CPO Semantics of Timed Interactive Actor Networks

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Abstract

We give a denotational framework for composing interactive components into closed or open systems and show how to adapt classical domain-theoretic approaches to open systems and to timed systems. For timed systems, instead of the usual metric-space-based approaches, we show that existence and uniqueness of behaviors are ensured by continuity with respect to a simply defined prefix order. Existence and uniqueness of behaviors, however, does not imply that a composition of components yields a useful behavior. The unique behavior could be empty or smaller than expected. We define liveness and show that appropriately defined causality conditions ensure liveness and freedom from Zeno conditions. In our formulation, causality does not require a metric and can embrace a wide variety of models of time.

Key words: semantics, CPOs, posets, interaction, actors, agents, timed systems, process networks, discrete events, actors, dataflow

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1 Introduction

Wegner [45] argues that interaction is more expressive than algorithms. Indeed there is a family of approaches to computing being studied by diverse communities that are distinctly interactive rather than algorithmic. These go under the names of coordination languages, agents, actors, and process networks. They all refactor software into components that co-exist and engage in dialog with one another. Key to the expressiveness of such component interactions is "entanglement" [46], where outputs from a component depend on previous outputs, in dialog with the environment. This is distinct from classical models of computing based on the Turing-Church thesis, which do not model such interaction.

In today's prevailing software engineering technologies, components are object oriented. In practice, the components of object-oriented design interact principally through transfer of control (method calls). The components are passive, and things get done to them, much like physical "objects" from which the name arises. [1] Methods implement classical algorithmic transformations of system state, and compositions of objects simply provide convenient architectural partitioning of such algorithmic transformations. Object-oriented techniques offer little or no help in designing interactive systems.

In this paper, we will use the term "actors" for interactive components. [2] In contrast to objects, actors are concurrent, in charge of their own actions. Their environment (which can include other actors) provides them with data, and they react and provide the environment with additional data. Actors engage in dialog with their environment, whereas objects are passive slaves to it. An immediate consequence is that actor-oriented designs tend to be highly concurrent.

The term "actors" has, of course, been used for models of this type. In the classical actor model of Hewitt and Agha [3,25], components have their own thread of control and interact via message passing. The term "actors" was also used by the dataflow community [21] to refer to chunks of computation that would react to the availability of input data by "firing" and producing output data.

We are using the term "actors" more broadly, inspired by the analogy with the physical world, where actors control their own actions. In fact, the most widespread use of interactive models fitting our notion of actors is not rooted in any of these classical communities, but is rather focused on embedded software (where interaction is intrinsic). For example, the synchronous/reactive languages [9] are "actor-oriented"

[1] So called "active objects" add to the basic object-oriented model threads, and their concurrency is largely orthogonal to their object-oriented nature.

[2] The term "agents" is equally good, but we avoid it because in the mind of many researchers, agents include a notion of mobility, which is orthogonal to interaction and irrelevant to our current discussion.

in our sense. Components react at ticks of a global clock, rather than reacting when other components invoke their methods. In the synchronous language Esterel [11], components exchange data through variables whose values are determined by solving fixed point equations. The Lustre [24] and Signal [10] languages focus more on the flow of data, but are semantically similar. Asynchronous dataflow models based on Kahn process networks [26] are also actor-oriented in our sense, and are used for media intensive embedded signal processing software [20]. Discrete-event (DE) systems are also actor oriented, and are commonly used in hardware design (VHDL and Verilog are DE languages) and in modeling and design of networked systems [14,6]. In DE, components interact via timed events, which carry data and a time stamp, and reactions are chronologically ordered by time stamp.

Wegner argues that interactive models are less amenable to formalism than algorithmic ones [45]. This is debatable, however. While the formalisms may be more complex (this should be expected), they are no less rigorous. Surrounding the actor-oriented approach are a number of semantic formalisms that complement traditional Turing-Church theories of computation by emphasizing interaction of concurrent components rather than sequential transformation of data. These include stream formalisms [26,13,41] and discrete-event formalisms [48,28]. A few such formalisms are rich enough to embrace a significant variety of actor-oriented models of computation, including interaction categories [1], behavioral types [31,5], interaction semantics [43], and the tagged-signal model [30].

As in object-oriented design, *composition* and *abstraction* are two central concepts in actor-oriented design. Actors can be composed to form new actors, which we call **composite actors**. We call actors that are not composite actors **atomic actors**; they may be predefined as language primitives (as is typical, for example, in the synchronous languages), or they may be user-defined, as is typical in coordination languages [4,38,16]. In a compositional formalism, a composite actor is itself an actor.

This paper builds on domain theory [2], developed for the denotational semantics of programming languages [47,42]. But unlike many semantics efforts that focus on system state and transformation of that state, we focus on concurrent interactions, and do not even assume that there is a well-defined notion of "system state."

