Workshop 10

March 31, 2014

Topics: Sequences and Series

1. **Explicit Sequences** Determine if the following sequences diverge or converge as $n \to \infty$. If they converge, give the limit (with proof!). If they diverge, prove that they diverge.
   
   (a) $a_n = \frac{3n^2-1}{10n^3+5n^2}$
   
   (b) $(-1)^n$
   
   (c) $\frac{(-1)^n}{n}$
   
   (d) $\frac{n^n}{n!}$
   
   (e) $\frac{n + 47}{\sqrt{n^2 + 3n}}$
   
   (f) $\sqrt{n + 47} - \sqrt{n}$

2. **Recursive Sequences.** Use the “Bounded Monotonic Theorem” to prove convergence. You will need to use mathematical induction to do this.

   (a) $a_1 = 1$ and $a_{n+1} = 3 - \frac{1}{a_n}$ for $n \geq 1$. Prove that $1 \leq a_n < 3$ for all $n$ and that the sequence is increasing. Find $\lim_{n \to \infty} a_n$.

   (b) $a_1 = \sqrt{2}$ and $a_n = \sqrt{2 + a_{n-1}}$. Prove that $a_n \leq 3$, that it is increasing and converges. Find $\lim_{n \to \infty} a_n$.

3. **Series.** Using basic properties, sum of geometric series and comparison test, see if the series converges or diverges.

   **Geometric Series:** $\sum_{n=0}^{\infty} \frac{1}{r^n} = \frac{1}{1 - 1/r}$.

   (a) $\sum_{n=1}^{\infty} \frac{1}{2^n}$
   
   Calculate the following Geometric Series:

   (b) $\sum_{n=0}^{\infty} e^{1-2n}$. Calculate the limit sum.

   (c) $\sum_{n=1}^{\infty} \frac{e^{n-1}}{3^n}$. Calculate the limit sum.

   Use comparison

   (d) $\sum_{n=1}^{\infty} \frac{2+n}{n^2}$

   (e) $\sum_{n=1}^{\infty} \frac{2}{n^2+1}$

   (f) $\sum_{n=1}^{\infty} \frac{1+n}{n^3+n^2}$

   (g) $\sum_{n=1}^{\infty} \frac{1+n}{3^n}$

   (h) $\sum_{n=1}^{\infty} \frac{1}{n!}$

   (i) $\sum_{n=1}^{\infty} 1/\ln(2 + 2n)$
4. Series. Using basic properties, alternating series, comparison, ratio tests, see if it converges or diverges

- **Alternating Test:** If \( \sum_{n=1}^{\infty} (-1)^n a_n \) is such that \( \lim_{n \to \infty} a_n = 0 \) and \( 0 \leq a_{n+1} \leq a_n \) then \( \sum_{n=1}^{\infty} (-1)^n a_n \) converges.

- **Ratio test:** If \( \sum_{n=1}^{\infty} a_n \) is such that
  \[
  \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L
  \]
  The ratio test states that:
  - if \( L < 1 \) then the series converges absolutely;
  - if \( L > 1 \) then the series does not converge;
  - if \( L = 1 \) or the limit fails to exist, then the test is inconclusive, because there exist both convergent and divergent series that satisfy this case.

- **Root test:** If \( \sum_{n=1}^{\infty} a_n \) is such that
  \[
  \lim_{n \to \infty} \sqrt[n]{a_n} = C
  \]
  The root test states that:
  - if \( C < 1 \) then the series converges absolutely;
  - if \( C > 1 \) then the series does not converge;
  - if \( C = 1 \) or the limit fails to exist, then the test is inconclusive, because there exist both convergent and divergent series that satisfy this case.

(a) Apply Alternating test: \( \sum_{n=1}^{\infty} \left(-\frac{1}{2n}\right)^n \)

(b) Apply Alternating test or Comparison? \( \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{2n} \)

(c) \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1/10}} \)

(d) Apply the Ratio Test:
  \( \sum_{n=1}^{\infty} \frac{2^{n-1}}{2n+1} \). (Was the Ratio test necessary here? )

(e) \( \sum_{n=1}^{\infty} \frac{2^n}{n!} \)

(f) \( \sum_{n=1}^{\infty} \frac{n^4}{(2n)!} \)

(g) \( \sum_{n=1}^{\infty} \frac{3^n}{n!} \)

(h) Try Root Test:
  \( \sum_{n=1}^{\infty} \frac{(-3n)^n}{(2n\sqrt{n+2})^n} \)

(i) \( \sum_{n=1}^{\infty} (1 + 1/n)^n \). **TIP:** \( \lim_{n \to \infty} (1 + 1/n)^n = e. \)

(j) \( \sum_{n=1}^{\infty} (1 + 1/n)^{n^2} \)