Polymorphic Types with Polynomial Sizes

Size tracking without dependent types

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ARRAY 2023, Orlando, Florida, United States

June 18, 2023

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-- Generic scalar product [valid only if n=p]
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returns (w:'T) where 'T numeric
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-- Monomorphic instantiation
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✔ (first example) ✘ (second example)

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Infer & check sizes together with types

Agenda

1. A sized type system

2. Inference

3. Perspectives and conclusion

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Type safety (ML)

'***Well-typed programs cannot go wrong***'

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Array access: indexes must respect bounds

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Array combinators[2]: predefined valid access schemes

- First orders (FOACs): `reverse`, `transpose`, ...
- Second orders (SOACs): `map`, `fold`, ...

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⋆⋆ under *size consistency* assumptions (`map2`, ...)

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Extensions of the ML type system

Type families		Refinement types
$\tau\ i_1\ \ldots\ i_k$	**Description**	$\{\ x : \tau \mid P(x)\ \}$
Polymorphism	**Genericity**	Dependent types
✗	**Sub-typing**	✓
✓	**Type checking decidability**	?
✓	**Type erasable semantics**	✗

Dimension types (1994) [5]
Sized types (1996) [3]
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ML

$$e \ ::= \ x \mid n \mid e\ e \mid \lambda x.\ e \mid \texttt{let}\ x \texttt{=} e\ \texttt{in}\ e$$ ***Expressions***

ML + Types

$\tau \ ::= \ \alpha \mid \texttt{int} \mid \tau \to \tau$ ***Types***

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ML + Types + Polymorphism

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$\boldsymbol{V} ::= \varepsilon \mid \alpha \cdot \boldsymbol{V}$ ***Generalization***

$\boldsymbol{S} ::= \varepsilon \mid \tau \cdot \boldsymbol{S}$ ***Instantiation***

ML + Types + Polymorphism + Sizes

$\eta \ ::= \ \iota \mid n \mid \eta + \eta \mid \eta * \eta$ ***Sizes***

$\tau \ ::= \ \alpha \mid \texttt{int} \mid \tau \to \tau \mid \texttt{<}\eta\texttt{>} \mid [\eta]$ ***Types***

$e \ ::= \ {}^{S}x \mid n \mid e\ e \mid \lambda x{:}\tau.\ e \mid \texttt{let}_{V}\ x{:}\tau = e\ \texttt{in}\ e \mid e \rhd \tau \mid \texttt{<}\eta\texttt{>}$ ***Expressions***

$V \ ::= \ \varepsilon \mid \alpha \cdot V \mid \iota \cdot V$ ***Generalization***

$S \ ::= \ \varepsilon \mid \tau \cdot S \mid \eta \cdot S$ ***Instantiation***

Integer refinements

- $\texttt{<}\eta\texttt{>}$: singleton type $\{\ x : \texttt{int} \mid x = \eta\ \}$ (*size* η)
- $[\eta]$: interval type $\{\ x : \texttt{int} \mid 0 \leq x < \eta\ \}$ (*index* η)

ML + Types + Polymorphism + Sizes

η ::= ι … **Sizes**

τ ::= α … **Types**

e ::= ***s*** … ***…ressions***

V ::= ε … ***…alization***

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Where are arrays?

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Where are arrays?

For typing purposes, arrays are functions on bounded domains:

$$[\eta]\tau \equiv [\eta] \to \tau$$

Array access: application

Array definition: abstraction

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η ::= ι … ***Sizes***

τ ::= α … ***Types***

e ::= *s* … ***…ressions***

V ::= ε … ***…alization***

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Array definition: abstraction

$$\texttt{let length}_{\iota\cdot\alpha} : [\iota]\alpha \to \texttt{<}\iota\texttt{>} \;=\; \lambda x : [\iota]\alpha.\ \texttt{<}\iota\texttt{>}$$

Integer …

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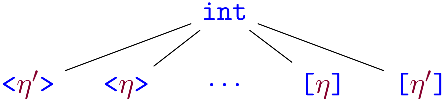
A flat hierarchy of types

Rudimentary sub-typing. No comparison between refinements



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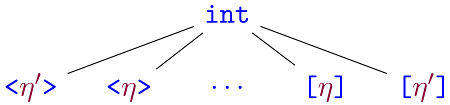

In particular

$$[\eta] \not<: [\eta + 1]$$

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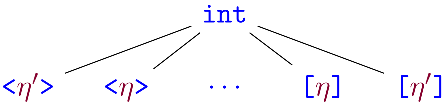

In particular

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No size inequalities

- Checking is decidable and simple (normalized polynomials)
- Inference is practical (although incomplete)

Second Order Array Combinators

$$\texttt{val map} : \forall \iota \cdot \alpha \cdot \beta.\ (\alpha \to \beta) \to \texttt{<}\iota\texttt{>} \to \texttt{[}\iota\texttt{]}\alpha \to \texttt{[}\iota\texttt{]}\beta$$