Our objective is to provide domain-theoretic semantics to timed concurrent systems, which traditionally rely instead on metric space approaches [48,28,7,8,19]. Our approach does not require a metric and embraces a wide variety of models of time (for another treatment that does not rely on metric spaces, see [12]). Wegner argues that temporal properties cannot be modeled with the rigor of mathematical functions [45]. We show here that they can be. In particular, we develop a timed version of the fixed-point semantics for process networks as introduced by Kahn [26]. Our version uses the tagged-signal model [30]. We focus on timed systems because in embedded systems, software engages in dialog with the physical world, and time is an essential part of the semantics of the physical world.

In the next section, we review the tagged signal model and define signals, which encompass the communication histories between actors. In section 3, we define compositions of interacting actors and open systems. We show that familiar fixed-point semantics, which are traditionally applied to closed systems, can be extended to open systems. In section 4, we specialize to timed systems, and show that the same fixed point semantics give conditions for existence and uniqueness of behaviors. In contrast to other authors [12,37,48], we do not require causality for existence and uniqueness of behaviors. Causality, however, is useful for liveness, the timed analog of freedom from deadlock. We define strict causality without the use of a metric, and like Naundorf [37], show that strict causality in a feedback loop is sufficient for liveness. This contrasts with other authors [12,48], who require a stronger form of causality called delta causality or time guardedness. Moreover, we extend Naundorf by including open systems, by giving conditions for freedom from Zeno behaviors, and by showing that the fixed point is constructive. We close with a discussion of Zeno conditions in timed systems.

2 Tagged Signals

The tagged-signal model [30] provides a formal framework for considering and comparing actor-oriented models of computation. It is similar in objectives to the coalgebraic formalism of abstract behavior types in [5], interaction categories [1], and interaction semantics [43]. As with all three of these, the tagged signal model seeks to model a variety of interaction styles between concurrent components.

In the tagged-signal model, each discrete communication between actors is called an **event**. An event is defined to be a pair (t, v), where $t \in T$ is a **tag** and $v \in V$ is a value. A **signal** is a set of events that typically represents the sum total of the communication between two actors along some communication path. For the systems we are interested in, these sets are very likely infinite. Most applications of the tagged-signal model impose structure on the tag set T and study the consequences of that structure. For example, T might represent causality properties, time, or activation orders.

To be sufficiently expressive to capture these properties of concurrent computation, we use a partially ordered set (**poset**). A poset $(T, \leq)$ is a set T and a binary relation $\leq$ that is reflexive ($t \leq t$), antisymmetric ($t_1 \leq t_2$ and $t_2 \leq t_1 \Rightarrow t_1 = t_2$), and transitive ($t_1 \leq t_2$ and $t_2 \leq t_3 \Rightarrow t_1 \leq t_3$).

In this paper, we constrain the tagged signal model of [30] in a subtle but important way. Specifically, we assume that the tag set is a poset $(T, \leq)$, and that a signal is a partial function defined on a down set of T (a similar restriction is made in [37]).

Formally,

Definition 1 (Down Set) *Let $(T, \leq)$ be a poset. A subset T' of T is a down set if for all $t' \in T'$ and $t \in T$, $t \leq t'$ implies $t \in T'$.*

Down sets are also called initial segments in the literature [23].

Definition 2 (Signal) *Let $(T, \leq)$ be a poset of tags, and V a non-empty set of values. A signal $s : T \rightharpoonup V$ is a partial function from T to V such that $\mathrm{dom}(s)$ is a down set of T.*

In the above definition, $\mathrm{dom}(s)$ is defined to be the subset of T on which s is defined.

Let $\mathcal{S}$ denote the set of all signals with tag set T and value set V. That is, this is the set of partial functions with domain T and codomain V that are defined on a down set of T. $\mathcal{S}$ is a poset under the **prefix order**, defined next.

Definition 3 (Prefix Order) *For any $s_1, s_2 \in \mathcal{S}$, s_1 is a prefix of s_2, denoted by $s_1 \sqsubseteq s_2$, if and only if $\mathrm{dom}(s_1) \subseteq \mathrm{dom}(s_2)$, and $s_1(t) = s_2(t),\ \forall t \in \mathrm{dom}(s_1)$.*

That is, a signal s_1 is a prefix of another signal s_2 if the graph of the function s_1 is a subset of the graph of the function s_2. The prefix order on signals is a natural generalization of the prefix order on strings or sequences, and the extension order on partial functions [44].

A **complete partial order** (CPO) $(P, \leq)$ is a poset where P has least element $\perp_P \in P$, and where every directed subset of P has a least upper bound. A subset $D \subseteq P$ is directed if for all $d_1, d_2 \in D$, $\{d_1, d_2\}$ has an upper bound in D.

A signal set with the prefix order $(\mathcal{S}, \sqsubseteq)$ is a CPO [33]. The least element of $\mathcal{S}$ is $s_\perp : \emptyset \to V$, an **empty signal** (it has no events). If a signal is defined for all tags in T, then it is a maximal element of $\mathcal{S}$, and is called a **total signal**.

