerate}

\begin{enumerate}
    \item the program \textbf{terminates}
    \item and it \textbf{computes the sum of the elements in the array}
\end{enumerate}

Application of the decomposition principle

Conjunction of two proofs:

\begin{enumerate}
    \item proved with a \textbf{ranking function}
    \item proved with \textbf{local invariants}
\end{enumerate}
Safety and Liveness Decomposition Example

We consider a very simple greatest common divider code function:

```c
int f(int a, int b)
{
    while(a > 0)
    {
        int d = b/a;
        int r = b - a * d;
        b = a;
        a = r;
    }
    return b;
}
```

Specification

When applied to positive integers, function \( f \) should always return their GCD.
Safety and Liveness Decomposition Example

We consider a very simple greatest common divider code function:

\[
\begin{align*}
l_0 & : \textbf{int } f(\textbf{int } a, \textbf{int } b) \{ \\
l_1 & : \quad \textbf{while}(a > 0) \{ \\
l_2 & : \quad \textbf{int } d = b/a; \\
l_3 & : \quad \textbf{int } r = b - a \times d; \\
l_4 & : \quad b = a; \\
l_5 & : \quad a = r; \\
l_6 & : \quad \} \\
l_7 & : \quad \textbf{return } b; \\
l_8 & : \quad \}
\end{align*}
\]

Specification

When applied to positive integers, function \( f \) should always return their GCD.

Safety part

For all trace starting with positive inputs, a conjunction of two properties:

- no runtime errors
- the value of \( b \) is the GCD

Liveness part

Termination, on all traces starting with positive inputs
The Zoo of semantic properties: current status

- **Trace properties**
  - total correctness

- **Safety properties**
  - never reach $s_0$ before $s_1$

- **State properties**
  - absence or runtime errors
  - partial correctness

- **Liveness properties**
  - termination

- **Safety**: if wrong, can be refuted with a finite trace
  proof done by **invariance**

- **Liveness**: if wrong, has to be refuted with an infinite trace
  proof done by **variance**
Outline
Notion of specification language

- Ultimately, we would like to **verify or compute** properties.
- So far, we simply describe properties with **sets of executions** or worse, with English / French / ... statements.
- Ideally, we would prefer to use a mathematical language for that:
  - to **gain in concision**, avoid ambiguity
  - to define sets of properties to consider, fix the form of inputs for verification tools...

**Definition: specification language**

A specification language is a set of terms \( \mathbb{L} \) with an **interpretation function** (or semantics)

\[
[.] : \mathbb{L} \rightarrow \mathcal{P}(\mathbb{S}^\infty) \quad (\text{resp.}, \mathcal{P}(\mathbb{S}))
\]

- We are now going to consider specification languages **for states**, **for traces**...
A State specification language

A first **example** of a (simple) specification language:

**A state specification language**

- **Syntax:** we let terms of $L_S$ be defined by:

  $$ p \in L_S ::= @l \mid x < x' \mid x < n \mid \neg p' \mid p' \land p'' \mid \Omega $$

- **Semantics:** $[p] \subseteq S_\Omega$ is defined by

  \[
  \begin{align*}
  [\@l] &= \{l\} \times M \\
  [x \leq x'] &= \{(l, m) \in S \mid m(x) \leq m(x')\} \\
  [x \leq n] &= \{(l, m) \in S \mid m(x) \leq n\} \\
  [\neg p] &= S_\Omega \setminus [p] \\
  [p \land p'] &= [p] \cap [p'] \\
  [\Omega] &= \{\Omega\}
  \end{align*}
  \]

**Exercise:** add $=, \lor, \implies$...
State properties: examples

Unreachability of control state $l_0$:
- specification: $\Omega \lor \neg @l_0$
- property: $[[\Omega \lor \neg @l_0]] = S_\Omega \setminus \{(l_0, m) \mid m \in M\}$

Absence of runtime errors:
- specification: $\neg \Omega$
- property: $[\neg \Omega] = S_\Omega \setminus \{\Omega\} = S$

Intermittent invariant:
- principle: attach a local invariant to each control state
- example:

\[
\begin{align*}
l_0 : & \quad \text{if}(x \geq 0) \{ \\
l_1 : & \quad y = x; \quad \quad \quad \quad @l_1 \implies x \geq 0 \\
l_2 : & \quad } \text{else}\{ \\
l_3 : & \quad y = -x; \quad \quad \quad \quad \land \quad @l_3 \implies x < 0 \\
l_4 : & \quad } \} \\
l_5 : & \quad \ldots \\
\end{align*}
\]
We now consider the **specification of trace properties**

- **temporal logic**: specification of properties in terms of events that occur at distinct times in the execution (hence, the name “temporal”)
- there are **many** instances of temporal logic
- we study a simple one: **Pnueli’s Propositional Temporal Logic**

