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A CONSTRUCTIVE CHARACTERIZATION OF THE LATTICES OF ALL RETRACTIONS, PRECLOSURE, QUASI-CLOSURE AND CLOSURE OPERATORS ON A COMPLETE LATTICE

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1. Introduction

We give a constructive characterization of the complete lattices of all retractions, preclosure, quasi-closure and closure operators on a complete lattice. Our general approach is the following: in order to study the structure of the set $\Gamma \subseteq (L \to L)$ of operators ρ on a complete lattice L satisfying a given axiom A, we show that ρ has property A if and only if it is a fixed point of some monotone operator F on the complete lattice $(L \to L)$ proving that Γ is the set of fixed points of F. Then using Cousot & Cousot's constructive version of Tarski's lattice theoretical fixed point theorem, we constructively characterize the infimum, supremum, union and intersection of the complete lattice Γ which are defined by means of limits of stationary transfinite iteration sequences for F. Variants of this argument are used when F is a clousre operator (in which case the constructive version of Tarski's theorem amounts to Ward's theorem) or when the operators with property A are the postfixed points of F or the common fixed points of two functionals. The reasoning is repeated when Γ is characterized by means of more than one axiom.

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2. Preliminaries

1.1. Let $L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ be a non-empty *complete lattice* with *partial ordering* $\sqsubseteq$, *least upper bound* $\sqcup$, *greatest lower bound* $\sqcap$. The *infimum* $\bot$ of L is $\sqcap L$, the *supremum* $\top$ of L is $\sqcup L$. (Birkhoff's standard reference book [2] provides the necessary background material).

1.2. Let $\theta \in (L \to M)$ be a total function from the complete lattice $L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ to the complete lattice $M(\sqsubseteq', \bot', \top', \sqcup', \sqcap')$. θ is a *join-morphism* when $(\forall x,y \in L, \theta(x \sqcup y) = \theta(x) \sqcup' \theta(y))$. θ is a *complete join-morphism* when $(\forall S \subseteq L, \theta(\sqcup S) = \sqcup' \theta(S))$. The dual notions are the ones of *meet-morphism* and *complete meet-morphism.*

1.3. Using Church[3]'s lambda notation (so that $f \in (L \to M)$ is $\lambda x.f(x)$) let us recall that the set $(L \to L)$ of all operators on the complete lattice $L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ is a complete lattice $(\sqsubseteq', \bot', \top', \sqcup', \sqcap')$ where $((f \sqsubseteq' g) \Leftrightarrow (\forall x \in L, f(x) \sqsubseteq g(x)))$, $\bot' = \lambda x. \bot$, $\top' = \lambda x. \top$, $\sqcup' = \lambda S.(\lambda x. \sqcup\{f(x) : f \in S\})$, $\sqcap' = \lambda S.(\lambda x. \sqcap\{f(x) : f \in S\})$. In the following we will omit the primes so that the distinction between $\sqsubseteq$, $\bot$, $\top$, $\sqcup$, $\sqcap$ and $\sqsubseteq'$, $\bot'$, $\top'$, $\sqcup'$, $\sqcap'$ will be contextual.

1.4. A *retraction* ρ on L is an operator on L (i. e. $\rho \in (L \to L)$) which is *monotone* (i. e. $\forall x, y \in L, (x \sqsubseteq y) \Rightarrow (\rho(x) \sqsubseteq \rho(y))$) and *idempotent* (i. e. $\rho = \rho \circ \rho$ that is $\forall x \in L, \rho(x) = \rho(\rho(x))$).

1.5. An *upper preclosure operator* $\bar{\rho}$ on L is monotone, idempotent and satisfies the *upper connectivity axiom* $(\forall x \in L, \bar{\rho}(x \sqcup \bar{\rho}(x)) = \bar{\rho}(x))$. The dual notion is the one of *lower preclosure operator* (Ladegaillerie, [6, Def. 1]).

1.6. A *quasi-closure operator* ρ on L is monotone, *comparing* (i. e. $\forall x \in L$, either $x \sqsubseteq \rho(x)$ or $\rho(x) \sqsubseteq x$) and satisfies the *connectivity axiom* $(\forall x \in L, \rho(x \sqcap \rho(x)) = \rho(x \sqcup \rho(x))$, Bernard[1, p. 6]). Notice that monotony and connectivity axiom imply $(\forall x \in L, \rho(x) = \rho(x \sqcap \rho(x)) = \rho(x \sqcup \rho(x)))$ and using the comparing hypothesis this in turn implies idempotence.

1.7. An *upper closure operator* $\bar{\rho}$ on L is monotone, idempotent and *extensive* (i. e. $\lambda x.x \sqsubseteq \bar{\rho}$), (Moore[9]). Dually a *lower closure operator* $\underline{\rho} \in (L \to L)$ is monotone, idempotent and *reductive* (i e. $\underline{\rho} \sqsubseteq \lambda x.x$).

1.8. Let $\bar{\rho}$ be an upper closure operator on $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$. For all $x \in L$, the set $\{y \in \bar{\rho}(L) : x \sqsubseteq y\}$ is not empty and $\bar{\rho}(x)$ is its least element (Monteiro & Ribeiro[8, Th. 5.2]).

1.9. Let $R \subseteq L$ and $\bar{\rho} \in (L \to R)$ be such that for any $x \in L$, $\bar{\rho}(x)$ is the least element of the set $\{y \in R : x \sqsubseteq y\}$ then $\bar{\rho}$ is an upper closure operator and $R = \bar{\rho}(L)$, (Monteiro & Ribeiro[8, Th. 5.3]).

1.10. Let $\bar{\rho}$ be an upper closure operator on L, then the image $\bar{\rho}(L)$ of L by $\bar{\rho}$ is a complete lattice $\bar{\rho}(L)(\sqsubseteq, \bar{\rho}(\perp), \top, \lambda S.\bar{\rho}(\sqcup S), \sqcap)$ which is a complete sublattice of L if and only if $\bar{\rho}$ is a complete join-morphism, (Ward[14, Th. 4.1]).

1.11. Applying the duality principle it follows that if $R \subseteq L$ and $\bar{\rho}$, $\underline{\rho} \in (L \to L)$ are such that for any $x \in L$, $\bar{\rho}(x)$ is the least element of the set $\{y \in R : x \sqsubseteq y\}$ and $\underline{\rho}(x)$ is the greatest element of the set $\{y \in R : y \sqsubseteq x\}$ then $R = \bar{\rho}(L) = \underline{\rho}(L)$ is a complete sublattice $R(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ of L whereas $\bar{\rho}$ is a complete join-morphism and $\underline{\rho}$ is a complete meet-morphism.

