Confluence, local confluence, cirtical pair lemma, orthogonal systems

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Outline

1. Confluence and local confluence
2. Critical pair lemma
3. Confluence of orthogonal systems

Term rewriting system (TRS)

- Let $\mathcal{X}$ be a set of variables,
- Let $\mathcal{F}$ be a set of function symbols disjoint from $\mathcal{X}$, each symbol $f \in \mathcal{F}$ being equipped with a fixed arity $ar(f) \geq 0$,
- Let $\mathcal{R}$ be a set of rewrite rules over the set $\mathcal{T}(\mathcal{F}, \mathcal{X})$ of first-order terms.

Definition

A TRS $\mathcal{R}$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is a set of pairs $(l, r) \in \mathcal{T}(\mathcal{F}, \mathcal{X}) \times \mathcal{T}(\mathcal{F}, \mathcal{X})$ for which

- $l \notin \mathcal{V}$ and
- all variables of r occur in l

Pairs (l, r) are called rewrite rules and are usually written as $l \to r$

Confluence and local confluence

Definition

A TRS $\mathcal{R}$ is called confluent if and only if for every two reduction sequences $t_1 \leftarrow^*_{\mathcal{R}} s \rightarrow^*_{\mathcal{R}} t_2$ and there is a term u such that there are two reduction sequences $t_1 \rightarrow^*_{\mathcal{R}} u \leftarrow^*_{\mathcal{R}} t_2$.

Definition

A TRS $\mathcal{R}$ is called locally confluent or weakly confluent if and only if for every two one-step reductions $t_1 \leftarrow_{\mathcal{R}} s \rightarrow_{\mathcal{R}} t_2$ there is a term u such that there are two reduction sequences $t_1 \rightarrow^*_{\mathcal{R}} u \leftarrow^*_{\mathcal{R}} t_2$.

Check confluence

- Difficult to check confluence "directly":
  1. must check for infinitely many start terms t
  2. must check for arbitrarily many steps from each t to t_1 and t_2
- Solutions:
  1. Newman's lemma: sufficient to check w.r.t terms t_1 and t_2 that can be reached in one step from start term t.
  2. Critical pairs lemma: consider a finite set of start terms t

Relation between confluence and local confluence

Confluence implies local confluence, but not vice versa.

Example $f(x, x) \to a$
$f(x, g(x)) \to b$
$c \to g(c)$
Term c has no normal form, but $f(c, c)$ has two a and b

However, confluence may be derived from local confluence if the TRS is also terminating, a result that is known as Newman's lemma.

Newman's lemma

Let $\mathcal{R}$ be a TRS, $\mathcal{R}$ is called terminating or noetherian if there is no infinite sequence of terms $t_1, t_2, \ldots$ such that $t_i \rightarrow_{\mathcal{R}} t_{i+1}$ for all $i \geq 1$

Lemma

If $\mathcal{R}$ is terminating then it is confluent if and only if it is locally confluent.

Remark: Thus, if termination can be proved, local confluence suffices for proving confluence.
In practical critical pairs may help in determining whether a TRS is locally confluent.

Unifier and most general unifier

A term t matches a term s if there exists a substitution σ such that $t\sigma = s$.

Definition

A unifier of two terms t and s is a substitution σ such that $t\sigma = s\sigma$

- If σ is a unifier of t, s then each instance of σ is also a unifier for the terms.
- σ is a most general unifier (mgu) for t, s if there is a θ such that $\rho = \theta \circ \sigma$ (ρ is an instance or extension of σ)

Critical pair

Problem: If $t_1 \leftarrow_{\mathcal{R}} s \rightarrow_{\mathcal{R}} t_2$ does there exists a term u such that $t_1 \rightarrow^*_{\mathcal{R}} u \leftarrow^*_{\mathcal{R}} t_2$?

Answer:

- If the two rewrite steps happen in different subtress (disjont redexes): YES
- If the two rewrite steps happen below each other (overlap at or below a variable position) : YES
- If the left-hand sides of the two rules overlap at a non-variable position: needed further investigation

Critical pair(Cont.)