**Definition: syntax of PTL (Propositional Temporal Logic)**

Properties over traces are defined as terms of the form

\[
\begin{align*}
t \in \mathbb{L}_{PTL} & ::= \ p & \text{state property, i.e., } p \in \mathbb{L}_S \\
t' \lor t'' & \text{disjunction} \\
\neg t' & \text{negation} \\
\bigcirc t' & \text{"next"} \\
t' \mathrel{U} t'' & \text{"until"}, \text{i.e., } t' \text{ until } t''
\end{align*}
\]
Propositional temporal logic: semantics

Some operators on traces:
- $|\sigma|$ denotes the **length** of trace $\sigma$ (either an integer or $\infty$)
- "tail" operator $\cdot_i$:
  $$\sigma_i = \epsilon \quad \text{if } |\sigma| < i$$
  $$\langle s_0, \ldots, s_i \rangle \cdot \sigma_{i} := \sigma \quad \text{otherwise}$$

Semantics of Propositional Temporal Logic formulae

\[
\begin{align*}
\llbracket p \rrbracket & = \{ s \cdot \sigma \mid s \in \llbracket p \rrbracket \land \sigma \in S^\infty \} \\
\llbracket t_0 \lor t_1 \rrbracket & = \llbracket t_0 \rrbracket \cup \llbracket t_1 \rrbracket \\
\llbracket \neg t_0 \rrbracket & = S^\infty \setminus \llbracket t_0 \rrbracket \\
\llbracket \Box t_0 \rrbracket & = \{ s \cdot \sigma \mid s \in S \land \sigma \in \llbracket t_0 \rrbracket \} \\
\llbracket t_0 \mathcal{U} t_1 \rrbracket & = \{ \sigma \in S^\infty \mid \exists n \in \mathbb{N}, \forall i < n, \sigma_{i} \in \llbracket t_0 \rrbracket \land \sigma_{n} \in \llbracket t_1 \rrbracket \} 
\end{align*}
\]
Temporal logic operators as syntactic sugar

Many useful operators can be added:

- **Boolean constants:**

  $$\text{true} ::= (x < 0) \lor \neg(x < 0)$$
  $$\text{false} ::= \neg\text{true}$$

- **Sometime:**

  $$\bigcirc t ::= \text{true} \uplus t$$

  *intuition*: there exists a rank \( n \) at which \( t \) holds

- **Always:**

  $$\Box t ::= \neg(\bigcirc(\neg t))$$

  *intuition*: there is no rank at which the negation of \( t \) holds

**Exercise**: what do \( \bigcirc \Box t \) and \( \Box \bigcirc t \) mean?
Propositional temporal logic: examples

We consider the program below:

\[
\begin{align*}
\ell_0 & : \text{ int } x = \text{input}(); \\
\ell_1 & : \text{ if}(x < 8) \\
\ell_2 & : \quad x = 0; \\
\ell_3 & : \quad \text{else} \\
\ell_4 & : \quad x = 1; \\
\ell_5 & : \quad \} \\
\ell_6 & : \ldots
\end{align*}
\]

Examples of properties:

- “when \( \ell_4 \) is reached, \( x \) is positive”
  \[ \square(\neg \ell_4 \implies x \geq 0) \]

- “if the value read at point \( \ell_0 \) is negative, and when \( \ell_6 \) is reached, \( x \) is equal to 0”
  \[(\neg \ell_1 \land x < 0) \implies \square(\neg \ell_6 \implies x = 0)\]
Outline
We now consider other interesting properties of programs, and show that they do not all reduce to trace properties.

**Security**

- collects many kinds of properties
- so we consider just one:

  an unauthorized observer should not be able to guess anything about private information by looking at public information

- **example:** another user should not be able to guess the content of an email sent to you
- we need to formalize this property
A few definitions

Assumptions:
- we let $S = (S, \rightarrow, S_I)$ be a transition system
- states are of the form $(\ell, m) \in L \times M$
- memory states are of the form $X \rightarrow V$
- we let $\ell, \ell' \in L$ (program entry and exit)
  and $x, x' \in X$ (private and public variables)

Security property we are looking at

Observing the value of $x'$ at $\ell'$ gives no information on the value of $x$ at $\ell$.