1.12. Let $\bar{\rho}_1$ and $\bar{\rho}_2$ be upper closure operators on L. Then according to Ore[12, p. 525], $\bar{\rho}_1 \circ \bar{\rho}_2$ and $\bar{\rho}_2 \circ \bar{\rho}_1$ are upper closure operators on L if and only if $\bar{\rho}_1$ and $\bar{\rho}_2$ are *commuting* (i. e. $\bar{\rho}_1 \circ \bar{\rho}_2 = \bar{\rho}_2 \circ \bar{\rho}_1$) in which case $\bar{\rho}_1 \circ \bar{\rho}_2(L) = \bar{\rho}_2 \circ \bar{\rho}_1(L) = \bar{\rho}_1(L) \cap \bar{\rho}_2(L)$.

1.13. Let $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ be a complete lattice, μ the smallest ordinal such that the class $\{\delta : \delta \in \mu\}$ has a cardinality greater than the cardinality of L and $f \in (L \to L)$. The *upper iteration sequence for f starting with* $d \in L$ is the μ-termed sequence $< X^\delta, \delta \in \mu >$ of elements of L defined by transfinite recursion in the following way:

— $X^0 = d$

— $X^\delta = f(X^{\delta-1})$ for every successor ordinal $\delta \in \mu$

— $X^\delta = \bigsqcup_{\alpha < \delta} X^\alpha$ for every limit ordinal $\delta \in \mu$

(The dual *lower iteration sequence* is defined by:

— $X^\delta = \bigsqcap_{\alpha < \delta} X^\alpha$ for every limit ordinal $\delta \in \mu$)

1.14. We say that the sequence $< X^\delta, \delta \in \mu >$ is *stationary* if and only if $(\exists \varepsilon \in \mu : (\forall \beta \in \mu, (\varepsilon \leqslant \beta) \Rightarrow (X^\varepsilon = X^\beta)))$ in which case the *limit* of the sequence is X^{ε}. We denote by *luis* $(f)(d)$ the limit of a stationary upper iteration sequence for f starting with d (dually *llis* $(f)(d)$).

1.15. Let f be a monotone operator on $L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$. The sets of *prefixed points*, *fixed points* and *postfixed points* of f are respectively $prefp(f) = \{x \in L : x \sqsubseteq f(x)\}$, $fp(f) = \{x \in L : x = f(x)\}$, $postfp(f) = \{x \in L : f(x) \sqsubseteq x\}$.

1.16. The set of postfixed points of a monotone operator f on a complete lattice $L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ is a complete lattice $postfp(f)$ $(\sqsubseteq,\ lfp(f),\ \top,\ \lambda S.luis(\lambda z.z \sqcup f(z))(\sqcup S), \sqcap)$ which is the image of L by the upper closure operator $\lambda d.luis(\lambda z.z \sqcup f(z))(d)$. The *least fixed point* of f is $lfp(f) = luis(f)(\bot)$, (Cousot & Cousot[4, Th. 4.2]).

1.17. Let f be a monotone operator on the complete lattice $L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$. Then $\lambda x.luis(\lambda y.y \sqcup f(y))(x) = \lambda x.luis(\lambda y.x \sqcup f(y))(x) = \lambda x.lfp(\lambda y.x \sqcup f(y))$ is an upper closure operator on L greater than or equal to f, (Cousot & Cousot[4,Th.4.1. & 4.2]).

1.18. Cousot & Cousot[4,Th.5.1]'s constructive version of Tarski[13,Th.1]'s lattice theoretical fixed point theorem states that the set $fp(f)$ of fixed points of the monotone operator f on $L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ is a complete lattice $fp(f)(\sqsubseteq, luis(f)(\bot), llis(f)(\top), \lambda S.luis(f)(\sqcup S), \lambda S.llis(f)(\sqcap S))$. $fp(f)$ is the image of L by the upper preclosure operator $llis(f) \circ luis(\lambda z.z \sqcup f(z))$ and the image of L by the lower preclosure operator $luis(f) \circ llis(\lambda z.z \sqcap f(z))$. Moreover $luis(f) \circ llis(\lambda z.z \sqcap f(z)) \sqsubseteq llis(f) \circ luis(\lambda z.z \sqcup f(z))$.

1.19. Let $\underline{F}$ and $\overline{F}$ be monotone operators on $L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ such that $\underline{F} \sqsubseteq \overline{F}$ and $\overline{F} \circ \underline{F} \sqsubseteq \underline{F} \circ \overline{F}$. Then the set $fp(\underline{F}, \overline{F}) = \{x \in L : x = \underline{F}(x) = \overline{F}(x)\}$ of common fixed points of $\overline{F}$ and $\underline{F}$ is a non-empty complete lattice $fp(\overline{F}, \underline{F})(\sqsubseteq,\ luis(\overline{F})(\bot),\ llis(\underline{F})(\top),\ \lambda S.luis(\overline{F})(\sqcup S),\ \lambda S.llis(\underline{F})(\sqcap S))$ which is the image of L by the lower preclosure operator $luis(\overline{F}) \circ llis(\lambda z.z \sqcap \underline{F}(z))$ and the image of L by the upper preclosure operator $llis(\underline{F}) \circ luis(\lambda z.z \sqcup \overline{F}(z))$. Moreover $luis(\overline{F}) \circ llis(\lambda z.z \sqcap \underline{F}(z)) \sqsubseteq llis(\underline{F}) \circ luis(\lambda z.z \sqcup \overline{F}(z))$, (Cousot & Cousot [4, Th. 6.1]).

2. The lattice of all monotone operators on a complete lattice

DEFINITION 2.1 $\overline{\mathrm{M}}, \underline{\mathrm{M}} \in ((\mathrm{L} \to \mathrm{L}) \to (\mathrm{L} \to \mathrm{L}))$

$$\overline{\mathrm{M}} = \lambda f.(\lambda x. \sqcup \{f(y) : (y \in \mathrm{L}) \text{ and } (y \sqsubseteq x)\})$$
$$\underline{\mathrm{M}} = \lambda f.(\lambda x. \sqcap \{f(y) : (y \in \mathrm{L}) \text{ and } (x \sqsubseteq y)\})$$

THEOREM 2.2. *If* $f \in (\mathrm{L} \to \mathrm{L})$ *then* $\overline{\mathrm{M}}(f)$ *is the least monotone operator on* L *greater than or equal to* f *and* $\underline{\mathrm{M}}(f)$ *is the greatest monotone operator on* L *less than or equal to* f.

