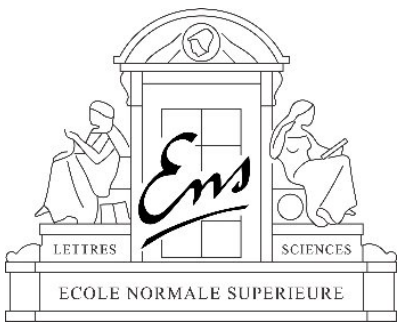
Quadratic Time, Linear Space Algorithms for Gram-Schmidt Orthogonalization and Gaussian Sampling in Structured Lattices

Vadim Lyubashevsky and Thomas Prest



THALES

Key Sizes in Cryptography

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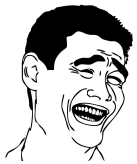
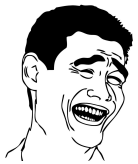


RSA

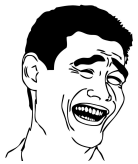
Key Sizes in Cryptography



RSA

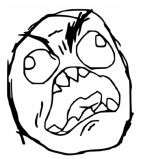
Elliptic Curves

Key Sizes in Cryptography

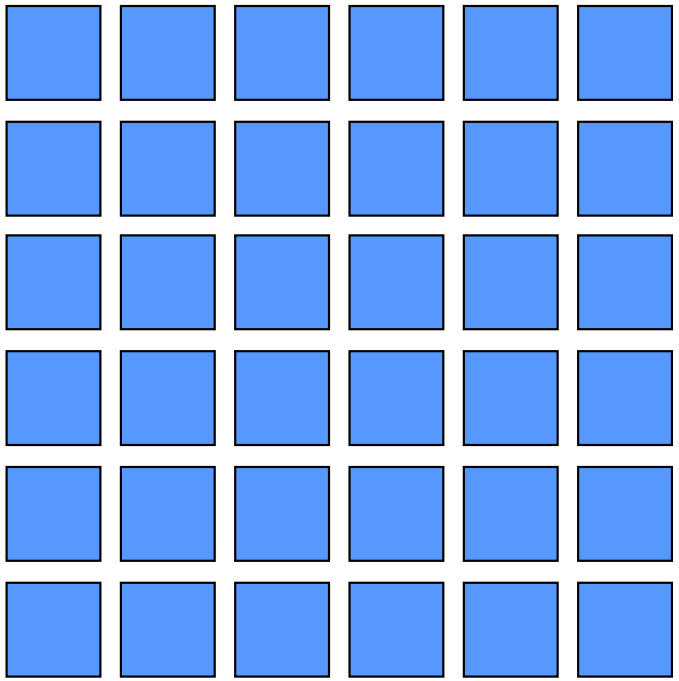


RSA

Elliptic Curves

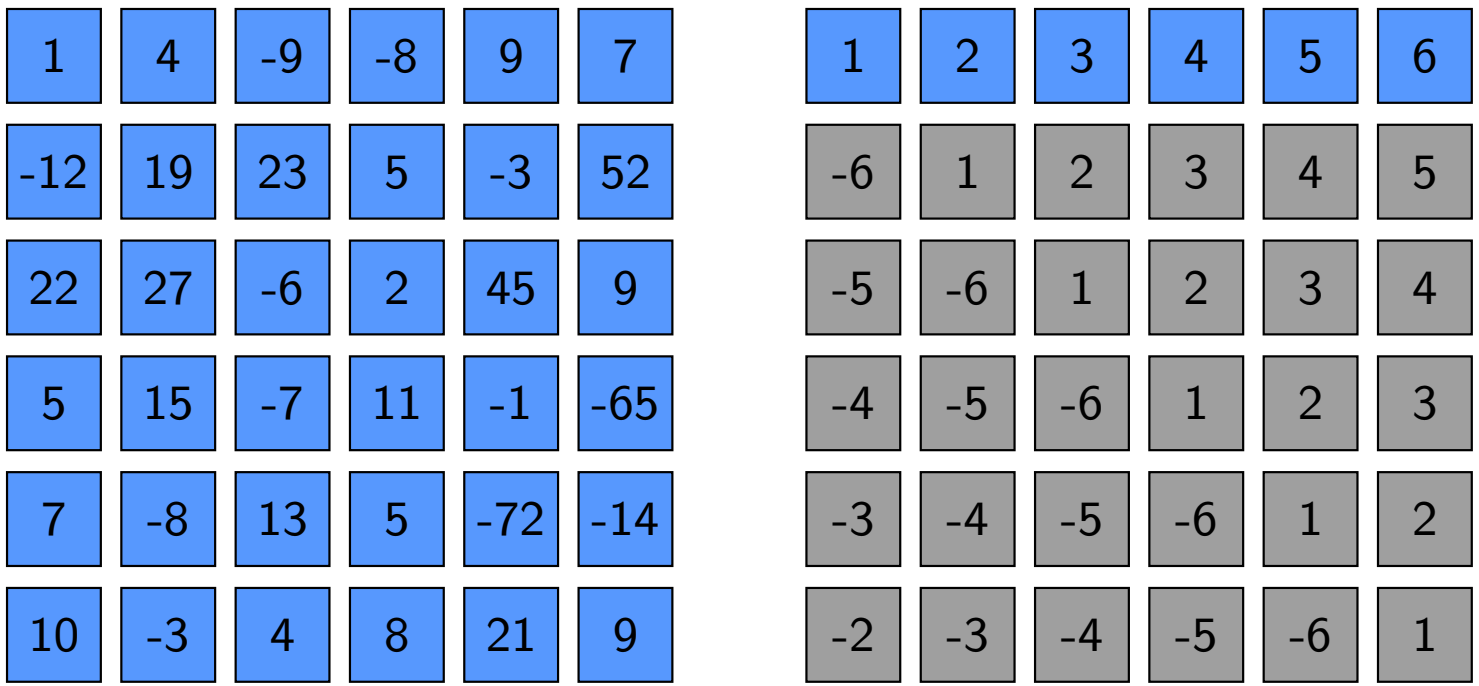


Lattices



Ideal lattices [LM06]

Figure: Size of the keys in lattice-based cryptography



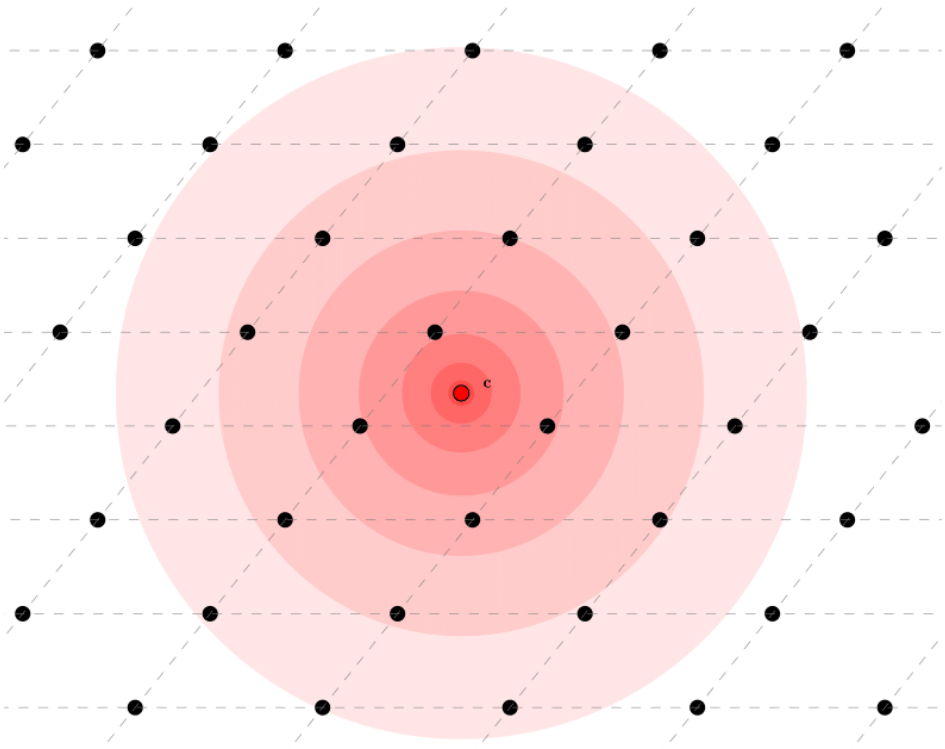
Space requirement goes from $O(n^2)$ to $O(n)$, where n is the dimension.