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```
let repeat = λn:<ι>. λa. (λi:[ι]. a)        [ ∀ι·α. <ι> → α → [ι]α ]

let inc_array = λa.
    map2 (+) <_> a (repeat <_> 1)
```

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Remark 1. Non type-erasable semantics

- Refinements[3] $\qquad (\lambda x : \texttt{[}\eta\texttt{]}.\ e)\ 8 \leadsto e\{8 \triangleright \texttt{[}\eta\texttt{]} / x\}$ $\qquad \Rightarrow$ stops reduction
- Size expressions $\qquad \texttt{<}\eta\texttt{>} \leadsto [\![\eta]\!]_\rho$ $\qquad \Rightarrow$ defines result

[3] Cormac Flanagan. "Hybrid type checking". (2006)

Type erasable semantics?

$\mathtt{let}\ \mathtt{repeat}\ =\ \lambda n{:}\mathtt{<}\iota\mathtt{>}.\ \lambda a.\ (\lambda i : \mathtt{[}\iota\mathtt{]}.\ a)$ $\qquad$ $[\ \forall \iota \cdot \alpha.\ \mathtt{<}\iota\mathtt{>} \rightarrow \alpha \rightarrow \mathtt{[}\iota\mathtt{]}\alpha\ ]$

$\mathtt{let}\ \mathtt{inc\_array}\ =\ \lambda a.$
$\quad \mathtt{map2}\ (+)\ \mathtt{<\_>}\ a\ (\mathtt{repeat}\ \mathtt{<\_>}\ 1)$ $\qquad$ $[\ \forall \iota.\ \mathtt{[}\iota\mathtt{]int} \rightarrow \mathtt{[}\iota\mathtt{]int}\ ]$

$\mathtt{let}\ \mathtt{even}\ =$
$\quad \mathtt{fold}\ (+)\ \mathtt{<\_>}\ 0\ (\mathtt{repeat}\ \mathtt{<\_>}\ 2)$ $\qquad$ (Error: *Unconstrained size*)

Remark 1. Non type-erasable semantics

- Refinements[3] $\quad (\lambda x : \mathtt{[}\eta\mathtt{]}.\ e)\ 8 \ \rightsquigarrow\ e\{8 \triangleright \mathtt{[}\eta\mathtt{]}\ / x\}$ $\quad \Rightarrow$ stops reduction
- Size expressions $\quad \mathtt{<}\eta\mathtt{>} \ \rightsquigarrow\ [\![\eta]\!]_\rho$ $\quad \Rightarrow$ defines result

Observational semantics does not depend on type variables

[3] Cormac Flanagan. "Hybrid type checking". (2006)

First Order Array Combinators

Simple combinators (available in SCADE)

Remark 1. Non type-erasable semantics

- Refinements[3] $(\lambda x : [\eta].\ e)\ 8 \rightsquigarrow e\{8 \triangleright [\eta] / x\}$ $\Rightarrow$ stops reduction
- Size expressions $<\eta> \rightsquigarrow [\![\eta]\!]_\rho$ $\Rightarrow$ defines result

Non type-erasable semantics

First Order Array Combinators

Simple combinators (available in Scade)

$\texttt{val reverse} : \forall \iota \cdot \alpha.\ \texttt{[}\iota\texttt{]}\alpha \rightarrow \texttt{[}\iota\texttt{]}\alpha$

Non type-erasable semantics

First Order Array Combinators

Simple combinators (available in SCADE)

$\texttt{val reverse} : \forall \iota \cdot \alpha.\ [\iota]\alpha \to [\iota]\alpha$

$\texttt{val transpose} : \forall \iota \cdot \kappa \cdot \alpha.\ [\iota][\kappa]\alpha \to [\kappa][\iota]\alpha$

First Order Array Combinators

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$\texttt{val reverse} : \forall \iota \cdot \alpha.\ [\iota]\alpha \rightarrow [\iota]\alpha$

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$\texttt{val concat} : \forall \iota \cdot \kappa \cdot \alpha.\ [\iota]\alpha \rightarrow [\kappa]\alpha \rightarrow [\iota + \kappa]\alpha$

Non type-erasable semantics

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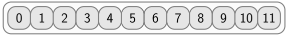
Equivalent type schemes

$\texttt{val concat} : \forall \iota \cdot \kappa \cdot \alpha.\ [\iota]\alpha \to [\kappa - \iota]\alpha \to [\kappa]\alpha$
$\texttt{val concat} : \forall \iota \cdot \kappa \cdot \alpha.\ [\iota - \kappa]\alpha \to [\kappa]\alpha \to [\iota]\alpha$

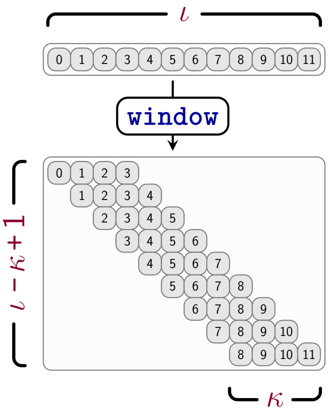
First Order Array Combinators

Non type-erasable semantics



Non type-erasable semantics

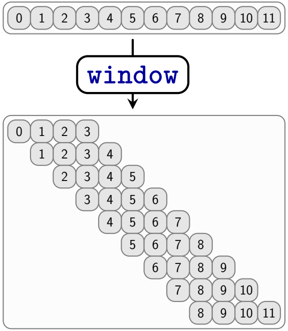
First Order Array Combinators



`val window` : $\forall \iota \cdot \kappa \cdot \alpha.\ \texttt{<}\kappa\texttt{>} \to [\iota]\alpha \to [\iota - \kappa + 1][\kappa]\alpha$

Non type-erasable semantics

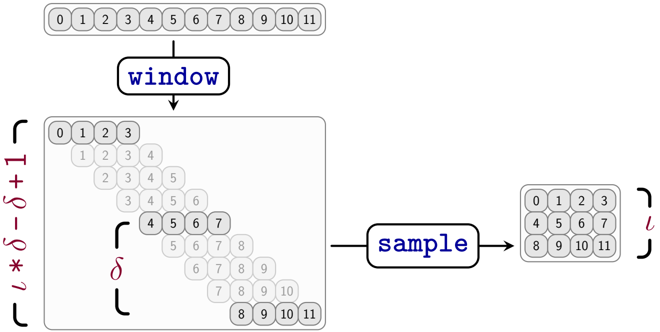
First Order Array Combinators