PROOF: Let $a, b \in \mathrm{L}$ be such that $a \sqsubseteq b$. For all $y \in \mathrm{L}$, $(y \sqsubseteq a)$ implies $(y \sqsubseteq b)$ so that $\sqcup \{f(y) : y \sqsubseteq a\} \sqsubseteq \sqcup \{f(y) : y \sqsubseteq b\}$ proving that $\overline{\mathrm{M}}(f)(a) \sqsubseteq \overline{\mathrm{M}}(f)(b)$. Since $\sqsubseteq$ is reflexive, $\forall a \in \mathrm{L}$ we have $f(a) \sqsubseteq \sqcup \{f(y) : y \sqsubseteq a\} = \overline{\mathrm{M}}(f)(a)$ that is $f \sqsubseteq \overline{\mathrm{M}}(f)$ proving that $\forall f \in (\mathrm{L} \to \mathrm{L})$, $\overline{\mathrm{M}}(f)$ is a monotone operator on L greater than or equal to f.

Let g be a monotone operator on L greater than or equal to f. For all $y \in \mathrm{L}$, $f(y) \sqsubseteq g(y)$ so that $\forall a \in \mathrm{L}$, $\overline{\mathrm{M}}(f)(a) = \sqcup \{f(y) : y \sqsubseteq a\} \sqsubseteq \sqcup \{g(y) : y \sqsubseteq a\} \sqsubseteq \sqcup \{g(y) : g(y) \sqsubseteq g(a)\} = g(a)$ proving that $\overline{\mathrm{M}}(f)$ is the least monotone operator on L greater than or equal to f.

By duality, $\underline{\mathrm{M}}(f)$ is the greatest monotone opeartor on L less than or equal to f. *End of proof.*

From 2.2 and 1.11 we get:

COROLLARY 2.3 $\overline{\mathrm{M}}$ *is an upper closure operator on* $(\mathrm{L} \to \mathrm{L})$ *which is a complete join-morphism and* $\underline{\mathrm{M}}$ *is a lower closure operator on* $(\mathrm{L} \to \mathrm{L})$ *which is a complete meet-morphism.*

The set of all monotone operators on the complete lattice L *is a complete sublattice* $\overline{\mathrm{M}}(\mathrm{L} \to \mathrm{L}) = \underline{\mathrm{M}}(\mathrm{L} \to \mathrm{L})(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ *of* $(\mathrm{L} \to \mathrm{L})$ $(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$.

3. The lattice of all extensive operators on a complete lattice

DEFINITION 3.1 $\underline{\mathrm{E}}, \overline{\mathrm{E}} \in ((\mathrm{L} \to \mathrm{L}) \to (\mathrm{L} \to \mathrm{L}))$

$$\overline{\mathrm{E}} = \lambda f.(\lambda x.(x \sqcup f(x)))$$
$$\underline{\mathrm{E}} = \lambda f.(\lambda x.(x \sqcap f(x)))$$

LEMMA 3.2 *If* $f \in (L \to L)$ *then* $\overline{E}(f)$ *is the least extensive operator on* L *greater than or equal to* f. $\overline{E}$ *is a complete join-morphism.*

COROLLARY 3.3 $\overline{E}$ *is an upper closure operator on* $(L \to L)$. *The set of all extensive operators on the complete lattice* L *is a complete sublattice* $\overline{E}(L \to L)(\sqsubseteq, \lambda x.x, \top, \sqcup, \sqcap)$, *of* $(L \to L)$.

LEMMA 3.4

— $\overline{M} \circ \overline{E} = \overline{E} \circ \overline{M}$ *is an upper closure operator on* $(L \to L)$

— $\overline{M}(L \to L) \cap \overline{E}(L \to L) = \overline{M} \circ \overline{E}(L \to L) = \overline{E} \circ \overline{M}(L \to L)$ *is a complete sublattice* $(\sqsubseteq, \lambda x.x, \top, \sqcup, \sqcap)$ *of* $(L \to L)$.

PROOF: Since $\sqsubseteq$ is reflexive, for any $f \in (L \to L)$ and $x \in L$ we have $x \sqsubseteq \sqcup \{y \sqcup f(y) : y \sqsubseteq x\}$ so that $\overline{M}(\overline{E}(f))(x) = \sqcup \{y \sqcup f(y) : y \sqsubseteq x\} = x \sqcup (\sqcup \{y \sqcup f(y) : y \sqsubseteq x\}) = \sqcup \{x \sqcup y \sqcup f(y) : y \sqsubseteq x\} = \sqcup \{x \sqcup f(y) : y \sqsubseteq x\} = x \sqcup (\sqcup \{f(y) : y \sqsubseteq x\}) = \overline{E}(\overline{M}(f))(x)$. Therefore $\overline{M}$ and $\overline{E}$ are commuting and the lemma follows from 1.12 and 1.10. *End of proof.*

4. The lattice of all upper closure operators on a complete lattice

LEMMA 4.1 *Let* F *be the operator on* $(\overline{M}(L \to L) \cap \overline{E}(L \to L))$ *defined by* $\lambda g.g \circ g$. *Then*

1 — F *is monotone*

2 — $fp(F) = postfp(F)$

PROOF: 1. If $f \sqsubseteq g$ then $f \in \overline{M}(L \to L)$ implies $F(f) = f \circ f \sqsubseteq f \circ g$. Also $f \circ g \sqsubseteq g \circ g = F(g)$ proving that F is monotone.

2. Let f be a postfixed point of F then $f \circ f \sqsubseteq f$. Since $f \in \overline{E}(L \to L)$ we have $\lambda x.x \sqsubseteq f$ so that $f \in \overline{M}(L \to L)$ implies $f \sqsubseteq f \circ f$. By antisymmetry $f \circ f = f$. *End of proof.*

DEFINITION 4.2

$\overline{I} \in ((\overline{M}(L \to L) \cap \overline{E}(L \to L)) \to (\overline{M}(L \to L) \cap \overline{E}(L \to L)))$

$\overline{I} = \lambda f.luis(\lambda g.g \circ g)(f) = \lambda f.luis(\lambda g.g \sqcup g \circ g)(f)$

$\overline{C} \in ((L \to L)) \to (L \to))$

$\overline{C} = \overline{I} \circ \overline{M} \circ \overline{E} = \overline{I} \circ \overline{E} \circ \overline{M}$