Gaussian Sampling [Kle00, GPV08]

Very important primitive in lattice-based cryptography:

- Hash-and-sign signatures [GPV08]
- (H)IBE [GPV08, CHKP10, ABB10]
- Standard-model signatures [ABB10]
- Attribute-based encryption [BGG$^+$14]
- ...

Current "best" one: variation [BLP$^+$13] of Gaussian Sampler [GPV08]



The Gaussian Sampler of [Kle00, GPV08]

What is the data required by the Gaussian Sampler for ideal lattices?

Basis $\mathbf{B}$

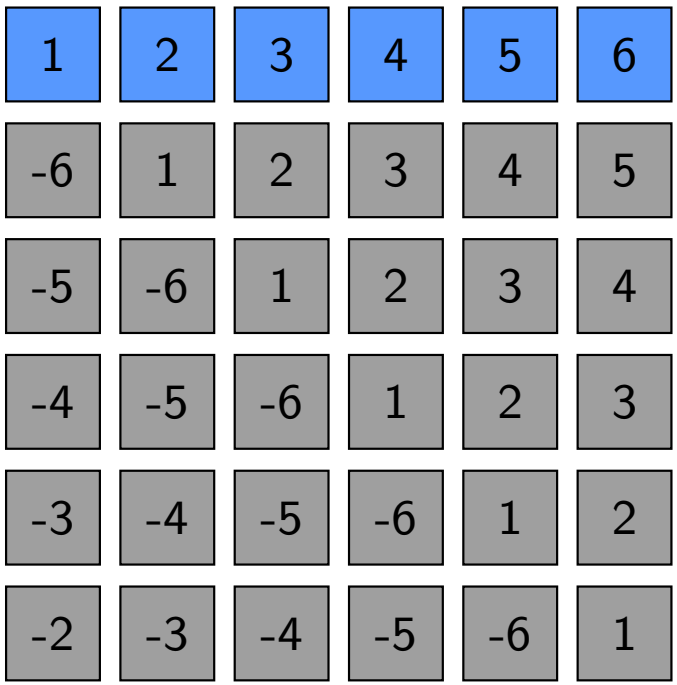
1	2	3	4	5	6
-6	1	2	3	4	5
-5	-6	1	2	3	4
-4	-5	-6	1	2	3
-3	-4	-5	-6	1	2
-2	-3	-4	-5	-6	1

Highly structured!

The Gaussian Sampler of [Kle00, GPV08]

What is the data required by the Gaussian Sampler for ideal lattices?

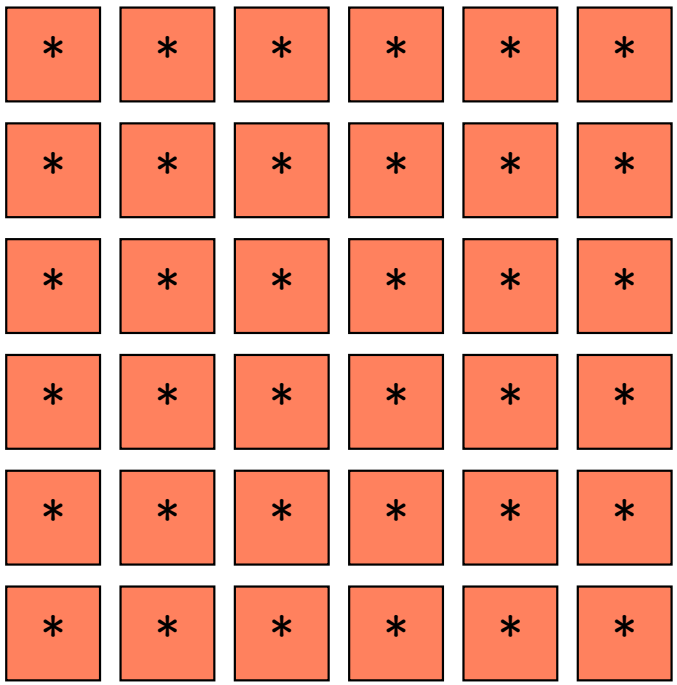
Basis $\mathbf{B}$



Highly structured!

$+$

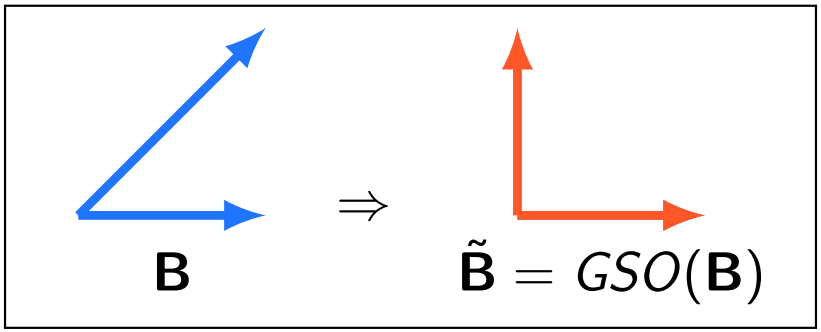
$\tilde{\mathbf{B}} = \texttt{GramSchmidt}(\mathbf{B})$



No space-saving structure!

Gram-Schmidt Orthogonalization (GSO)

What is the Gram-Schmidt Orthogonalization (GSO)?



The GSO for two vectors

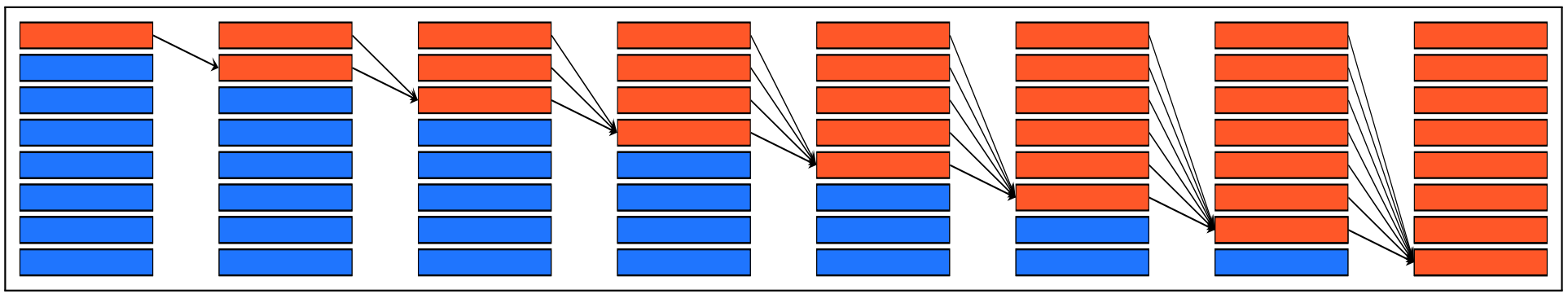


Figure: Computing the GSO $\tilde{\mathbf{B}} = \{\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2, ..., \tilde{\mathbf{b}}_n\}$ for n vectors.
One $\longrightarrow$ arrow takes time $O(n)$. Total time complexity $= O(n^3)$

The Geometry of Ideal Lattices

For the bases we study, we have $\forall k, \mathbf{b}_k = r(\mathbf{b}_{k-1})$:

Where r is an easily computable isometry: $\|r(\mathbf{v})\| = \|\mathbf{v}\|$.