```
let convolution = λk. λi.
    map (dot <_> k) <_> (window <_> i)      [ ∀ι·κ. [κ]int → [ι + κ − 1]int → [ι]int ]
```

First Order Array Combinators

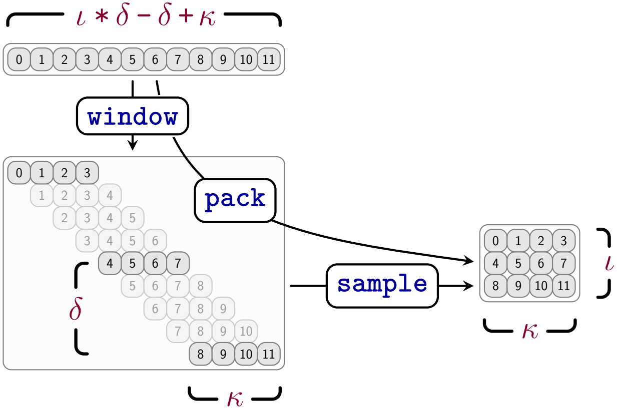
Non type-erasable semantics



`val sample` : $\forall\iota\cdot\delta\cdot\alpha.\ \langle\delta\rangle \to [\iota * \delta - \delta + 1]\alpha \to [\iota]\alpha$

Non type-erasable semantics

First Order Array Combinators

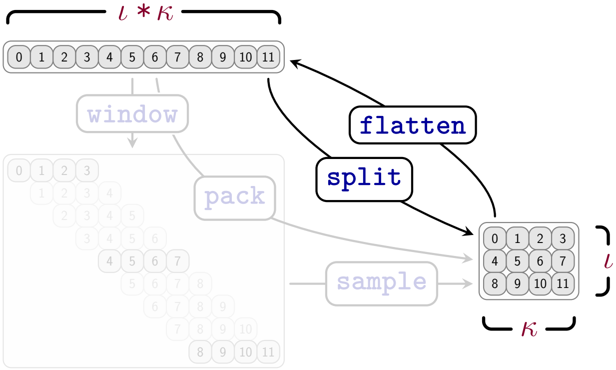



```
let pack = λs. λx.
    sample s (window <_> x)
```

$[\ \forall \iota \cdot \kappa \cdot \delta \cdot \alpha.\ \langle\delta\rangle \rightarrow [\iota * \delta - \delta + \kappa]\alpha \rightarrow [\iota]\,[\kappa]\alpha\ ]$

Non type-erasable semantics

First Order Array Combinators

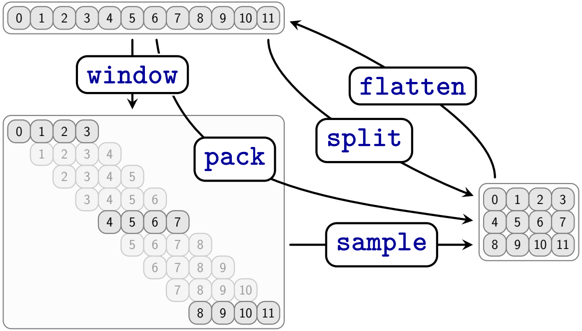


val split : $\forall \iota \cdot \kappa \cdot \alpha.\ \langle\kappa\rangle \to [\iota * \kappa]\alpha \to [\iota][\kappa]\alpha$

val flatten : $\forall \iota \cdot \kappa \cdot \alpha.\ \langle\kappa\rangle \to [\iota][\kappa]\alpha \to [\iota * \kappa]\alpha$

Non type-erasable semantics

First Order Array Combinators



Principal types?

```
let mat : [_][_]int =
    split <_> (λi : [4]. 0)
```

Non type-erasable semantics

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let mat : [_][_]int =
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$$\left\{ \begin{array}{l} \texttt{[1][4]int} \\ \texttt{[2][2]int} \\ \texttt{[4][1]int} \end{array} \right.$$

Principal types?

Non type-erasable semantics

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let mat : [_][_]int =
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```

$\left\{\begin{array}{l}\texttt{[1][4]int}\\ \texttt{[2][2]int}\\ \texttt{[4][1]int}\end{array}\right.$

```
let slope = λf. λi : [_]. λj : [_].
    (f i - f j) / (i - j)
```

Non type-erasable semantics

Principal types?

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let mat : [_][_]int =
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```

$$\left\{ \begin{array}{l} [1]\,[4]\,\texttt{int} \\ [2]\,[2]\,\texttt{int} \\ [4]\,[1]\,\texttt{int} \end{array} \right.$$

```
let slope = λf. λi : [_]. λj : [_].
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$$\left\{ \begin{array}{r} \forall \iota.\ ([\iota] \to \texttt{int}) \to [\iota] \to [\iota] \to \texttt{int} \\ \forall \iota \cdot \kappa.\ (\texttt{int} \to \texttt{int}) \to [\iota] \to [\kappa] \to \texttt{int} \end{array} \right.$$

Non type-erasable semantics

Principal types?