We consider the transformer $\Phi$ defined by:

$$\Phi : M \rightarrow \mathcal{P}(M)$$

$$m \mapsto \{m' \in M \mid \exists \sigma = \langle (\ell, m), \ldots, (\ell', m') \rangle \in \llbracket S \rrbracket \}$$
Non-interference

Definition: non-interference

There is no interference between \((l, x)\) and \((l', x')\) and we write \((l', x') \not\rightarrow (l, x)\) if and only if the following property holds:

\[
\forall m \in M, \forall v_0, v_1 \in V,
\{ m'(x') \mid m' \in \Phi(m[x \leftarrow v_0]) \} = \{ m'(x') \mid m' \in \Phi(m[x \leftarrow v_1]) \}
\]

Intuition:

- if two observations at point \(l\) differ only in the value of \(x\), there is no difference in observation of \(x'\) at \(l'\)
- in other words, observing \(x'\) at \(l'\) (even on many executions) gives no information about the value of \(x\) at point \(l\)...
Beyond safety and liveness

Non-interference is not a trace property

- we assume $\mathbb{V} = \{0, 1\}$ and $\mathbb{X} = \{x, x'\}$ (store $m$ is defined by the pair $(m(x), m(x'))$, and denoted by it)
- we assume $\mathbb{L} = \{\ell, \ell'\}$ and consider two systems such that all transitions are of the form $(\ell, m) \rightarrow (\ell', m')$
  (i.e., system $S$ is isomorphic to its transformer $\Phi[S]$)

\[
\Phi[S_0] : 
\begin{align*}
(0, 0) & \mapsto M & (0, 0) & \mapsto M \\
(0, 1) & \mapsto M & (0, 1) & \mapsto M \\
(1, 0) & \mapsto M & (1, 0) & \mapsto \{(1, 1)\} \\
(1, 1) & \mapsto M & (1, 1) & \mapsto \{(1, 1)\}
\end{align*}
\]

- $S_1$ has fewer behaviors than $S_0$: $[S_1]^* \subset [S_0]^*$
- $S_0$ has the non-interference property, but $S_1$ does not
- if non interference was a trace property, $S_1$ should have it (monotony)

Thus, the non interference property is not a trace property
Beyond safety and liveness

Dependence properties

Dependence property

- many notions of dependences
- so we consider just one:

  what inputs may have an impact on the observation of a given output

Applications:

- **reverse engineering**: understand how an input gets computed
- **slicing**: extract the fragment of a program that is relevant to a result

This corresponds to the **negation** of non-interference
Interference

**Definition: interference**

There is interference between \((l, x)\) and \((l', x')\) and we write \((l', x') \rightsquigarrow (l, x)\) if and only if the following property holds:

\[
\exists m \in M, \exists v_0, v_1 \in V, \{ m'(x') \mid m' \in \Phi(m[x \leftarrow v_0]) \} \neq \{ m'(x') \mid m' \in \Phi(m[x \leftarrow v_1]) \}
\]

- This expresses that there is at least one case, where the value of \(x\) at \(l\) has an impact on that of \(x'\) at \(l'\).
- It may not hold even if the computation of \(x'\) reads \(x\):

\[
\begin{align*}
l & : x' = 0 \ast x; \\
l' & : \ldots
\end{align*}
\]
Interference is not a trace property

- we assume $\mathcal{V} = \{0, 1\}$ and $\mathcal{X} = \{x, x'\}$ (store $m$ is defined by the pair $(m(x), m(x'))$, and denoted by it)

- we assume $\mathcal{L} = \{l, l'\}$ and consider two systems such that all transitions are of the form $(l, m) \rightarrow (l', m')$
  (i.e., system $S$ is isomorphic to its transformer $\Phi[S]$)

  $\Phi[S_0] : (0, 0) \mapsto M$
  $\quad (0, 1) \mapsto M$
  $\quad (1, 0) \mapsto \{(1, 1)\}$
  $\quad (1, 1) \mapsto \{(1, 1)\}$

  $\Phi[S_1] : (0, 0) \mapsto \{(1, 1)\}$
  $\quad (0, 1) \mapsto \{(1, 1)\}$
  $\quad (1, 0) \mapsto \{(1, 1)\}$
  $\quad (1, 1) \mapsto \{(1, 1)\}$

- $S_1$ has fewer behavior than $S_0$: $[S_1]^* \subset [S_0]^*$
- $S_0$ has the interference property, but $S_1$ does not
- if interference was a trace property, $S_1$ should have it (monotony)

Thus, the interference property is not a trace property
The Zoo of semantic properties

Sets of sets of executions
non-interference, dependency

Trace properties
total correctness

Safety properties
never reach $s_0$ before $s_1$

State properties
absence or runtime errors partial correctness

Liveness properties
termination
Summary

To sum-up:

- **trace properties** allow to express a large range of program properties
- **safety** = absence of bad behaviors
- **liveness** = existence of good behaviors
- trace properties can be **decomposed** as conjunctions of safety and liveness properties, with **dedicated proof methods**
- some interesting properties are **not trace properties**
  - security properties are **sets of sets of executions**
- notion of **specification languages** to describe program properties