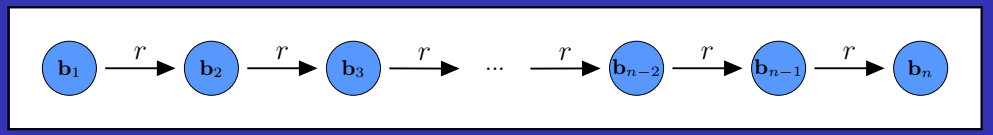
Example:

1	2	3	4
-4	1	2	2
-3	-4	1	2
-2	-3	-4	1

$$f \mod x^4 + 1$$
$$xf \mod x^4 + 1$$
$$x^2 f \mod x^4 + 1$$
$$x^3 f \mod x^4 + 1$$

(It still is the case when replacing $x^4 + 1$ with any cyclotomic polynomial)

Exploiting the Relationship



The idea: compute $\tilde{\mathbf{b}}_{k+1}$ from $\tilde{\mathbf{b}}_k$

$\tilde{\mathbf{b}}_k$ is the reduction of $\mathbf{b}_k$ w.r.t. all the previous vectors

$\Downarrow$ $\qquad$ $\Downarrow$

$$\begin{aligned} & \tilde{\mathbf{b}}_k \perp \mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_{k-1} \\ \Rightarrow\ & r(\tilde{\mathbf{b}}_k) \perp r(\mathbf{b}_1), r(\mathbf{b}_2), ..., r(\mathbf{b}_{k-1}) \\ \Rightarrow\ & r(\tilde{\mathbf{b}}_k) \perp \cancel{\mathbf{b}_1}, \mathbf{b}_2, \mathbf{b}_3, ..., \mathbf{b}_k \end{aligned}$$

$$\begin{aligned} & \mathbf{b}_k - \tilde{\mathbf{b}}_k \in \mathit{Span}(\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_{k-1}) \\ \Rightarrow\ & r(\mathbf{b}_k) - r(\tilde{\mathbf{b}}_k) \in \mathit{Span}(r(\mathbf{b}_1), r(\mathbf{b}_2), ..., r(\mathbf{b}_{k-1})) \\ \Rightarrow\ & \mathbf{b}_{k+1} - r(\tilde{\mathbf{b}}_k) \in \mathit{Span}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, ..., \mathbf{b}_k) \end{aligned}$$

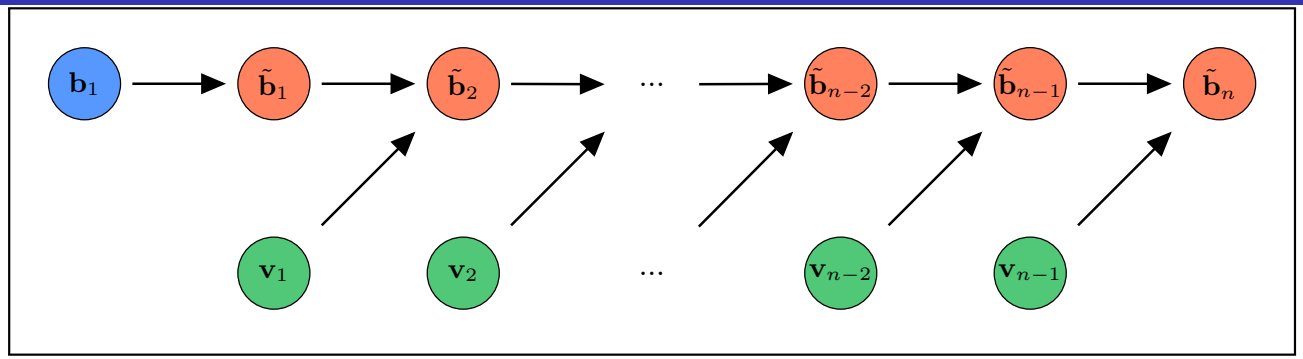
$\Downarrow$ $\qquad$ $\Downarrow$

$r(\tilde{\mathbf{b}}_k)$ is "almost" the reduction of $\mathbf{b}_{k+1}$ w.r.t. all the previous vectors

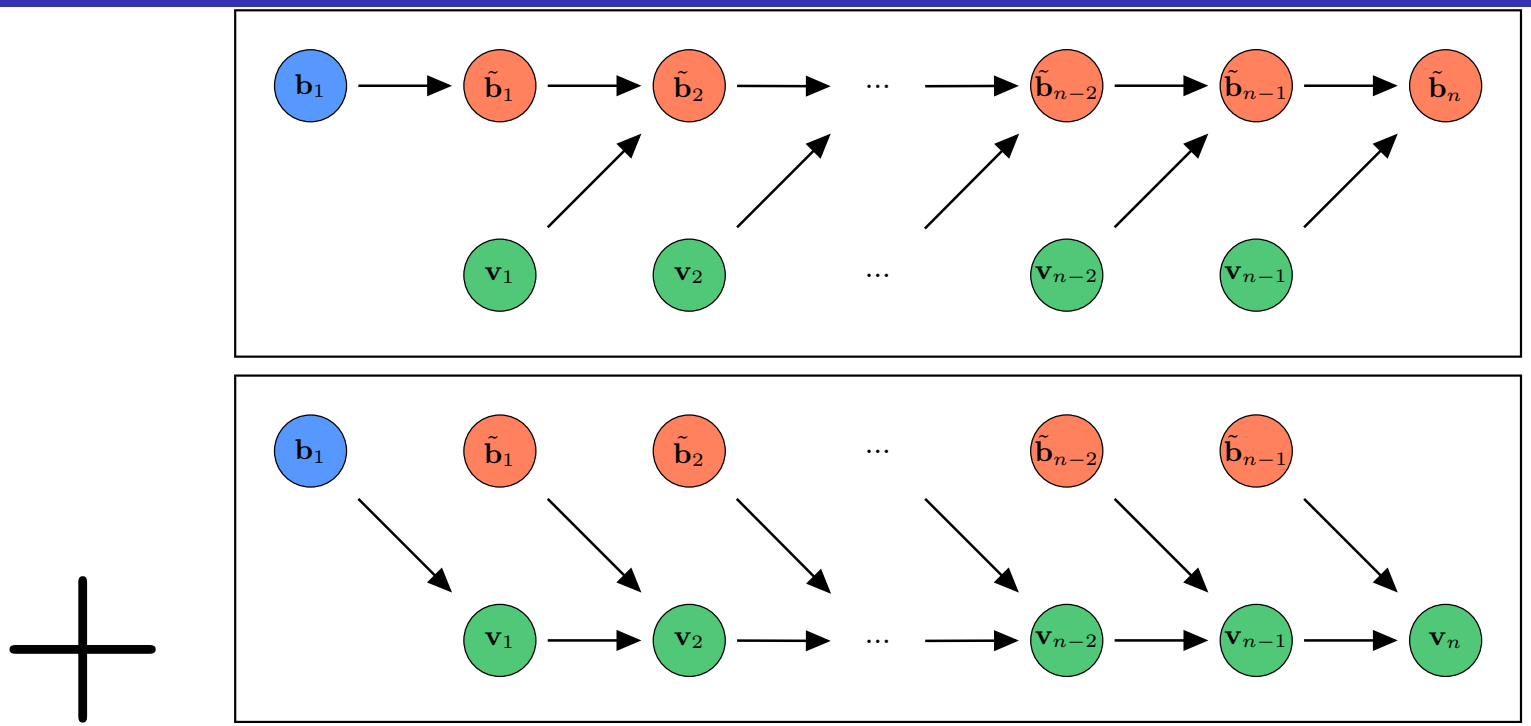
How to turn $r(\tilde{\mathbf{b}}_k)$ into a complete reduction of $\mathbf{b}_{k+1}$?