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let mat : [_] [_] int =
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$$\left\{\begin{array}{l}\texttt{[1] [4] int}\\ \texttt{[2] [2] int}\\ \texttt{[4] [1] int}\end{array}\right.$$

```
let slope = λf. λi : [_]. λj : [_].
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```

$$\left\{\begin{array}{r}\forall\iota.\ (\texttt{[}\iota\texttt{]} \to \texttt{int}) \to \texttt{[}\iota\texttt{]} \to \texttt{[}\iota\texttt{]} \to \texttt{int}\\ \forall\iota\cdot\kappa.\ (\texttt{int} \to \texttt{int}) \to \texttt{[}\iota\texttt{]} \to \texttt{[}\kappa\texttt{]} \to \texttt{int}\end{array}\right.$$

Remark 2. No principal types

- Polynomial sizes constraints — $\iota * \kappa - 4 = 0$
- Simple polymorphism — Extra refinements add size constraints

Non type-erasable semantics

Principal types?

$$\texttt{let mat} : [\_]\,[\_]\,\texttt{int} = \texttt{split} \mathrel{<\_>} (\lambda i : [4].\ 0)$$

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$$\texttt{let slope} = \lambda f.\ \lambda i : [\_].\ \lambda j : [\_].\ (f\ i - f\ j)\ /\ (i - j)$$

$$\begin{cases} \forall \iota.\ ([\iota] \to \texttt{int}) \to [\iota] \to [\iota] \to \texttt{int} \\ \forall \iota \cdot \kappa.\ (\texttt{int} \to \texttt{int}) \to [\iota] \to [\kappa] \to \texttt{int} \end{cases}$$

Remark 2. No principal types

- Polynomial sizes constraints — $\iota * \kappa - 4 = 0$
- Simple polymorphism — Extra refinements add size constraints

Constrained polymorphism[4] $\Rightarrow$ delayed size resolution

$$\forall \iota \cdot \kappa \cdot \alpha \mid [\iota], [\kappa] <: \alpha.\ (\alpha \to \texttt{int}) \to [\iota] \to [\kappa] \to \texttt{int}$$

4 John C. Mitchell. “Coercion and Type Inference”. (1984)

Non type-erasable semantics

Agenda

1. A sized type system

2. Inference

3

Remark 2. No principal types

- Polynomial sizes constraints $\iota * \kappa - 4 = 0$
- Simple polymorphism Extra refinements add size constraints

Inference

Non type-erasable semantics

No principal types

Purpose. Fill missing *redundant* information

Inference

Non type-erasable semantics

No principal types

Purpose. Fill missing *redundant* information

t^*

Untyped term

Inference

💡 Non type-erasable semantics

💡 No principal types

Purpose. Fill missing *redundant* information

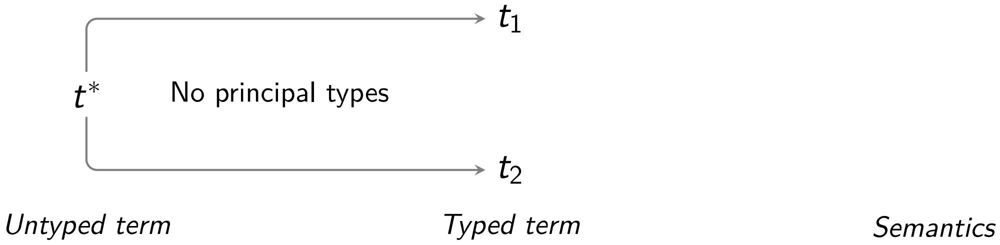
t^*

Untyped term

Typed term

Inference

Purpose. Fill missing *redundant* information



Inference

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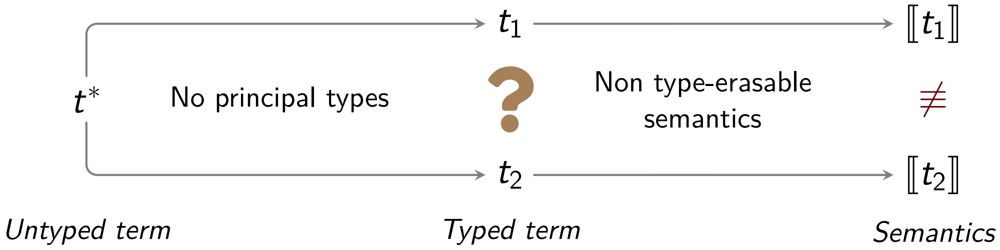

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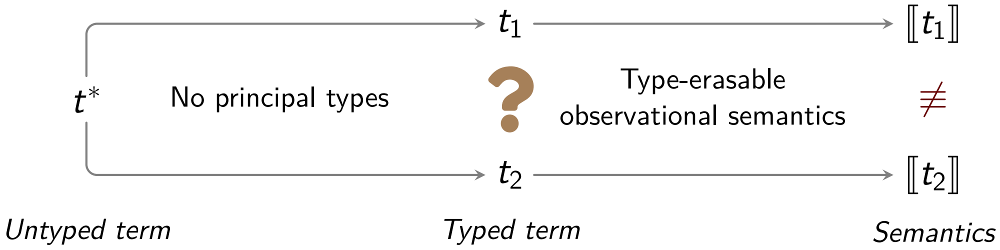
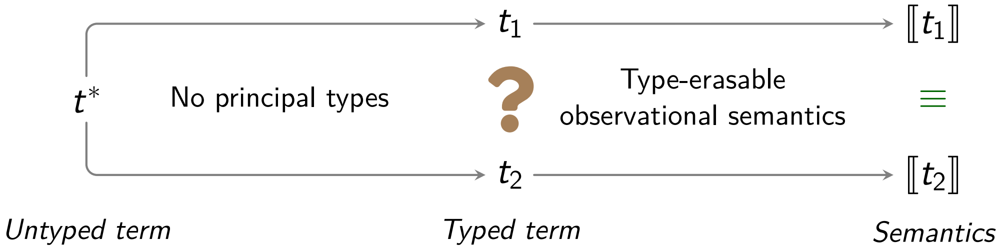


Inference

Purpose. Fill missing *redundant* information, i.e. **without *specializing* terms**



No arbitrary size substitutions

Algorithm

One-pass resolution.

1. Collect sub-typing constraints: $\{ \tau_1 <: \tau_2 \}$
2. At `let` declarations, solve them and generalize size and type variables

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Constraint solving. Build type & size substitution

Type unification	Refinement propagation	Size resolution	Refinement saturation