Ward[14,Th.4.2]'s theorem states that the set Γ of all upper closure operators on a complete lattice $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ is a complete lattice $\Gamma(\sqsubseteq, \lambda x.x, \top, \lambda S. \sqcap \{\eta \in \Gamma : (\forall \rho \in S, \rho \sqsubseteq \eta)\}, \sqcap)$. We give now a constructive version of this theorem:

THEOREM 4.3

1 — $\overline{C}(L \to L)(\sqsubseteq, \lambda x.x, \top, \lambda S.luis(\lambda g.g \circ g)(\sqcup S), \sqcap)$ *is the complete lattice of all uper closure operators on the complete lattice* L

2 — $\overline{C}$ *is an upper closure operator on* $(L \to L)$

3 — $\forall f \in (L \to L)$, $\overline{C}(f)$ *is the least upper closure operator on* L *greater than or equal to f.*

PROOF: According to definition 1.7 the set of all upper closure operators on the complete lattice $L(\sqsubseteq. \perp, \top, \sqcup, \sqcap)$ is the set of elements of $(\overline{M}(L \to L) \cap \overline{E}(L \to L))$ which are idempotent that is to say which are fixed points of $\lambda g.g \circ g$. In compliance with lemmas 4.1 and 3.4 $\lambda g.g \circ g$ is a monotone operator on the complete lattice $(\overline{M}(L \to L) \cap \overline{E}(L \to L))$ $(\sqsubseteq, \lambda x.x, \perp, \sqcup, \sqcap)$ and $fp\ (\lambda g.g \circ g) = postfp\ (\lambda g.g \circ g)$. Therefore 1.16 implies that the set of all upper closure operators on L is the complete lattice $postfp(\lambda g.g \circ g)(\sqsubseteq, luis(\lambda g.g \circ g)(\lambda x.x) = \lambda x.x, \top, \lambda S.luis(\lambda g.g \circ g)(\sqcup S), \sqcap)$ which is the image of $\overline{M}(L \to L) \cap \overline{E}(L \to L)$ by the upper closure operator $\overline{I}$ that is (lemma 3.4 and definition 4.2) $\overline{I}(\overline{M}(L \to L) \cap \overline{E}(L \to L)) = \overline{I} \circ \overline{M} \circ \overline{E}(L \to L) = \overline{C}(L \to L)$. $\overline{C}$ is monotone and extensive since it is the composition of upper closure operators. $\forall f \in (L \to L)$, $\overline{C}(f)$ is extensive, monotone and idempotent, hence $\overline{C}(f) = \overline{E} \circ \overline{C}(f) = \overline{M} \circ \overline{C}(f) = \overline{I} \circ \overline{C}(f) = \overline{I} \circ \overline{M} \circ \overline{E} \circ \overline{C}(f) = \overline{C} \circ \overline{C}(f)$ proving that $\overline{C}$ is idempotent whence an upper closure operator. Then 4.3.3 follows from 1.8. *End of proof.*

COROLLARY 4.4

1 — $\overline{C} = \lambda f.(luis(\overline{E}(\overline{M}(f))))$

2 — $\overline{C} = \lambda f.(\lambda x.lfp(\lambda y.x \sqcup \overline{M}(f)(y)))$

3 — $\overline{C}(L \to L)(\sqsubseteq, \lambda x.x, \top, \lambda S.luis(\sqcup S), \sqcap)$

4 — $\overline{C}(L \to L)(\sqsubseteq, \lambda x.x, \top, \lambda S.(\lambda x.lfp(\lambda y.x \sqcup (\sqcup S)(y))), \sqcap)$

PROOF: It follows from 1.17 that $\forall f \in (L \to L)$, $luis(\overline{E}(\overline{M}(f))) = \lambda x.lfp(\lambda y.x \sqcup \overline{M}(f)(y))$ is an upper closure operator on L greater than or equal to $\overline{\mathrm{M}}(f)$ whence according to theorem 2.2 greater than or equal to f. Let $\bar{\rho} \in \overline{\mathrm{C}}(\mathrm{L} \to \mathrm{L})$ be such that $f \sqsubseteq \bar{\rho}$. Then $\overline{\mathrm{E}}(\overline{\mathrm{M}}(f)) \sqsubseteq \overline{\mathrm{E}}(\overline{\mathrm{M}}(\bar{\rho})) = \bar{\rho}$ so that by monotony $luis(\overline{\mathrm{E}}(\overline{\mathrm{M}}(f))) \sqsubseteq luis(\bar{\rho})$. Moreover $luis(\bar{\rho}) = \bar{\rho}$ since $\bar{\rho}$ is idempotent proving that $luis(\overline{\mathrm{E}}(\overline{\mathrm{M}}(f)))$ is the least upper closure operator on L greater than or equal to f and therefore $\overline{\mathrm{C}} = \lambda f.(luis(\overline{\mathrm{E}}(\overline{\mathrm{M}}(f)))) = \lambda f.(\lambda x.lfp(\lambda y.x \sqcup \overline{\mathrm{M}}(f)(y)))$, (1.17).

4.4.3 and 4.4.4 result from 1.10 remarking that whenever $\mathrm{S} \subseteq \overline{\mathrm{C}}(\mathrm{L} \to \mathrm{L})$ then $\sqcup \mathrm{S}$ is monotone and extensive (theorems 2.3 and 3.3) so that $luis(\overline{\mathrm{E}}(\overline{\mathrm{M}}(\sqcup \mathrm{S}))) = luis(\sqcup \mathrm{S})$ and $\lambda x.lfp(\lambda y.x \sqcup \overline{\mathrm{M}}(\sqcup \mathrm{S})(y)) = \lambda x.lfp(\lambda y.x \sqcup (\sqcup \mathrm{S})(y))$. *End of proof.*

It should be noted that given a complete lattice $\mathrm{L}(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ and $f \in (\mathrm{L} \to \mathrm{L})$ there does not necessarily exists an upper closure operator on L less than or equal to f. A counter-example is given by $\mathrm{L} = \{\perp, \top\}$ where $\perp \sqsubseteq \perp \sqsubseteq \top \sqsubseteq \top$ and f defined by $f(\perp) = \top$ and $f(\top) = \perp$. The upper closure operators on L are $\bar{\rho}_1$ (such that $\bar{\rho}_1(\perp) = \perp$ and $\bar{\rho}_1(\top) = \top$) and $\bar{\rho}_2$ (such that $\bar{\rho}_2(\perp) = \bar{\rho}_2(\top) = \top$) and neither $\bar{\rho}_1 \sqsubseteq f$ nor $\bar{\rho}_2 \sqsubseteq f$.