- Reduce $r(\tilde{\mathbf{b}}_k)$ w.r.t. to $\mathbf{b}_1$? Breaks orthogonality ✗
- Reduce $r(\tilde{\mathbf{b}}_k)$ w.r.t. to $\mathbf{v}_k \triangleq \mathbf{b}_1 - \mathbf{Proj}(\mathbf{b}_1, \mathit{Span}(\mathbf{b}_2, ..., \mathbf{b}_k))$? ✓

GSO for Isometric Bases



GSO for Isometric Bases



GSO for Isometric Bases

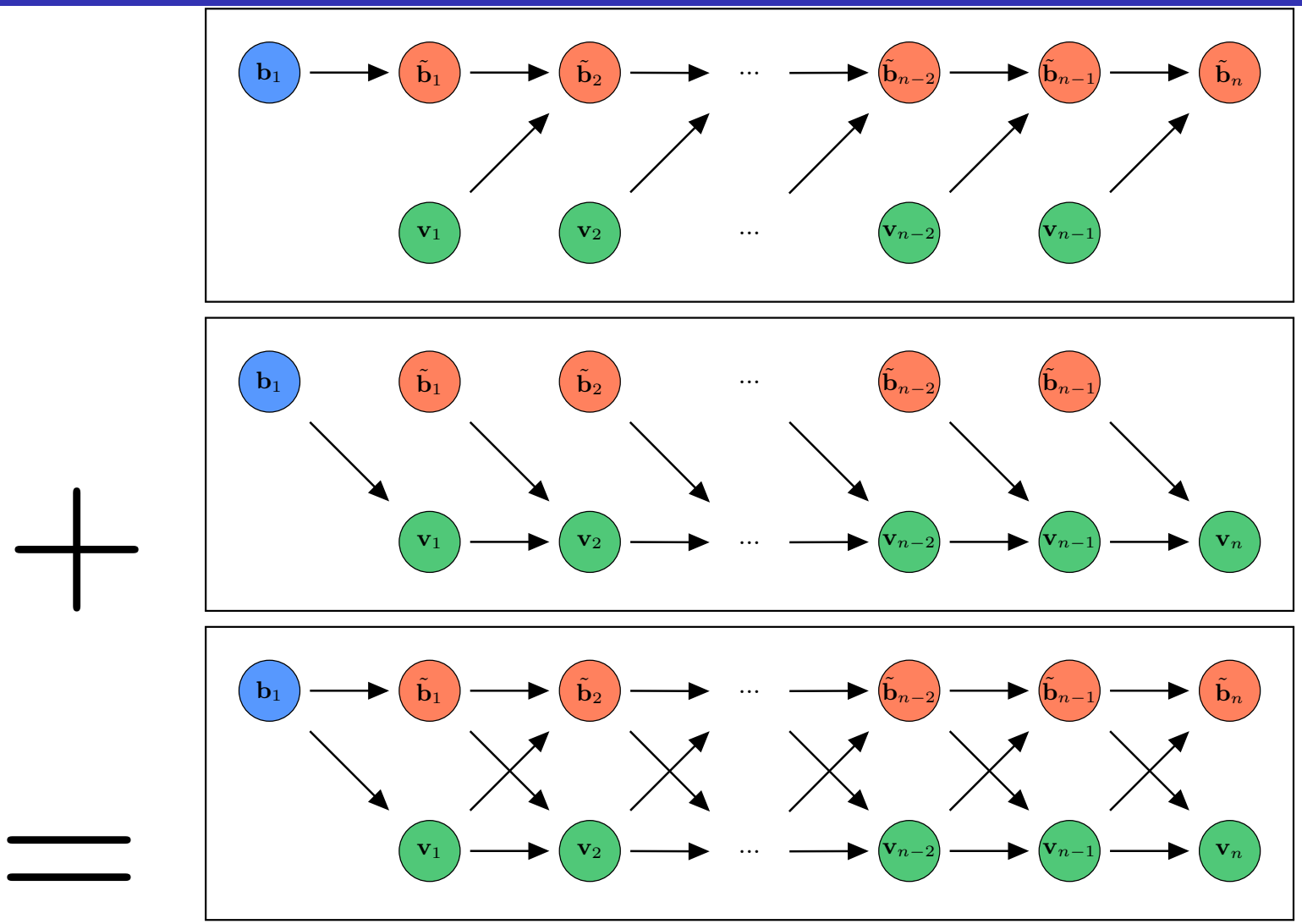
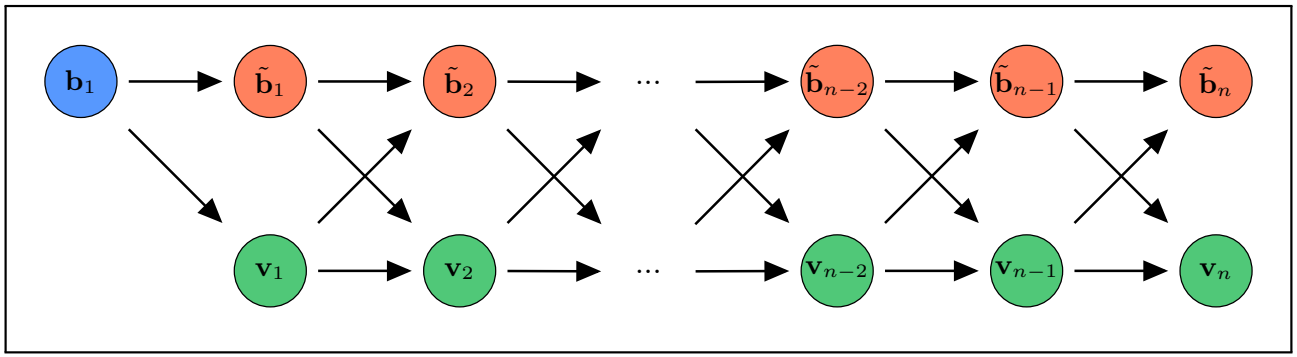


Figure: Computing the Gram-Schmidt Orthogonalization with a double recursion
One $\longrightarrow$ arrow takes time $O(n)$. Total time complexity $= O(n^2)$

GSO for Isometric Bases



Algorithm `Isometric_GSO`$(\mathbf{B})$

Require: Basis $\mathbf{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$

Ensure: GSO basis $\tilde{\mathbf{B}} = \{\tilde{\mathbf{b}}_1, ..., \tilde{\mathbf{b}}_n\}$

$\tilde{\mathbf{b}}_1 \leftarrow \mathbf{b}_1$

$\mathbf{v}_1 \leftarrow \mathbf{b}_1$

for $k = 1, ..., n-1$ **do**

$$\tilde{\mathbf{b}}_{k+1} \leftarrow r(\tilde{\mathbf{b}}_k) - \frac{\langle \mathbf{v}_k, r(\tilde{\mathbf{b}}_k) \rangle}{\|\mathbf{v}_k\|^2} \mathbf{v}_k$$

$$\mathbf{v}_{k+1} \leftarrow \mathbf{v}_k - \frac{\langle \mathbf{v}_k, r(\tilde{\mathbf{b}}_k) \rangle}{\|\tilde{\mathbf{b}}_k\|^2} r(\tilde{\mathbf{b}}_k)$$

end for

return $\tilde{\mathbf{B}} = \{\tilde{\mathbf{b}}_1, ..., \tilde{\mathbf{b}}_n\}$

Algorithm `Classical_GSO`$(\mathbf{B})$

Require: Basis $\mathbf{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$

Ensure: GSO basis $\tilde{\mathbf{B}} = \{\tilde{\mathbf{b}}_1, ..., \tilde{\mathbf{b}}_n\}$

$\tilde{\mathbf{b}}_1 \leftarrow \mathbf{b}_1$

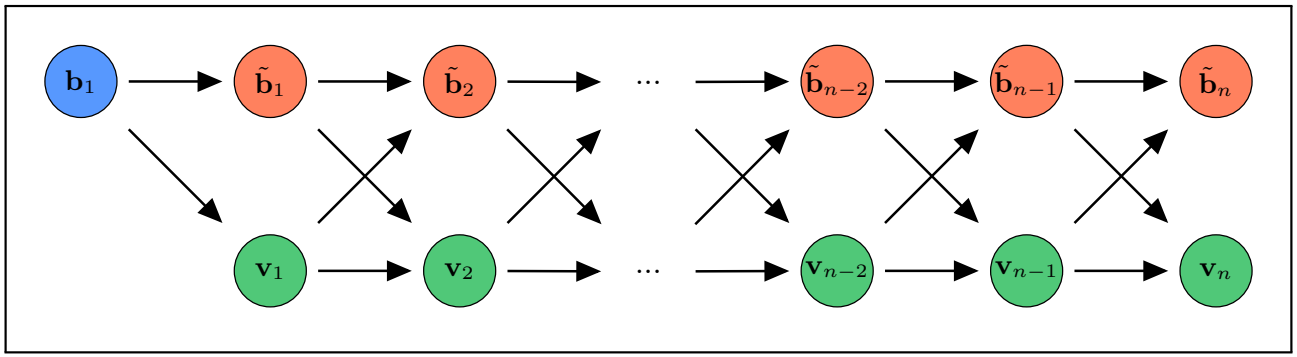
for $k = 1, ..., n-1$ **do**

$$\tilde{\mathbf{b}}_{k+1} \leftarrow r(\mathbf{b}_k) - \sum_{j<k} \frac{\langle \tilde{\mathbf{b}}_j, r(\mathbf{b}_k) \rangle}{\|\tilde{\mathbf{b}}_j\|^2} \tilde{\mathbf{b}}_j$$

end for

return $\tilde{\mathbf{B}} = \{\tilde{\mathbf{b}}_1, ..., \tilde{\mathbf{b}}_n\}$