```
let slope : α =
    λf. λi : [_]. λj : [_]. (f i - f j) / (i - j)
```

Algorithm

One-pass resolution.

1. Collect sub-typing constraints: $\{ \tau_1 <: \tau_2 \}$
2. ...

Cons

First order unification (Hindley-Milner) $\tau = \tau$

Type unification | *Refinement propagation* | *Size resolution* | *Refinement saturation*

```
let slope : (int → int) → int → int → int =
    λf. λi:[_]. λj:[_]. (f i - f j) / (i - j)
```

Algorithm

One-pass resolution.

1. Collect sub-typing constraints: $\{ \tau_1 <: \tau_2 \}$
2. .

Cons

Select needed refinements (regardless sizes) $r <: r$

Type unification

Refinement propagation

Size resolution

Refinement saturation

```
let slope : (α⁺ → α⁻) → [_] → [_] → int =
    λf. λi : [_]. λj : [_]. (f i - f j) / (i - j)
```

Algorithm

One-pass resolution.

1. Collect sub-typing constraints: $\{ \tau_1 <: \tau_2 \}$
2. .

Simplify polynomial system (keeping all solutions) $\eta = \eta$

Cons

Type unification | *Refinement propagation* | *Size resolution* | *Refinement saturation*

$\texttt{let slope} : (\alpha^+ \to \alpha^-) \to [\iota] \to [\kappa] \to \texttt{int} =$

$\quad \lambda f.\ \lambda i : [\_].\ \lambda j : [\_].\ (f\ i\ -\ f\ j)\ /\ (i\ -\ j)$

Algorithm

One-pass resolution.

1. Collect sub-typing constraints: $\{ \tau_1 <: \tau_2 \}$
2.

Cons

Propagate refinements if sizes match $\tau <: \tau$

Type unification | *Refinement propagation* | *Size resolution* | ***Refinement saturation***

$\texttt{let}\ \texttt{slope} : (\texttt{int} \to \texttt{int}) \to [\iota] \to [\kappa] \to \texttt{int}\ =$
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```
let slope : α =
    λf. λi:[6]. λj:[6]. (f i - f j) / (i - j)
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$$\texttt{let slope} : (\texttt{int} \to \texttt{int}) \to \texttt{int} \to \texttt{int} \to \texttt{int} =$$
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```
let slope : (α⁺ → α⁻) → [_] → [_] → int =
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$$\texttt{let slope} : (\alpha^+ \rightarrow \alpha^-) \rightarrow \texttt{[6]} \rightarrow \texttt{[6]} \rightarrow \texttt{int} =$$
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Type unification | *Refinement propagation* | *Size resolution* | *Refinement saturation*

$\texttt{let slope} : (\texttt{[6]} \rightarrow \texttt{int}) \rightarrow \texttt{[6]} \rightarrow \texttt{[6]} \rightarrow \texttt{int} =$
$\quad \lambda f.\, \lambda i : \texttt{[6]}.\, \lambda j : \texttt{[6]}.\, (f\ i\ \texttt{-}\ f\ j)\ \texttt{/}\ (i\ \texttt{-}\ j)$

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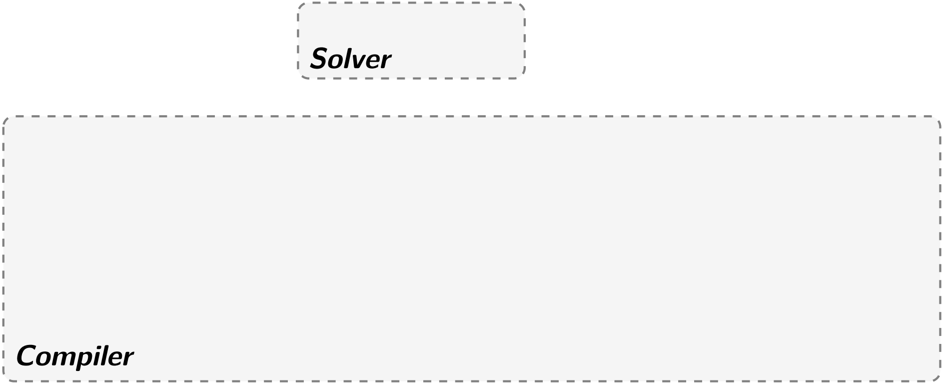