Monteiro [7], Iseki [5] and Morgado ([10, 11] have given several characterizations of the upper closure operators on a partially ordered set by means of a single axiom. We now formulate a characterization of the upper closure operators on a complete lattice using only one axiom:

COROLLARY 4.5 *If* $\mathrm{L}(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ *is a complete latt'ce and* $\bar{\rho}$ *is an operator on* L, *then* $\bar{\rho}$ *is an upper closure operator on* L *if and only if*:

$$\bar{\rho} = \lambda x.\sqcup\{y \sqcup \bar{\rho}(\bar{\rho}(y)) : (y \in \mathrm{L}) \text{ and } (y \sqsubseteq x)\}$$

PROOF: If $\bar{\rho}$ is a closure operator then it is monotone (so that $\overline{\mathrm{M}}(\bar{\rho}) = \bar{\rho}$ (2.2)), extensive (so that $\overline{\mathrm{E}}(\bar{\rho}) = \bar{\rho}$) and idempotent (so that $\bar{\rho} = \bar{\rho} \circ \bar{\rho}$) therefore $\bar{\rho} = \overline{\mathrm{M}}(\overline{\mathrm{E}}(\bar{\rho} \circ \bar{\rho})) = \lambda x.\sqcup\{y \sqcup \bar{\rho}(\bar{\rho}(y)) : (y \in \mathrm{L}) \text{ and } (y \sqsubseteq x)\}$. Reciprocally, if $\bar{\rho}$ is equal to the monotone operator $\overline{\mathrm{M}}(\overline{\mathrm{E}}(\bar{\rho} \circ \bar{\rho}))$ then it is monotone. Also since $\overline{\mathrm{M}}$ and $\overline{\mathrm{E}}$ commute (3.4) $\bar{\rho}$ is equal to the extensive operator $\overline{\mathrm{E}}(\overline{\mathrm{M}}(\bar{\rho} \circ \bar{\rho}))$ whence it is extensive. Since $\bar{\rho} \circ \bar{\rho}$ is the composition of monotone and extensive operators it is monotone and extensive and therefore $\bar{\rho} = \overline{\mathrm{M}}(\overline{\mathrm{E}}(\bar{\rho} \circ \bar{\rho})) = \bar{\rho} \circ \bar{\rho}$ (3.2,2.2) proving that $\bar{\rho}$ is idempotent whence an upper closure operator. *End of proof.*

5. The lattice of all retractions on a complete lattice

DEFINITION 5.1

$\bar{\mathrm{J}}\,,\ \underline{\mathrm{J}} \in (\bar{\mathrm{M}}(\mathrm{L} \to \mathrm{L}) \to \bar{\mathrm{M}}(\mathrm{L} \to \mathrm{L}))$

$\bar{\mathrm{J}} = \lambda f.(llis(\lambda g.g{\circ}g){\circ}luis(\lambda g.g \sqcup g{\circ}g)(f))$

$\underline{\mathrm{J}} = \lambda f.(luis(\lambda g.g{\circ}g){\circ}llis(\lambda g.g \sqcap g{\circ}g)(f))$

$\bar{\mathrm{R}}\,,\ \underline{\mathrm{R}} \in ((\mathrm{L} \to \mathrm{L}) \to (\mathrm{L} \to \mathrm{L}))$

$\bar{\mathrm{R}} = \bar{\mathrm{J}}{\circ}\bar{\mathrm{M}}$

$\underline{\mathrm{R}} = \underline{\mathrm{J}}{\circ}\underline{\mathrm{M}}$

THEOREM 5.2 *The set of all retractions on a complete lattice* $\mathrm{L}(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ *is a complete lattice* $\bar{\mathrm{R}}(\mathrm{L} \to \mathrm{L})(\sqsubseteq, \bot, \top, \lambda \mathrm{S}.luis(\lambda g.g{\circ}g)(\sqcup \mathrm{S}), \lambda \mathrm{S}.llis(\lambda g.g{\circ}g)(\sqcap \mathrm{S}))$ *which is the image of* $(\mathrm{L} \to \mathrm{L})$ *by the retractions* $\underline{\mathrm{R}}$ *and* $\bar{\mathrm{R}}$ *such that* $\underline{\mathrm{R}} \sqsubseteq \bar{\mathrm{R}}$.

$\forall f \in (\mathrm{L} \to \mathrm{L})$, *if* ρ *is a retraction on* L *greater than or equal to (less than or equal to)* f, *then* $\bar{\mathrm{R}}(f)$ $(\underline{\mathrm{R}}(f))$ *is less than or equal to (greater than or equal to)* ρ.

PROOF: The set of all retractions on L is the set of monotone operators on L(i. e. elements of $\underline{\mathrm{M}}(\mathrm{L} \to \mathrm{L}) = \bar{\mathrm{M}}(\mathrm{L} \to \mathrm{L})$) which are idempotent that is to say fixed points of $\lambda g.g{\circ}g$. Let $f,h \in \bar{\mathrm{M}}(\mathrm{L} \to \mathrm{L})$ be such that $f \sqsubseteq h$. Then $h \sqsubseteq h$ so that $f{\circ}h \sqsubseteq h{\circ}h$ and by monotony $f{\circ}f \sqsubseteq f{\circ}h$ hence $f{\circ}f \sqsubseteq h{\circ}h$ proving that $\lambda g.g{\circ}g$ is a monotone operator on $\bar{\mathrm{M}}(\mathrm{L} \to \mathrm{L})$. Consequently the set of all retractions on L is $fp(\lambda g.g{\circ}g)$ which according to 1.18 is a complete lattice $fp(\lambda g.g{\circ}g)(\sqsubseteq, luis(\lambda g.g{\circ}g)(\bot) = \bot, llis(\lambda g.g{\circ}g)(\top) = \top, \lambda \mathrm{S}.luis(\lambda g.g{\circ}g)(\sqcup \mathrm{S}), \lambda \mathrm{S}.llis(\lambda g.g{\circ}g)(\sqcap \mathrm{S}))$ which is the image of $\bar{\mathrm{M}}(\mathrm{L} \to \mathrm{L})$ by the upper preclosure operator $\bar{\mathrm{J}}$ and the image of $\underline{\mathrm{M}}(\mathrm{L} \to \mathrm{L})$ by the lower preclosure operator $\underline{\mathrm{J}}$ hence the image of $(\mathrm{L} \to \mathrm{L})$ by $\bar{\mathrm{R}}$ and $\underline{\mathrm{R}}$.