GSO for Isometric Bases



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Require: Basis $\mathbf{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$

Ensure: GSO basis $\tilde{\mathbf{B}} = \{\tilde{\mathbf{b}}_1, ..., \tilde{\mathbf{b}}_n\}$

$\tilde{\mathbf{b}}_1 \leftarrow \mathbf{b}_1$

$\mathbf{v}_1 \leftarrow \mathbf{b}_1$

for $k = 1, ..., n-1$ **do**

$\tilde{\mathbf{b}}_{k+1} \leftarrow r(\tilde{\mathbf{b}}_k) - \frac{\langle \mathbf{v}_k, r(\tilde{\mathbf{b}}_k) \rangle}{\|\mathbf{v}_k\|^2} \mathbf{v}_k$

$\mathbf{v}_{k+1} \leftarrow \mathbf{v}_k - \frac{\langle \mathbf{v}_k, r(\tilde{\mathbf{b}}_k) \rangle}{\|\tilde{\mathbf{b}}_k\|^2} r(\tilde{\mathbf{b}}_k)$

end for

return $\tilde{\mathbf{B}} = \{\tilde{\mathbf{b}}_1, ..., \tilde{\mathbf{b}}_n\}$

Algorithm `Classical_GSO`$(\mathbf{B})$

Require: Basis $\mathbf{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$

Ensure: GSO basis $\tilde{\mathbf{B}} = \{\tilde{\mathbf{b}}_1, ..., \tilde{\mathbf{b}}_n\}$

$\tilde{\mathbf{b}}_1 \leftarrow \mathbf{b}_1$

for $k = 1, ..., n-1$ **do**

$\tilde{\mathbf{b}}_{k+1} \leftarrow r(\mathbf{b}_k) - \sum_{j<k} \frac{\langle \tilde{\mathbf{b}}_j, r(\mathbf{b}_k) \rangle}{\|\tilde{\mathbf{b}}_j\|^2} \tilde{\mathbf{b}}_j$

end for

return $\tilde{\mathbf{B}} = \{\tilde{\mathbf{b}}_1, ..., \tilde{\mathbf{b}}_n\}$

Further optimizations: We can avoid 2 out 3 scalar products $\Rightarrow$ up to 67% faster

Gram-Schmidt Decomposition (GSD)

Definition (Gram-Schmidt Decomposition (GSD))

Write $\mathbf{B} = \mu \times \tilde{\mathbf{B}}$, where $\tilde{\mathbf{B}}$ is the GSO of $\mathbf{B}$

Applications:

- In cryptology: lattice reduction (LLL, BKZ...), Gaussian Sampling
- Outside cryptology: solving least square problems, linear systems, computing eigenvalues...

Speeding up GSD for Isometric Bases

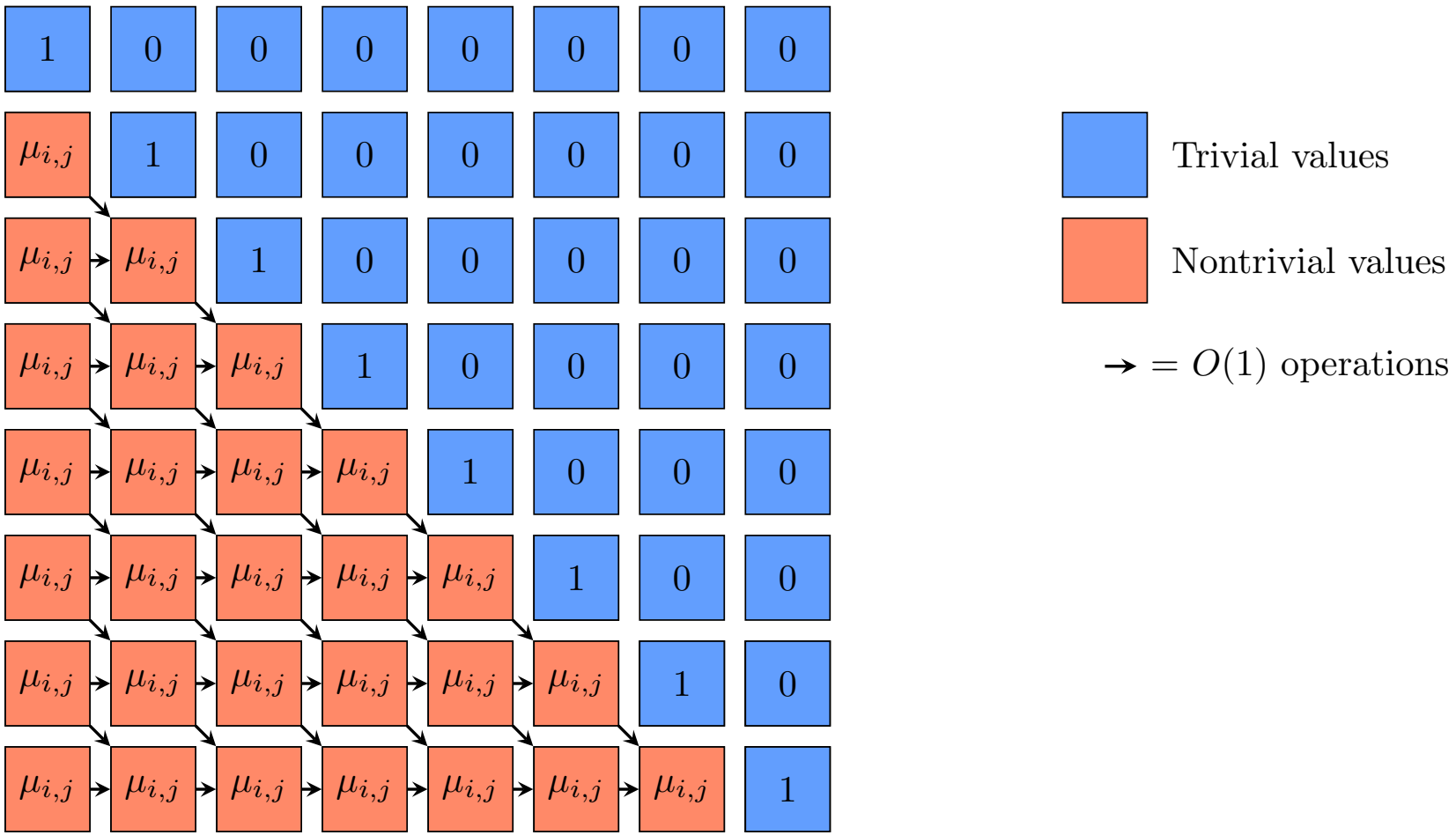
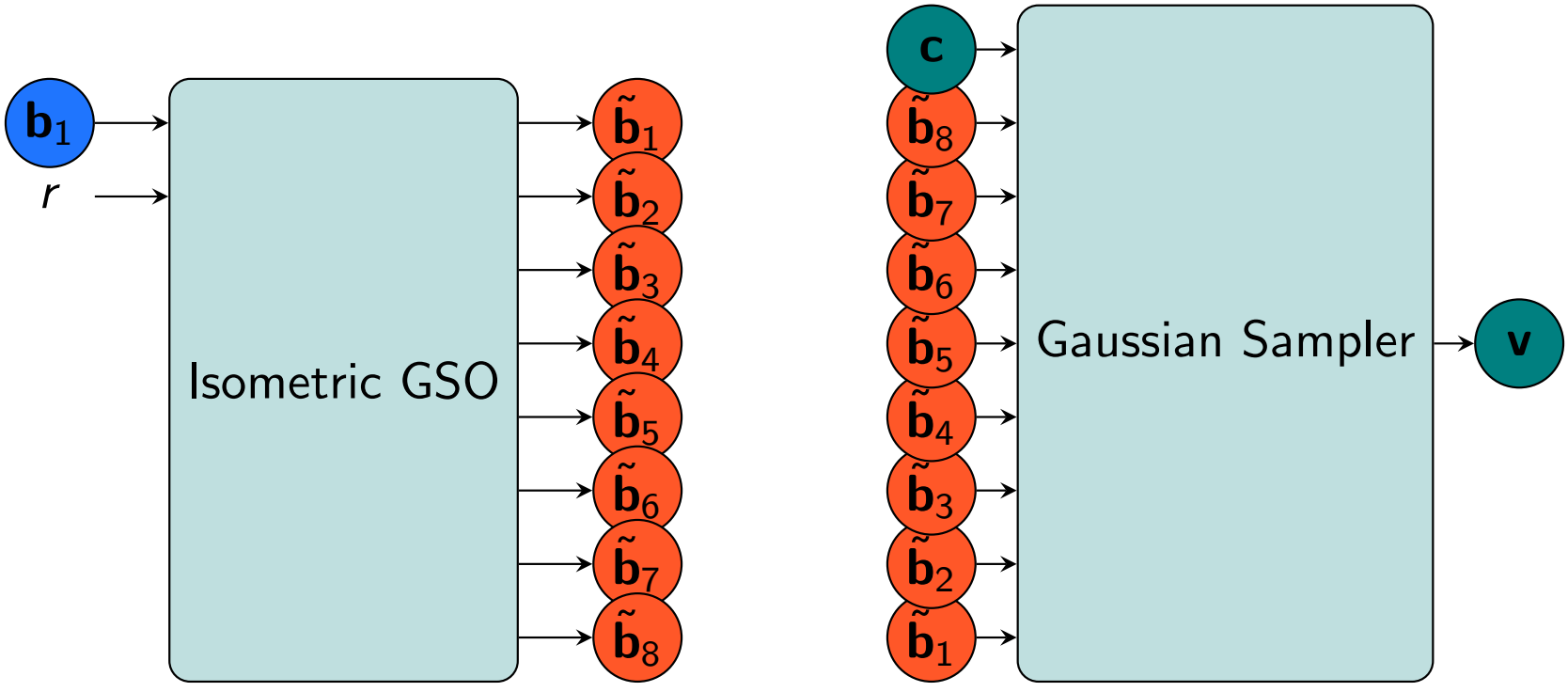