Type unification	*Refinement propagation*	*Size resolution*	*Refinement saturation*

$$\texttt{let}\ \texttt{slope} : (\texttt{[6]} \to \texttt{int}) \to \texttt{[6]} \to \texttt{[6]} \to \texttt{int} \ =$$
$$\lambda f.\ \lambda i : \texttt{[6]}.\ \lambda j : \texttt{[6]}.\ (f\ i\ \texttt{-}\ f\ j)\ \texttt{/}\ (i\ \texttt{-}\ j)$$

Generalization. Variables must...

- ... not appear in any remaining constraints (simple polymorphism)
- ... appear in the type of the declaration (implicit instantiation)

Using solvers



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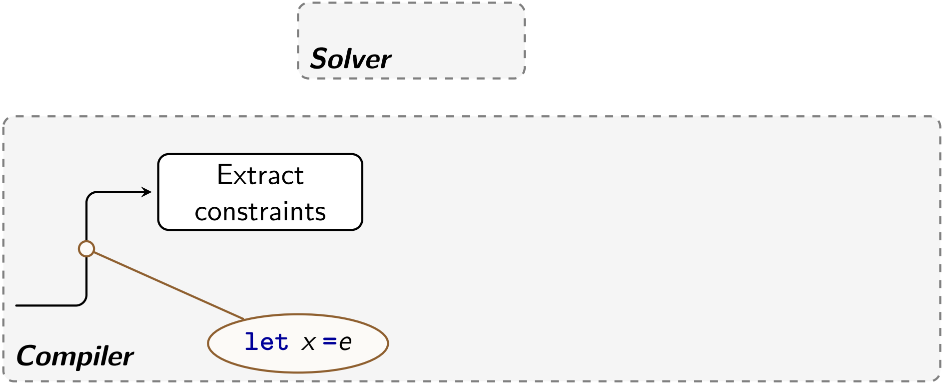

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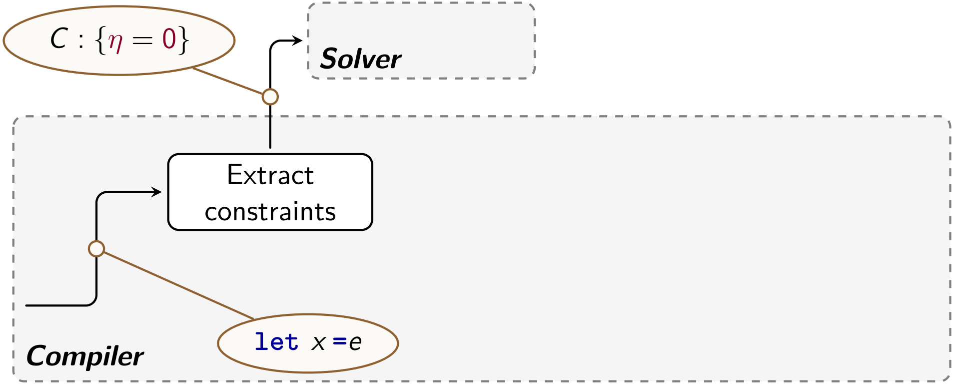

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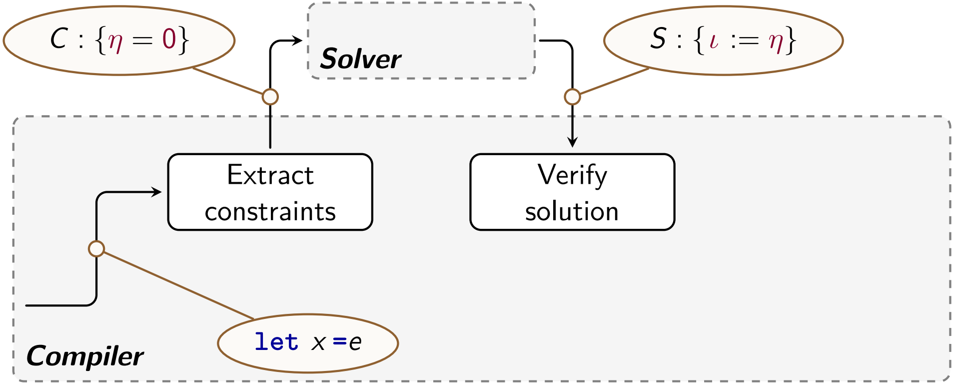

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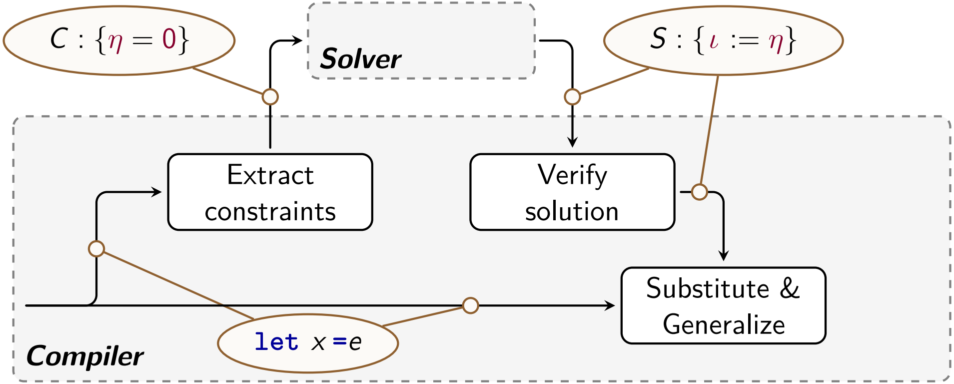

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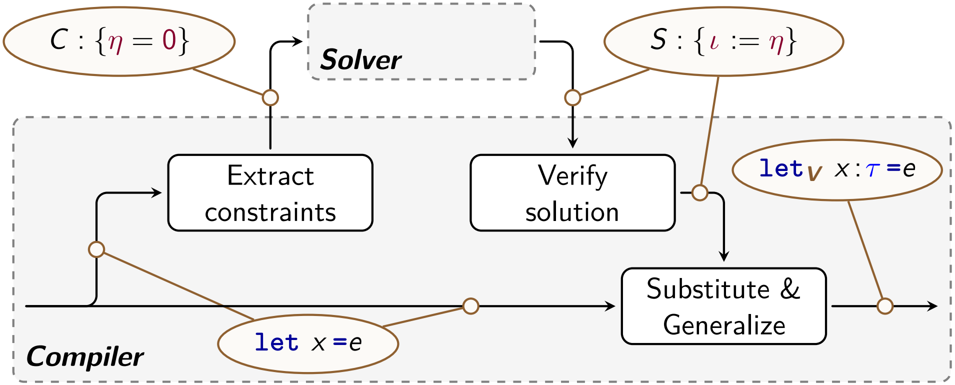

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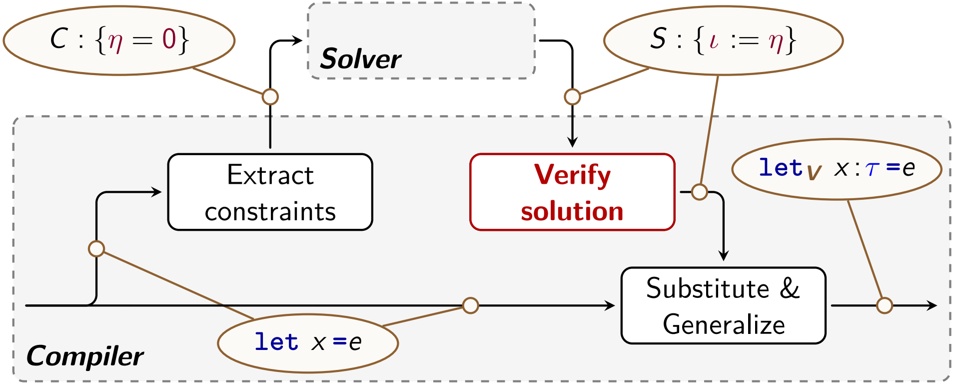
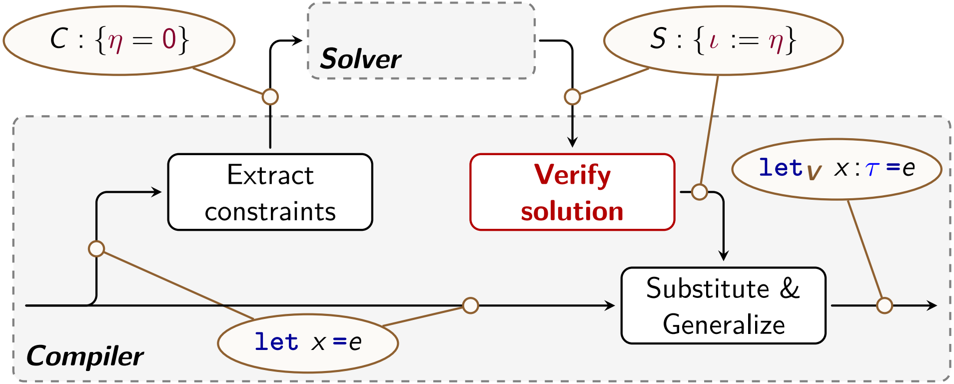


Solution checks

- *Soundness* $\vdash C\{S\}$ S solves C ✔ (Correction)

Using solvers



Solution checks

- *Soundness* $\vdash C\{S\}$ — S solves C ✔ (Correction)
- *Completeness* $C \implies S$ — S is necessary ✘ (Determinism)

Agenda

1. A sized type system

2. Inference

3. Perspectives and conclusion

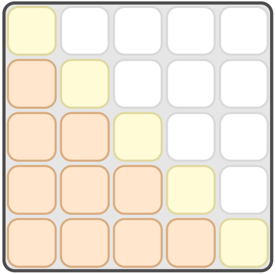
Locally abstract sizes (or types)

Locally abstract sizes (or types)

Local existential sizes.

Construct matrix lines by assembling arrays of varying size

Example: The Cholesky decomposition



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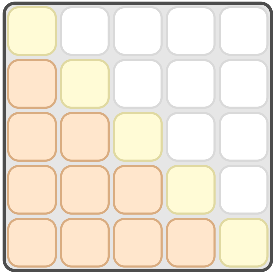

First class polymorphism. [5]
Handle dynamically sized arrays with existential sizes.

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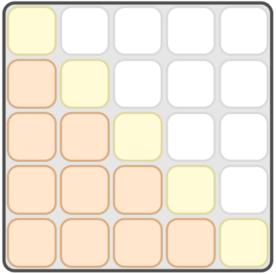

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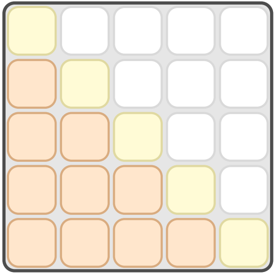

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Size resolution.
Polynomial decomposition

$$\eta_1 * \kappa + \eta_2 = 0 \overset{?}{\iff} \begin{cases} \eta_1 = 0 \\ \eta_2 = 0 \end{cases}$$

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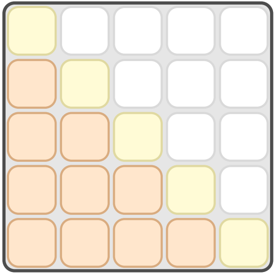

First class polymorphism.[5]
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Call recursively with varying size

Size resolution.
Polynomial decomposition