$\forall f \in (\mathrm{L} \to \mathrm{L})$, $\bar{\mathrm{J}}{\circ}\bar{\mathrm{M}}(f) \in \bar{\mathrm{M}}(\mathrm{L} \to \mathrm{L})$ hence $\bar{\mathrm{J}}{\circ}\bar{\mathrm{M}} = \bar{\mathrm{M}}{\circ}\bar{\mathrm{J}}{\circ}\bar{\mathrm{M}}$ (Th. 2.3.) Also $\bar{\mathrm{M}}{\circ}\bar{\mathrm{J}}{\circ}\bar{\mathrm{M}}(f)$ is idempotent so that it is a fixed point of $\lambda g.g{\circ}g$ and consequently a fixed point of $\bar{\mathrm{J}}$ proving that $\bar{\mathrm{J}}{\circ}\bar{\mathrm{M}}{\circ}\bar{\mathrm{J}}{\circ}\bar{\mathrm{M}}(f) = \bar{\mathrm{M}}{\circ}\bar{\mathrm{J}}{\circ}\bar{\mathrm{M}}(f) = \bar{\mathrm{J}}{\circ}\bar{\mathrm{M}}(f)$. $\bar{\mathrm{J}}{\circ}\bar{\mathrm{M}}$ is idempotent but also monotone since $\bar{\mathrm{M}}$ is a monotone operator on $(\mathrm{L} \to \mathrm{L})$ and $\bar{\mathrm{J}}$ a monotone operator on $\bar{\mathrm{M}}(\mathrm{L} \to \mathrm{L})$, hence $\bar{\mathrm{R}}$ and by duality $\underline{\mathrm{R}}$ are retractions on $(\mathrm{L} \to \mathrm{L})$.

Let $f \in (L \to L)$ and $\rho \in \overline{R}(L \to L)$ be such that $\rho \sqsubseteq f$. Then $\rho = \overline{R}(\rho) \sqsubseteq \overline{R}(f)$ since ρ is a retraction on L and $\overline{R}$ is monotone.

By duality if ρ is a retraction on L greater than or equal to f then $\underline{R}(f)$ is less than or equal to ρ. In general f and $\overline{R}(f)$ on one hand and f and $\underline{R}(f)$ on the other hand are not comparable. However $\underline{R}(f) \sqsubseteq \overline{R}(f)$ (1.18). *End of proof.*

6. The lattice of all upper preclosure operators on a complete lattice

DEFINITION 6.1

$\overline{A} \in (\overline{R}(L \to L) \to \overline{R}(L \to L))$

$\overline{A} = \lambda f. luis(\lambda g. g \circ \overline{E}(g))(f) = \lambda f. luis(\lambda g. g \sqcup g \circ \overline{E}(g))(f)$

$\overline{P}, \underline{P} \in ((L \to L) \to (L \to L))$

$\overline{P} = \overline{A} \circ \overline{R}$

$\underline{P} = \overline{A} \circ \underline{R}$

THEOREM 6.2 *The set of all upper preclosure operators on a complete lattice* $L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ *is a complete lattice* $\overline{P}(L \to L)$ $(\sqsubseteq, \bot, \top, \lambda S. luis(\lambda g. g \circ \overline{E}(g)) \circ luis(\lambda g. g \circ g)(\sqcup S), \lambda S. llis(\lambda g. g \circ g)(\sqcap S))$ *which is the image of* $(L \to L)$ *by the retractions* $\underline{P}$ *and* $\overline{P}$ *satisfying* $\underline{P} \sqsubseteq \overline{P}$.

$\forall f \in (L \to L)$ *if* $\bar{\rho}$ *is an upper preclosure operator on* L *greater than (less than) or equal to* f *then* $\overline{P}(f)$ $(\underline{P}(f))$ *is less than (greater than) or equal to* $\bar{\rho}$.

PROOF: The set of all upper preclosure operators on L is the set of monotone and idempotent operators on L (i. e. elements ρ of $\overline{R}(L \to L) = \underline{R}(L \to L))$ satisfying the upper connectivity axiom (i. e. $\rho = \lambda x.(\rho(x \sqcup \rho(x))))$ that is to say which are fixed points of $F = \lambda g.(\lambda x.(g(x \sqcup g(x)))) = \lambda g. g \circ \overline{E}(g)$. Let ρ be an element of $\overline{R}(L \to L)$. Then $F(\rho)$ is monotone. Also $F(\rho) \circ F(\rho) = \lambda x.(\rho(x \sqcup \rho(x))) \circ \lambda x.(\rho(x \sqcup \rho(x)))$ $= \lambda x . \rho(\rho(x \sqcup \rho(x)) \sqcup \rho(\rho(x \sqcup \rho(x)))) = \rho(\rho(x \sqcup \rho(x))) = \rho(x \sqcup \rho(x)) = F(\rho)$ since ρ is idempotent and proving that $F(\rho)$ is idempotent. Therefore F is an operator on $\overline{R}(L \to L)$. Let ρ and η be elements of $\overline{R}(L \to L)$ such that $\rho \sqsubseteq \eta$. Then $\overline{E}(\rho) \sqsubseteq \overline{E}(\eta)$ and by monotony $F(\rho) = \rho \circ \overline{E}(\rho) \sqsubseteq \rho \circ \overline{E}(\eta) \sqsubseteq \eta \circ \overline{E}(\eta) = F(\eta)$ proving that F is a monotone operator on

$\overline{R}(L \to L)$. $\forall \rho \in \overline{R}(L \to L)$ $\rho = F(\rho)$ implies $F(\rho) \sqsubseteq \rho$. Reciprocally if $F(\rho) \sqsubseteq \rho$ then $\lambda x.x \sqsubseteq \overline{E}(\rho)$ implies by monotony $\rho \sqsubseteq \rho \circ \overline{E}(\rho) = F(\rho)$ and by antisymmetry $\rho = F(\rho)$ proving that $fp(F) = postfp(F)$. It follows from 5.2 and 1.16 that the set of all upper preclosure operators on L is a complete lattice $postfp(F)(\sqsubseteq, luis(F)(\bot) = \bot, \top, \lambda S.luis(F) \circ luis(\lambda g.g \circ g)(\sqcup S), \lambda S.llis(\lambda g.g \circ g)(\sqcap S))$ which is the image of $\overline{R}(L \to L) = \underline{R}(L \to L)$ by the upper closure operator $\overline{A} = \lambda f.luis(F)(f)$ hence the image of $((L \to L)$ by $\overline{P}$ and $\underline{P}$.