Figure: Fast computation of the matrix μ for an isometric basis.
Overall time and space complexity: $O(n^2)$

GSO and Gaussian Sampler

Now both GSO and the Gaussian Sampler run in time $O(n^2)$.

Can we directly plug together the Isometric GSO and the Gaussian Sampler to both run on-the-fly?



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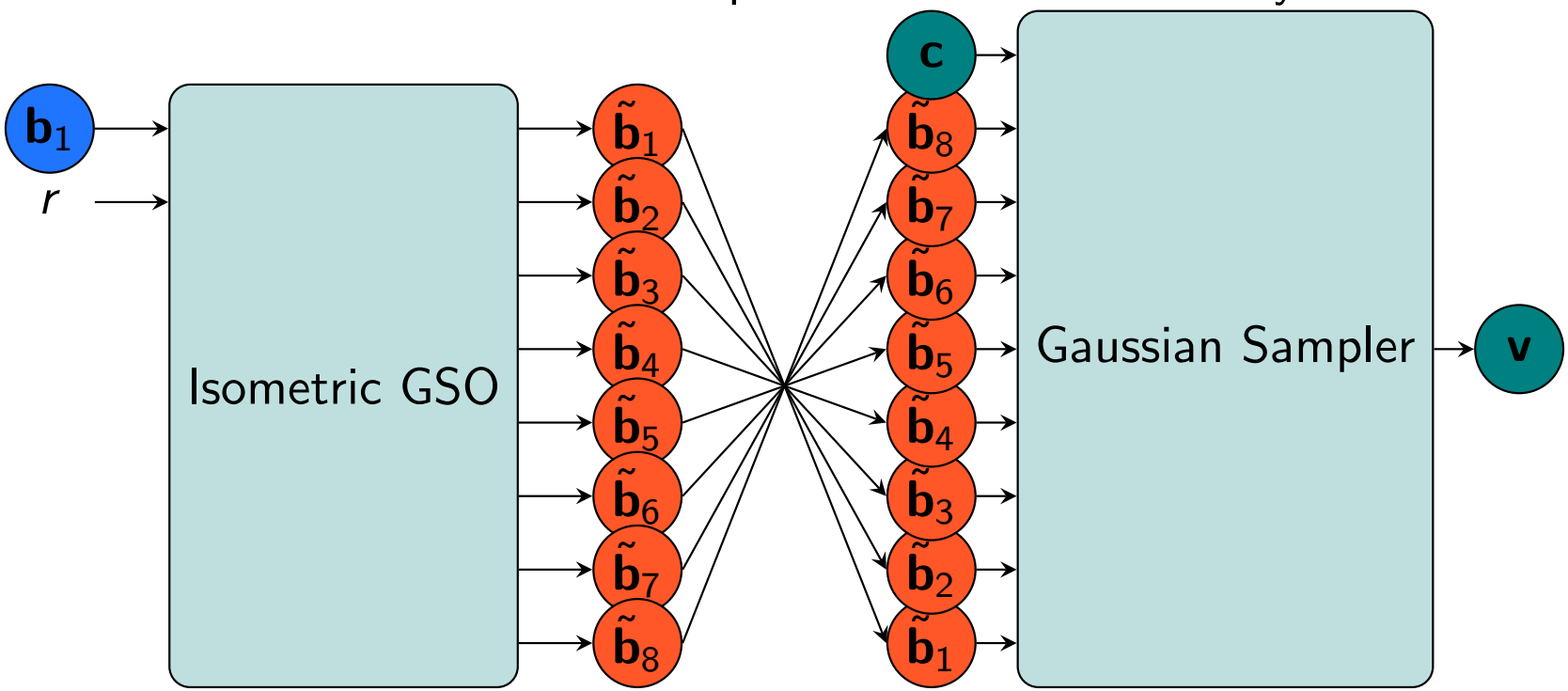


Not really.