$$\begin{cases} \eta_1 * \kappa + \eta_2 = 0 \\ \text{if } \eta_2 \text{ cannot capture } \kappa \end{cases} \overset{\checkmark}{\iff} \begin{cases} \eta_1 = 0 \\ \eta_2 = 0 \end{cases}$$

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Take-away

A modest extension of ML

- A restricted form of sub-typing
- Polymorphism is extended to sizes
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Compilation perspectives

- Unguarded dynamic array accesses `mapi`
- Improved memory placement `concat (A,B)`

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A modest extension of ML

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- Polynomial sizes: trade-off between expressiveness, verification and inference

Compilation perspectives

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Thanks for your attention

Size checking?

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$[\ \forall \iota \cdot \alpha.\ [\iota]\alpha \to [-\iota]\alpha\ ]$

Add post-typing constraints. Check...

- ... at run-time (exceptions)
- ... at instantiation, when sizes get constant values
- ... at declaration, with advanced formal tools

Constraint solving

$\texttt{let slope} : \alpha \;=\; \lambda f.\, \lambda i : \texttt{[\_]}.\, \lambda j : \texttt{[\_]}.\, (f\ i\ \texttt{-}\ f\ j)\ \texttt{/}\ (i\ \texttt{-}\ j)$

Constraint solving

`let slope` : $\alpha = \lambda f.\, \lambda i : [\_].\, \lambda j : [\_].\, (f\ i - f\ j)\ /\ (i - j)$

1. Simple type unification $\tau = \tau$

ML simple types (without refinements):

$\tau ::= \alpha \mid \texttt{int} \mid \tau \rightarrow \tau$

First order unification (Hindley-Milner)

Possible failures

- ✗ Incompatible types $\texttt{int} = \alpha \rightarrow \alpha$
- ✗ Recursive substitution $\alpha = \alpha \rightarrow \alpha$

Constraint solving

`let` slope : α $=$ $\lambda f.\ \lambda i : [\_].\ \lambda j : [\_].\ (f\ i\ -\ f\ j)\ /\ (i\ -\ j)$

1. Simple type unification — $\tau = \tau$ α

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1. Sim

α

$; \rightarrow \texttt{int}$

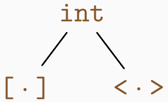
2. Refinement propagation

$r <: r$

Select needed refinements (regardless sizes)

$r ::= x \mid \texttt{int} \mid \texttt{<}\cdot\texttt{>} \mid \texttt{[}\cdot\texttt{]}$

$$
\begin{array}{lcl}
x^- <: \texttt{[}\cdot\texttt{]} & \rightarrow & x^- := \texttt{[}\cdot\texttt{]} \\
\texttt{int} <: x^+ & \rightarrow & x^+ := \texttt{int}
\end{array}
$$



Saturation is deferred after size resolution

Possible failures

✘ Incompatible refinements $\texttt{<}\cdot\texttt{>} <: \texttt{[}\cdot\texttt{]}$

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let slope : α = λf. λi:[_]. λj:[_]. (f i - f j) / (i - j)
```