Definition

- Let $l_i \to r_i (i = 1, 2)$ be two rewrite rules in a TRS $\mathcal{R}$ whose variables have been renamed such that: $var(\{l_1, r_1\}) \cap var(\{l_2, r_2\}) = \emptyset$
- Let $p \in \mathcal{P}os(l_1)$ be a position such that $l_1|_p$ is not a variable and σ is an mgu of $l_1|_p$ and l_2.
- Then $r_1\sigma \leftarrow l_1\sigma \to (l_1\sigma)[r_2\sigma]_p$, $< r_1\sigma, (l_1\sigma[r_2\sigma]_p) >$ is called a critical pair of $\mathcal{R}$
- If $l_1 \to r_1$ and $l_2 \to r_2$ are different rewrite rules such that $l_1\sigma = l_2\sigma$ for some subtitution σ, then the critical pair $< r_1\sigma, r_2\sigma >$ is called an overlay.

Critical pairs lemma

Definition

A critical pair $< s, t >$ is called joinable if there exist a term u such that $s \rightarrow^*_{\mathcal{R}} u \leftarrow^*_{\mathcal{R}} t$

Lemma

A TRS is locally confluent if and only if its critical pairs are joinable.

This proof can be easily be checked by going though all posible types of overlap.

Remark: The critical pair lemma states that a TRS is locally confluent iff it has no critical pairs.

Orthogonal systems

A TRS for which all critical pairs are overlays is called an overlay system.

A term is linear if every variable occurs at most once in the term.

A TRS for which the left-hand side of every rule is a linear term, is called a left-linear TRS.

A TRS without critical pairs, is called a non-overlapping TRS.

A left-linear and non-overlapping TRS is called an orthogonal TRS.

Confluence of orthogonal systems

Theorem

Every orthogonal system is confluent.

Remark: $\rightarrow_{\mathcal{R}} \subseteq \nrightarrow_{\mathcal{R}} \subseteq \rightarrow^*_{\mathcal{R}}$ hence, $\nrightarrow^*_{\mathcal{R}} = \rightarrow^*_{\mathcal{R}}$
Parallel reduction relation is a sufficient condition for the confluent of TRS $\mathcal{R}$.

We need to consider parallel rewriting because if $s \rightarrow t_1$ and $s \rightarrow t_2$ at position that are not disjoint then a subterm of s may appear many times in t_1 or t_2 and all of these occurrences may have to be rewritten in parallel to obtain a u to which both t_1 and t_2 rewrite in one (parallel) step.

Top relation

Definition

For a TRS $\mathcal{R}$ the top relation $\rightarrow^{\epsilon}_{\mathcal{R}}$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is defined by $t \rightarrow^{\epsilon}_{\mathcal{R}} u$ iff there is an rewrite rule $l \rightarrow r \in \mathcal{R}$ and a subtitution $\sigma : \mathcal{X} \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{X})$ such that $t = l\sigma$ and $u = r\sigma$

Parallel reduction relation

Parallel rewriting is rewriting at one or more disjoint redexes at the same time.

Definition

Let $\mathcal{R}$ be a TRS, the parallel reduction relation included by $\mathcal{R}$ is the smallest relation $\twoheadrightarrow_{\mathcal{R}}$ such that

- $t \in \mathcal{X} \cup \mathcal{C}$ then $t \twoheadrightarrow_{\mathcal{R}} t$
- $t_1 \twoheadrightarrow_{\mathcal{R}} u_1 \ldots t_n \twoheadrightarrow_{\mathcal{R}} u_n$ then $ft_1 \ldots t_n \twoheadrightarrow_{\mathcal{R}} fu_1 \ldots u_n$
- $t \twoheadrightarrow_{\mathcal{R}}^{\epsilon} u$ then $t \twoheadrightarrow_{\mathcal{R}} u$

Parallel reduction of a substitution

Definition

Let σ and θ be subtitutions and $\mathcal{X}$ be a set of variable symbols. We write: $\sigma \twoheadrightarrow\!\!\!\!\!/\ \theta$ if $\sigma(x) \twoheadrightarrow\!\!\!\!\!/\ \theta(x)$ for all $x \in \mathcal{X}$

Lemma

Let σ and θ be subtitutions and t be a term. If $\sigma \twoheadrightarrow\!\!\!\!\!/\ \theta$ and $\mathcal{V}ar(t) \subseteq \mathcal{X}$ then $t\sigma \twoheadrightarrow\!\!\!\!\!/\ t\theta$

Thank you for your attention!!!