Linear-Space Gaussian Sampler

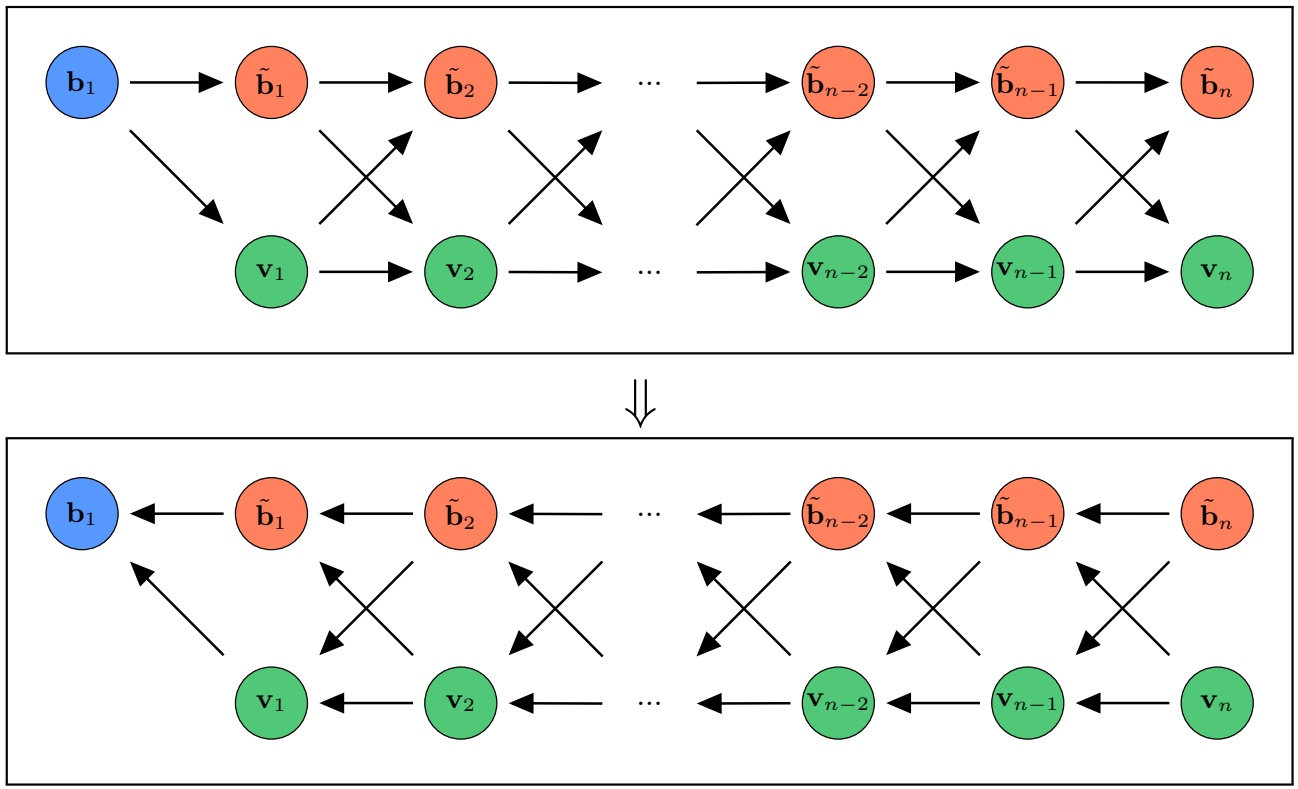
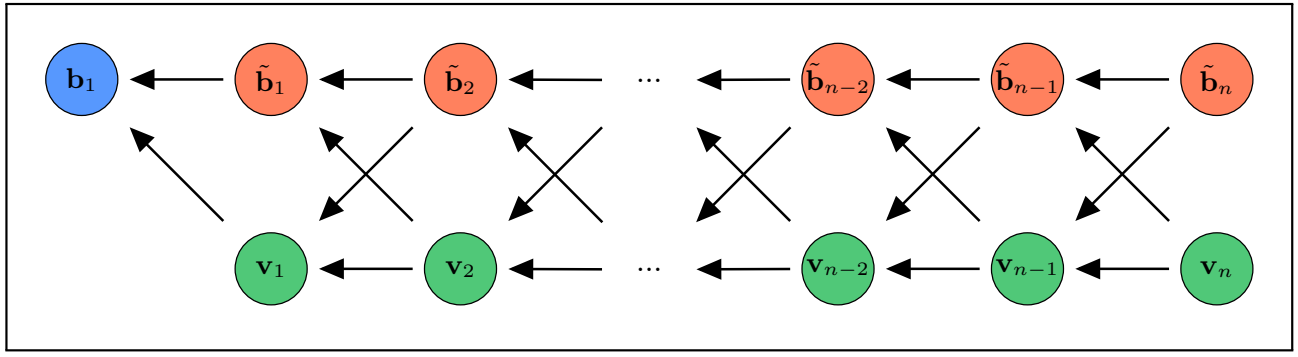


Figure: Reverting Isometric GSO in order to do linear-space Gaussian Sampling
One $\longrightarrow$ arrow takes time $O(n)$ and space $O(1)$ (besides the vectors $\tilde{\mathbf{b}}_k, \mathbf{v}_k$).

A Linear Space Gaussian Sampler



Algorithm

`Classic_Sampler`$(\mathbf{B}, \tilde{\mathbf{B}}, \sigma, \mathbf{c})$

Require: $\mathbf{B}$, $\tilde{\mathbf{B}}$, σ, $\mathbf{c}$

Ensure: $\mathbf{z}$ sampled in $\mathcal{D}_{\Lambda(\mathbf{B}),\sigma,\mathbf{c}}$

$\mathbf{c}_n \leftarrow \mathbf{c}$

for $k \leftarrow n, ..., 1$ **do**

$d_k \leftarrow \langle \mathbf{c}_k, \tilde{\mathbf{b}}_k \rangle / \|\tilde{\mathbf{b}}_k\|^2$

$\sigma_k \leftarrow \sigma / \|\tilde{\mathbf{b}}_k\|$

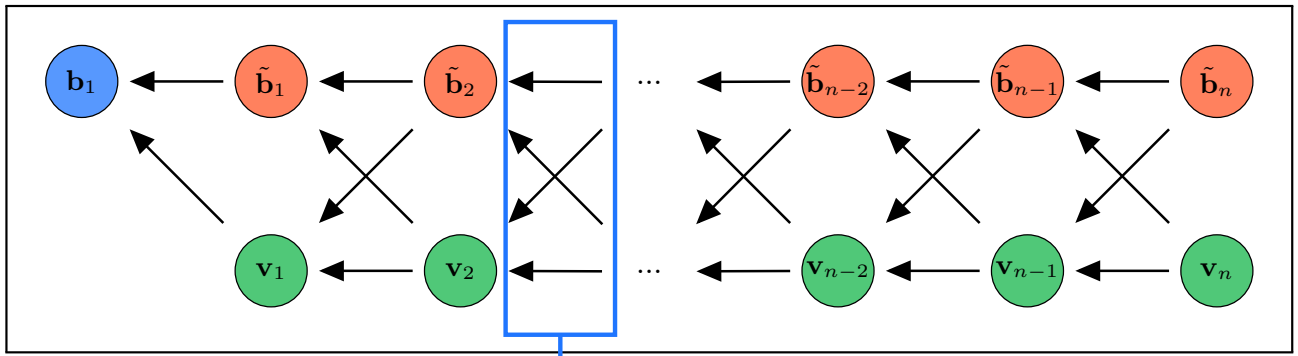
$z_k \leftarrow D_{\mathbb{Z},\sigma_k,d_k}$

$\mathbf{c}_{k-1} \leftarrow \mathbf{c}_k - z_k \mathbf{b}_k$

end for

return $\mathbf{c} - \mathbf{c}_0$

A Linear Space Gaussian Sampler



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for $k \leftarrow n, ..., 1$ **do**
  $d_k \leftarrow \langle \mathbf{c}_k, \tilde{\mathbf{b}}_k \rangle / \|\tilde{\mathbf{b}}_k\|^2$
  $\sigma_k \leftarrow \sigma / \|\tilde{\mathbf{b}}_k\|$
  $z_k \leftarrow D_{\mathbb{Z},\sigma_k,d_k}$
  $\mathbf{c}_{k-1} \leftarrow \mathbf{c}_k - z_k \mathbf{b}_k$
end for
return $\mathbf{c} - \mathbf{c}_0$

Algorithm
`Compact_Sampler`$(\mathbf{B}, \tilde{\mathbf{b}}_n, \mathbf{v}_n, \sigma, \mathbf{c}, (H_k, I_k)_k)$

Require: $\mathbf{B}$, $\tilde{\mathbf{b}}_n$, $\mathbf{v}_n$, σ, $\mathbf{c}$
Ensure: $\mathbf{z}$ sampled in $\mathcal{D}_{\Lambda(\mathbf{B}),\sigma,\mathbf{c}}$

$\mathbf{c}_n \leftarrow \mathbf{c}$
for $k \leftarrow n, ..., 1$ **do**
  $d_k \leftarrow \langle \mathbf{c}_k, \tilde{\mathbf{b}}_k \rangle / \|\tilde{\mathbf{b}}_k\|^2$
  $\sigma_k \leftarrow \sigma / \|\tilde{\mathbf{b}}_k\|$
  $z_k \leftarrow D_{\mathbb{Z},\sigma_k,d_k}$
  $\mathbf{c}_{k-1} \leftarrow \mathbf{c}_k - z_k \mathbf{b}_k$
  $\tilde{\mathbf{b}}_{k-1} = r^{-1}(H_k \tilde{\mathbf{b}}_k + I_k \mathbf{v}_k)$
  $\mathbf{v}_{k-1} = I_k \tilde{\mathbf{b}}_k + H_i \mathbf{v}_k$
end for
return $\mathbf{c} - \mathbf{c}_0$

Timings and Space Requirements

Table: Timings and space requirements of the classic and compact Gaussian Samplers (Classic GS and Compact GS)[1].