1. Simp

2. Refi

3. Size resolution

$\eta = \eta$

Simplification of a polynomial system: $\{\ \eta_i = 0\ \}_i$

Isolated variable elimination:

$$\iota - \eta = 0 \ \rightarrow\ \iota := \eta \qquad \text{where } \iota \notin FV(\eta)$$

More advanced strategy using *locally abstract variables*

Possible failures

✗ Incompatible sizes $6 = 0$

α

$; \rightarrow$ `int`

$\rightarrow \alpha_r^+$

$\rightarrow$ `int`

Constraint solving

`let slope` : α = $\lambda f.\ \lambda i : [\_].\ \lambda j : [\_].\ (f\ i\ -\ f\ j)\ /\ (i\ -\ j)$

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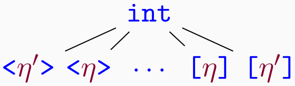
4. Refinement saturation $\tau <: \tau$

Propagate refinements to positive variables, if sizes match

$$\langle\eta\rangle <: \alpha^+ \ \rightarrow\ \alpha^+ := \langle\eta\rangle$$

$$\left\{\begin{array}{l}[\eta_1] <: \alpha^+ \\ [\eta_2] <: \alpha^+\end{array}\right. \ \rightarrow\ \alpha^+ := \texttt{int}$$



Possible failures

1. Sim

2. Refi

3. Size

α

; $\rightarrow$ `int`

$\rightarrow \alpha_r^+$

$\rightarrow$ `int`

$\rightarrow$ `int`

$\rightarrow$ `int`

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Issue 1: Size-dependent Semantics

Non type-erasable semantics

Coerced substitutions

$(\lambda x : [\eta].\ e)\ 8 \rightsquigarrow e\{8 \triangleright [\eta] / x\}$

Sizes may stop the reduction

Size expressions

$<\eta> \rightsquigarrow [\![\eta]\!]_\rho$

Sizes define the reduction result

Observational semantics does not depend on type variables:

$$e\{\overline{\tau}_1/\overline{\alpha}\} \equiv e\{\overline{\tau}_2/\overline{\alpha}\} \qquad \text{where } \overline{\alpha} = FV(e)$$

Intuition: Type variables...

- ... have no *computational content*
- ... may only impact the domain of definition

Issue 2: Principal types?

$$\texttt{let slope} = \lambda f.\ \lambda i : [\_].\ \lambda j : [\_].\ (f\ i\ \texttt{-}\ f\ j)\ /\ (i\ \texttt{-}\ j)$$

1. Simple polymorphism

$$\forall \iota.\ ([\iota] \to \texttt{int}) \to [\iota] \to [\iota] \to \texttt{int}$$

$$\forall \iota \cdot \kappa.\ (\texttt{int} \to \texttt{int}) \to [\iota] \to [\kappa] \to \texttt{int}$$

⇒ no principal types

Constrained polymorphism[6]

$$\forall \iota \cdot \kappa \cdot \alpha \mid [\iota], [\kappa] <: \alpha.\ (\alpha \to \texttt{int}) \to [\iota] \to [\kappa] \to \texttt{int}$$

⇒ delayed size resolution

2. Polynomial size constraints:

$$\iota * \kappa - 4 = 0$$

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