Workshop 7

March 7, 2014

Topics: polynomial division, partial fractions, trigonometric substitution, integration review

Practice exercises:

1. **Polynomial Division and Partial Fractions** We use these techniques to integrate ratios of polynomials

   \[
   \frac{f(x)}{g(x)} = \frac{f_n x^n + f_{n-1} x^{n-1} + \ldots + f_0}{g_m x^m + g_{m-1} x^{m-1} + \ldots + g_0}
   \]

   Firstly we check if we need to use long division by asking if \( \deg(f) \geq \deg(g) \), if it is we must use long division, if not we can skip straight to partial fractions.

   Integrate the following using these techniques.

   (a) \( \frac{x^3 + 7}{x^2 - x - 6} \)
   
   (b) \( \frac{x^3 + 4x^2 + 3}{x^2 + 2x + 1} \)
   
   (c) \( \frac{x - 8}{x^2 - x - 6} \)
   
   (d) \( \frac{x}{x^2 - 4x - 5} \)
   
   (e) \( \frac{2x + 1}{x^2 + 2x + 1} \)
   
   (f) \( \frac{1}{x^2 - 1} \)
   
   (g) \( \frac{x^4 - x^3 - x - 1}{x^2 - x^2} \)
   
   (h) \( \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 3)} \)

   **Solution:**

   (a) \( \frac{x + 7}{x^2 - x - 6} = \frac{2}{x - 3} - \frac{1}{x + 2} \) so answer is \( 2 \ln(x - 3) - \ln(x + 2) + C \)

   (b) Long division gives us that \( \frac{x^3 + 4x^2 + 3}{x^2 + 2x + 1} = x + 2 + \frac{-5x + 1}{x^2 + 2x + 1} = x + 2 + \frac{-5}{x + 1} + \frac{6}{(x + 1)^2} \) which integrates to \( x^2/2 + 2x - 5 \ln(x + 1) - 6/(x + 1) \)

   (c) partial fraction decomp = \( 2/(x + 2) - 1/(x - 3) \) which integrates to \( 2 \ln(x + 2) - \ln(x - 3) \)

   (d) partial fraction decomp = \( \frac{5}{6(x - 5)} + \frac{1}{6(x + 1)} \) which integrates to \( 5/6 \ln(x - 5) + 1/6 \ln(x + 1) \)

   (e) partial fraction decomp = \( 2/(x + 1) - 1/(x - 1)^2 \) which integrates to \( 2 \ln(x + 1) + 1/(x - 1) \)

   (f) partial fraction decomp = \( 1/4(\frac{1}{x - 1} - \frac{1}{x + 2}) \) which integrates to \( 1/4 \ln |\frac{x - 2}{x + 2}| \)

   (g) long division = \( x - \frac{x + 1}{x^2(x - 1)} \) partial fraction decomp = \( x + 2/x + 1/x^2 - 2/(x - 1) \) which integrates to \( x^2/2 + 2 \ln|x| - 1/x - 2 \ln|x - 1| + C \)

   (h) partial fraction decomp = \( 2/x - 2/(x - 1) + (2x + 4)/(x^2 + 4) \) which integrates to \( 2 \ln(x) - 2 \ln(x - 1) + \ln(x^2 + 4) + 2 \arctan(x/2) \)

2. **Trigonometric substitution** There is no trick here, it just takes lot’s of practice! Integrate the following using trigonometric substitution.
3. General Integration Techniques

Just integrate, using any appropriate technique.

- (a) \( \int_0^\pi \sin^2(\theta) \, d\theta \)
- (b) \( \int_1^2 \frac{dx}{x^2 - 2x + 1} \)
- (c) \( \int e^{-2t} \cos t \, dt \)
- (d) \( \int \frac{3x^3 - 17x^2 + 30x - 35}{x^3 - 4x^2 + 4} \, dx \)
- (e) \( \int \tan^4(s) \sec^4(s) \, ds \)
- (f) \( \int \frac{x}{x + 1} \, dx \)
- (g) \( \int \sqrt{9 - 4x^2} \, dx \)
- (h) \( \int \frac{2x - 1}{x^2 - 2x + 10} \, dx \)

Solution:

- (a) Standard use of \( \sin^2(u) = 1/2(1 - \cos(2u)) \) gives \( \pi/2 \)
- (b) By parts \( u = \ln(2x) , \quad dv = 1/x^2 \) gives \( -\ln(6)/3 + \ln(2) - 1/3 + 1 \)
- (c) Do by parts twice with \( u = e^{-2t} , \quad dv = \cos(t) \), and then the same except sin for cos, then we get a recursion which solves to give an answer of \( 1/5(e^{-2t} \sin(t) - 2e^{-2t} \cos(t)) \)
- (d) After division and partial fractionation \( 3x - 5 + 4/(x - 2) + 7/(x - 2)^2 \) integrates to \( 3/2x^2 - 5x + 4 \ln |x - 2| + 7/(x - 2) \)
- (e) \( u = \tan(s) \) and using some trig identities gives you \( \tan^5(s)/5 + \tan^7(s)/7 \)
- (f) This is a classic integration by parts (after the obvious substitution of \( u = x - 1 \)) to give \((x - 1) \arcsin(x - 1) + \sqrt{1 - (x - 1)^2} \)
- (g) Trig substitution \( x = 3/2 \sin(u) \) gives the integral as \( 9/4 \arcsin(2/3x) + 1/2x \sqrt{9 - 4x^2} \)
(h) Complete the square on the bottom, then rewrite the top as $2(x - 1) + 1$, this gives an answer of $\ln(x^2 - 2x + 10) + \frac{1}{3}\arctan(\frac{x - 1}{1})$.