Statistical distance from ideal		2^{-128}
Precision needed	Classic GS	163 bits
	Compact GS	193 bits
Running time	Classic GS	170 ms
	Compact GS	521 ms
Space requirement	Classic GS	163 Mb
	Compact GS	0.47 Mb

- Running time: $\times$**3.06**
- Space requirement: /**340**

[1] The implementation was done in `C++` using `GMP`. Timings were performed on an Intel Core i5-3210M laptop with a 2.5GHz CPU and 6GB RAM.

A few open questions:

- Precision analysis of Isometric GSO?
- Better precision analysis of Gaussian Sampler?
- Combine with ideas of [DN12]?
- How does this link to Arnoldi iteration and Lanczos Algorithm?
- Use same principles to improve other algorithms?

Conclusion

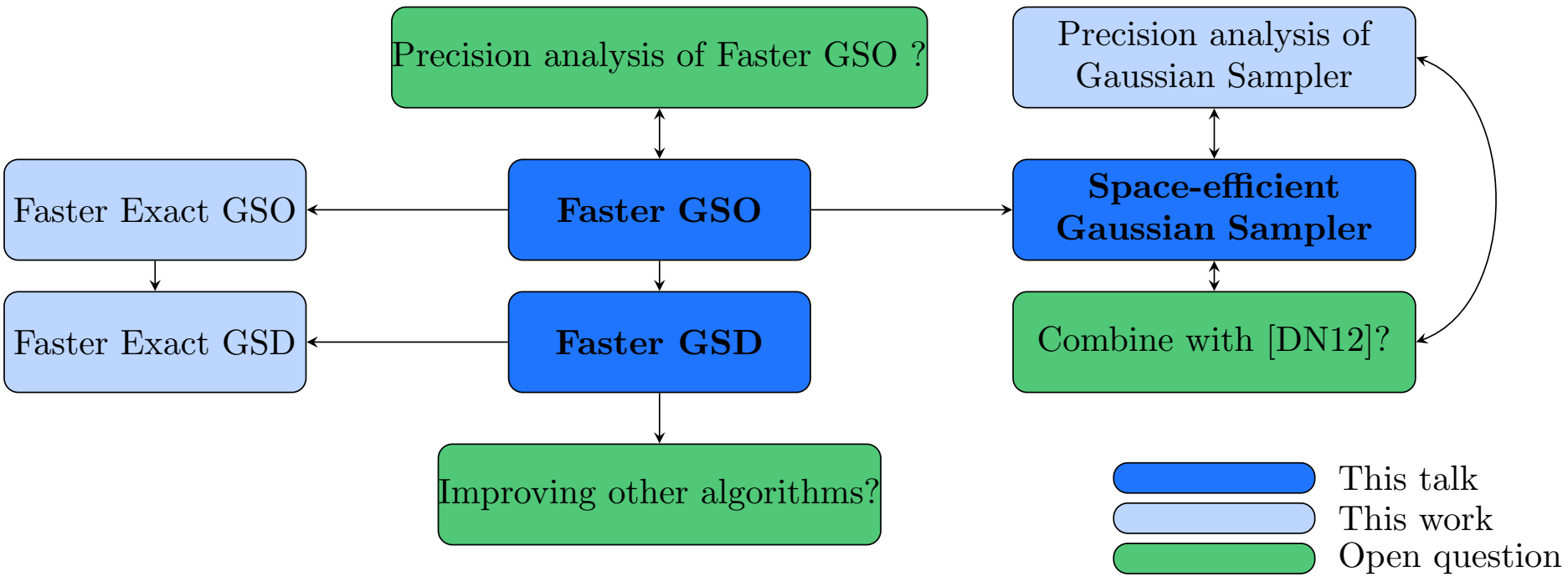

Figure: Present and future work

Conclusion

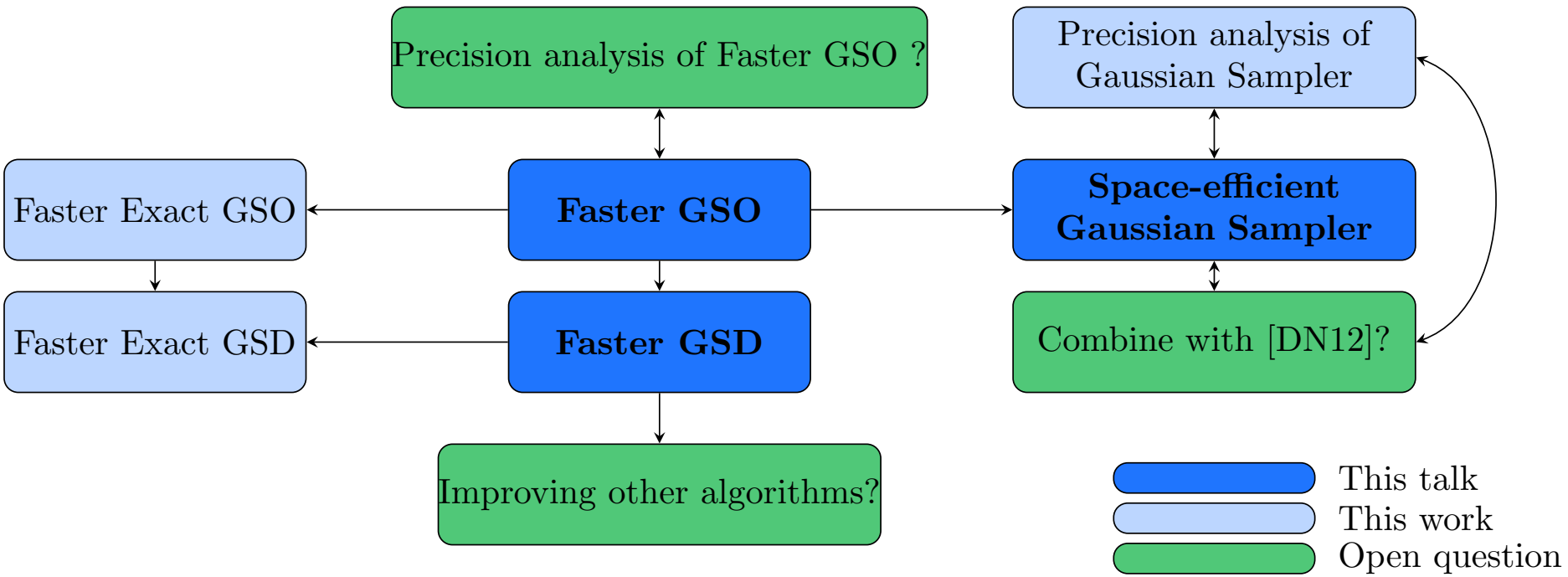

Figure: Present and future work

Thank you!

☞ Shweta Agrawal, Dan Boneh, and Xavier Boyen.
Lattice basis delegation in fixed dimension and shorter-ciphertext hierarchical IBE.
In *CRYPTO'10*, pages 98–115, 2010.

☞ Dan Boneh, Craig Gentry, Sergey Gorbunov, Shai Halevi, Valeria Nikolaenko, Gil Segev, Vinod Vaikuntanathan, and Dhinakaran Vinayagamurthy.
Fully key-homomorphic encryption, arithmetic circuit ABE and compact garbled circuits.
In *EUROCRYPT'14*, pages 533–556, 2014.

☞ Zvika Brakerski, Adeline Langlois, Chris Peikert, Oded Regev, and Damien Stehlé.
Classical hardness of learning with errors.
In *STOC*, pages 575–584, 2013.

☞ David Cash, Dennis Hofheinz, Eike Kiltz, and Chris Peikert.
Bonsai trees, or how to delegate a lattice basis.
In *EUROCRYPT*, pages 523–552, 2010.

☞ Léo Ducas and Phong Q. Nguyen.
Faster gaussian lattice sampling using lazy floating-point arithmetic.
In *ASIACRYPT'2012*, pages 415–432, 2012.

☞ Craig Gentry, Chris Peikert, and Vinod Vaikuntanathan.
Trapdoors for hard lattices and new cryptographic constructions.
In *STOC*, pages 197–206, 2008.

☞ Philip N. Klein.
Finding the closest lattice vector when it's unusually close.
In *SODA'00*, pages 937–941, 2000.

☞ Vadim Lyubashevsky and Daniele Micciancio.
Generalized compact knapsacks are collision resistant.
In *ICALP (2)*, pages 144–155, 2006.