$\underline{P}$ and $\overline{P}$ are monotone since $\overline{R}$ ánd $\underline{R}$ are monotone and $\overline{A}$ is an upper closure whence monotone operator on $\overline{R}(L \to L) = \underline{R}(L \to L)$. Also $\forall f \in (L \to L), \overline{A} \circ \overline{R}(f)$ is monotone and idempotent so that $\overline{A} \circ \overline{R}(f) = \overline{R} \circ \overline{A} \circ \overline{R}(f)$. $\overline{A} \circ \overline{R}(f)$ satisfies the upper connectivity axiom and therefore it is a fixed point of F and consequently a fixed point of $\overline{A}$ so that $\overline{A} \circ \overline{R}(f) = \overline{A} \circ \overline{A} \circ \overline{R}(f) = \overline{A} \circ \overline{R} \circ \overline{A} \circ \overline{R}(f)$ proving that $\overline{P}$ is idempotent. The same way $\underline{P}$ is a retraction on $(L \to L)$. Since $\underline{R} = \overline{R}$ and $\overline{A}$ is monotone we have $\underline{P} = \overline{A} \circ \underline{R} \sqsubseteq \overline{A} \circ \overline{R} = \overline{P}$.

$\forall f \in (L \to L)$, let $\bar{\rho} \in \overline{P}(L \to L)$ be such that $f \sqsubseteq \bar{\rho}$. Then $\bar{\rho} \in \overline{R}(L \to L)$ and 5.2 implies $\overline{R}(f) \sqsubseteq \bar{\rho}$. Since $\bar{\rho}$ is closed under $\overline{A}$ we have $\overline{P}(f) = \overline{A}(\overline{R}(f)) \sqsubseteq \overline{A}(\bar{\rho}) = \bar{\rho}$. Similarly $\bar{\rho} \sqsubseteq f$ implies $\bar{\rho} = \overline{A}(\bar{\rho}) \sqsubseteq \overline{A}(\overline{R}(f)) = \overline{P}(f)$. *End of proof.*

7. The lattice of all comparing operators on a complete lattice

DEFINITION 7.1

$\overline{K}, \underline{K} \in ((L \to L) \to (L \to L))$

$\overline{K} = \lambda f.(\lambda x.(\underline{if}\ f(x) \sqsubseteq x\ \underline{then}\ f(x)\ \underline{else}\ x \sqcup f(x)))$

$\underline{K} = \lambda f.(\lambda x.(\underline{if}\ x \sqsubseteq f(x)\ \underline{then}\ f(x)\ \underline{else}\ x \sqcap f(x)))$

LEMMA 7.2 *If* $f \in (L \to L)$ *then* $\overline{K}(f)$ *(respectively* $\underline{K}(f)$*) is the least (greatest) comparing operator on* L *greater than (less than) or equal to f.*

COROLLARY 7.3 $\overline{K}$ *is an upper closure operator on* $(L \to L)$, $\underline{K}$ *is a lower closure operator on* $(L \to L)$. *The set of all comparing operators on the complete lattice* L *is a complete sublattice* $\overline{K}(L \to L) = \underline{K}(L \to L)(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ *of* $(L \to L)$.

LEMMA 7.4

— $\overline{K}\circ\overline{M}$ *is an upper closure operator on* $(L \to L)$

— $\underline{K}\circ\underline{M}$ *is a lower closure operator on* $(L \to L)$

— $(\overline{M}(L \to L) \cap \overline{K}(L \to L)) = (\underline{M}(L \to L) \cap \underline{K}(L \to L)) = \overline{K}\circ\overline{M}(L \to L) = \underline{K}\circ\underline{M}(L \to L)$ *is a complete sublattice* $(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ *of* $(L \to L)$.

PROOF: $\overline{K}\circ\overline{M}$ is monotone and extensive since it is the composition of upper closure operators. $\forall f \in (L \to L)$, $\overline{K}(\overline{M}(f))$ is either equal to $\overline{M}(f)$ or to $\lambda x.(x \sqcup \overline{M}(f)(x))$ proving that $\overline{K}(\overline{M}(f))$ is monotone and therefore equal to $\overline{M}(\overline{K}(\overline{M}(f)))$. Hence $\overline{K}(\overline{M}(f)) = \overline{K}(\overline{K}(\overline{M}(f))) = \overline{K}(\overline{M}(\overline{K}(\overline{M}(f))))$ proving that $\overline{K}\circ\overline{M}$ is an upper closure operator and by duality $\underline{K}\circ\underline{M}$ is a lower closure operator on $(L \to L)$. Also $\overline{K}\circ\overline{M}(f)$ (respectively $\underline{K}\circ\underline{M}(f)$) is the least (greatest) monotone and comparing operator on L greater than (less than) or equal to f. Hence according to 1.11, $(\overline{M}(L \to L) \cap \overline{K}(L \to L)) = (\underline{M}(L \to L) \cap \underline{K}(L \to L))$ is the complete sublattice $\overline{K}\circ\overline{M}(L \to L) = \underline{K}\circ\underline{M}(L \to L)$ $(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ of $(L \to L)$. *End of proof.*

8. The lattice of all quasi-closure operators on a complete lattice

DEFINITION 8.1

$\overline{B}, \underline{B} \in (\overline{K}\circ\overline{M}(L \to L) \to \overline{K}\circ\overline{M}(L \to L))$

$\underline{B} = luis(\lambda g. g\circ\overline{E}(g))\circ llis(\lambda g. g \sqcap g\circ\underline{E}(g))$

$\overline{B} = llis(\lambda g. g\circ\underline{E}(g))\circ luis(\lambda g. g \sqcup g\circ\overline{E}(g))$

$\underline{Q}, \overline{Q} \in ((L \to L) \to (L \to L))$

$\underline{Q} = \underline{B}\circ\underline{K}\circ\underline{M}$

$\overline{Q} = \overline{B}\circ\overline{K}\circ\overline{M}$

Bernard [1,Th.V.II.1]'s theorem states that the set Γ of all quasi-closure operators on a complete lattice $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ is a complete lattice $\Gamma(\sqsubseteq, \perp, \top, \lambda S.(\underline{\sqcup}\,\underline{E}(S))\circ(\overline{\sqcup}\,\overline{E}(S)), \lambda S.(\overline{\sqcap}\,\overline{E}(S))\circ(\underline{\sqcap}\,\underline{E}(S)))$ where $(\overline{\sqcup}, \overline{\sqcap})$ and $(\underline{\sqcup}, \underline{\sqcap})$ respectively denote the union and intersection in the complete lattices of upper and lower closure operators on L. We now give a constructive and simpler version of this theorem:

THEOREM 8.2 *The set of all quasi-closure operators on a complete lattice* $L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ *is a complete lattice* $\overline{Q}(L \to L)$ $(\sqsubseteq, \bot, \top, \lambda S.\, luis(\lambda g.g\circ\overline{E}(g))(\sqcup S), \lambda S.\, llis(\lambda g.g\circ\underline{E}(g))(\sqcap S))$ *which is the image of* $(L \to L)$ *by the retractions* $\underline{Q}$ *and* $\overline{Q}$ *satisfying* $\underline{Q} \sqsubseteq \overline{Q}$.

$\forall f \in (L \to L)$, *if* ρ *is a quasi-closure operator on* L *greater than (respectively less than) or equal to* f *then* $\overline{Q}(f)$ $(\underline{Q}(f))$ *is less than (greater than) or equal to* ρ.

PROOF: The quasi-closure operators on L are the monotone and comparing operators on L (i. e. $\rho \in \overline{K}\circ\overline{M}(L \to L)$) satisfying the connectivity axiom $\rho = \lambda x.\rho(x \sqcap \rho(x)) = \lambda x.\rho(x \sqcup \rho(x))$ that is the elements of $\overline{K}\circ\overline{M}(L \to L)$ which are common fixed points of the functionals $\underline{F} = \lambda g.g\circ\underline{E}(g) = \lambda g.(\lambda x.(g(x \sqcap g(x))))$ and $\overline{F} = \lambda g.g\circ\overline{E}(g) = \lambda g.(\lambda x.(g(x \sqcup g(x))))$.

$\forall \rho \in \overline{K}\circ\overline{M}(L \to L)$, ρ is monotone and so are $\underline{F}(\rho)$ and $\overline{F}(\rho)$. Also ρ is comparing so that given $x \in L$ either $x \sqsubseteq \rho(x)$ in which case $x \sqsubseteq \rho(x) = \rho(x \sqcap \rho(x)) = \underline{F}(\rho)(x)$ or $\rho(x) \sqsubseteq x$ in which case $\underline{F}(\rho)(x) = \rho(x \sqcap \rho(x)) = \rho(\rho(x))$ and by monotony $\rho(\rho(x)) \sqsubseteq \rho(x) \sqsubseteq x$. Hence $\underline{F}(g)$ is comparing proving that $\underline{F}$ (and by duality $\overline{F}$) is an operator on $\overline{K}\circ\overline{M}(L \to L)$.

$\underline{F}$ and $\overline{F}$ are monotone operators on $\overline{K}\circ\overline{M}(L \to L)$ and $\underline{F} \sqsubseteq \overline{F}$. Let us show that $\overline{F}\circ\underline{F} \sqsubseteq \underline{F}\circ\overline{F}$ that is $\forall \rho \in \overline{K}\circ\overline{M}(L \to L)$, $\forall x \in L$, $\overline{F}(\underline{F}(\rho))(x) = \rho((x \sqcup \rho(x \sqcap \rho(x))) \sqcap \rho(x \sqcup \rho(x \sqcap \rho(x)))) \sqsubseteq \rho((x \sqcap \rho(x \sqcup \rho(x))) \sqcup \rho(x \sqcap \rho(x \sqcup \rho(x)))) = \underline{F}(\overline{F}(\rho))(x)$. If $x \sqsubseteq \rho(x)$ then $(x \sqcup \rho(x \sqcap \rho(x))) = \rho(x)$ and $(x \sqcap \rho(x \sqcup \rho(x))) = x$ so that we must show that $\rho(\rho(x) \sqcap \rho(\rho(x))) \sqsubseteq \rho(x \sqcap \rho(x))$ which is true since $(\rho(x) \sqcap \rho(\rho(x))) = \rho(x)$ and $(x \sqcup \rho(x)) = \rho(x)$. Else $\rho(x) \sqsubseteq x$ and then $(x \sqcup \rho(x \sqcap \rho(x))) = x$ and $(x \sqcap \rho(x \sqcup \rho(x))) = \rho(x)$ so that we must show that $\rho(x \sqcap \rho(x)) \sqsubseteq \rho(\rho(x) \sqcup \rho(\rho(x)))$ which is true since $(x \sqcap \rho(x)) = \rho(x)$ and $(\rho(x) \sqcup \rho(\rho(x))) = \rho(x)$. Applying 1.19 and using 7.4 we see that the set $fp(\underline{F}, \overline{F})$ of common fixed points of $\underline{F}$ and $\overline{F}$ is a complete lattice $fp(\underline{F}, \overline{F})(\sqsubseteq, luis(\overline{F})(\bot) = \bot, llis(\underline{F})(\top) = \top, \lambda S.\, luis(\overline{F})(\sqcup S), \lambda S.\, llis(\underline{F})(\sqcap S))$ which is the image of $\overline{K}\circ\overline{M}(L \to L) = \underline{K}\circ\underline{M}(L \to L)$ by the lower preclosure operator $\underline{B}$ and the upper preclosure operator $\overline{B}$ satisfying $\underline{B} \sqsubseteq \overline{B}$ hence the image of $(L \to L)$ by $\overline{Q}$ and $\underline{Q}$.

Since $\underline{M} \sqsubseteq \overline{M}$, $\underline{K} \sqsubseteq \overline{K}$, $\underline{B} \sqsubseteq \overline{B}$ and $\underline{K}$, $\underline{B}$ are monotone we have $\underline{Q} \sqsubseteq \overline{Q}$. $\underline{Q}$ and $\overline{Q}$ are composition of monotone operators and therefore monotone. Also $\overline{Q}$ (and dually $\underline{Q}$) is idempotent since $\forall f \in (L \to L)$, $\overline{Q}(f)$ is monotone (hence $\overline{M} \circ \overline{Q}(f) = \overline{Q}(f)$), comparing (hence $\overline{K} \circ \overline{Q}(f) = \overline{Q}(f)$) and satisfies the connectivity axiom (hence is a common fixed point of $\underline{F}$ and $\overline{F}$ therefore fixed point of $\overline{B}$) so that $\overline{B} \circ \overline{K} \circ \overline{M} \circ \overline{Q}(f) = \overline{Q}(f)$.

$\forall f \in (L \to L)$, $\forall \rho \in \overline{Q}(L \to L)$, if $f \sqsubseteq \rho$ then $\overline{Q}(f) \sqsubseteq \overline{Q}(\rho) = \rho$ and if $\rho \sqsubseteq f$ then $\rho = \underline{Q}(\rho) \sqsubseteq \underline{Q}(f)$. *End of proof.*

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