Note that any pair of signals $\{s_1, s_2\} \subset \mathcal{S}$ has a greatest lower bound $s_1 \wedge s_2 \in \mathcal{S}$. This greatest lower bound is the common prefix, which may be the empty signal if the two signals have nothing in common. In fact, any non-empty subset $S' \subseteq \mathcal{S}$ has a greatest lower bound, which makes $\mathcal{S}$ a **complete semilattice** in addition to a CPO [18].

3 Tagged Systems

Signals, defined in the previous section, represent communication between actors. Actors receive and produce events on **ports**. Thus, a port is associated with a signal, which is a set of events. In this section, we give a declarative definition of actors and show how actors can be composed.

3.1 Behaviors

Consider actor A with a finite set of ports $P_A = \{p_1, p_2, ..., p_n\}$. Assume each port sends or receives signals in a signal set $\mathcal{S}_i$ with tag set T_i and value set V_i. Let $\mathcal{S}_A = \mathcal{S}_1 \cup \mathcal{S}_2 \cup ... \cup \mathcal{S}_n$. A **behavior** of A is a function

$$\sigma \colon P_A \to \mathcal{S}_A,$$

with the constraint that $\sigma(p_i) \in \mathcal{S}_i$. A behavior for a set of ports assigns to each port a signal. The set of all behaviors for ports P_A is written $\Sigma_{P_A} \subseteq (P_A \to \mathcal{S}_A)$.

The prefix order can be generalized to behaviors. Given two behaviors $\sigma_1, \sigma_2 \in \Sigma_{P_A}$, we say $\sigma_1 \sqsubseteq \sigma_2$ if and only if for all $p \in P_A$, $\sigma_1(p) \sqsubseteq \sigma_2(p)$. It is then easy to see that $(\Sigma_{P_A}, \sqsubseteq)$ is a CPO and a complete semilattice.

Other definitions also generalize naturally. For instance, a behavior $\sigma \colon P_A \to \mathcal{S}_A$ is **total** if for all $p \in P_A$, $\sigma(p)$ is total.

3.2 Actors as Sets of Behaviors

An **actor** A with ports P_A is a set of behaviors $A \subseteq \Sigma_{P_A}$. That is, an actor can be viewed as constraints on the signals at its ports. A signal $s \in \mathcal{S}_i$ at port $p_i \in P_A$ is said to **satisfy** an actor A if there is a behavior $\sigma \in A$ such that $s = \sigma(p_i)$.

A **connector** C between ports in set P_C is also a set of behaviors $C \subseteq \Sigma_{P_C}$, but with the constraint that for each behavior $\sigma \in C$, there is a signal $s \in \mathcal{S}_C$ such that

$$\forall\, p \in P_C, \quad \sigma(p) = s.$$

That is, a connector asserts that the signals at a set of ports are identical.

3.3 Composition of Actors

Given two actors, A with ports P_A and B with ports P_B, the **composition behavior** is the intersection, defined as

$$A \wedge B \subseteq ((P_A \cup P_B) \to (\mathcal{S}_A \cup \mathcal{S}_B)),$$

where

$$A \wedge B = \{\sigma \mid \sigma \upharpoonright P_A \in A \text{ and } \sigma \upharpoonright P_B \in B\},$$

where $\sigma \upharpoonright P$ denotes the restriction of σ to the subset P of ports. Note that this is not the intersection of the graphs of the functions. It is larger.

A set of actors (each of which is a set of behaviors) and a set of connectors (each of which is also a set of behaviors) defines a *composite actor*. The composite actor is defined to be the composition behavior of the actors and connectors.

Notice that because all signals in a behavior of a connector must be identical, there is a type check that must be performed on actor composition. Moreover, whereas a classical type system would focus only on the value sets V, our type check has to also check the tag sets T. This means that actors communicating through connectors must have compatible semantics on their ports. For example, if an actor sends a stream, the receiving actor must accept a stream. If an actor sends timed events, the receiving actor must accept timed events.

In many actor-oriented formalisms, ports are either inputs or outputs to an actor but not both. Consider an actor A with ports $P_A = P_i \cup P_o$, where P_i are the input ports, P_o are the output ports, and $P_i \cap P_o = \emptyset$. The actor is said to be **functional** if

$$\forall\, \sigma_1, \sigma_2 \in A, \quad (\sigma_1 \upharpoonright P_i = \sigma_2 \upharpoonright P_i) \Rightarrow (\sigma_1 \upharpoonright P_o = \sigma_2 \upharpoonright P_o).$$

Such an actor can be viewed as a function from input signals to output signals. Specifically, given a functional actor A with input ports P_i and output ports P_o, we can define an **actor function**

$$F_A \colon (P_i \to \mathcal{S}_i) \rightharpoonup (P_o \to \mathcal{S}_o). \tag{1}$$

When it creates no confusion, we make no distinction between the actor (a set of behaviors) and the actor function. If the actor function is total, the actor is said to be **receptive**. A connector, of course, is functional and receptive, where its single input port is assumed to be an output port of an actor, and all other ports are assumed to be input ports of actors.

An actor with no input ports (only output ports) is functional if and only if its behavior set is a singleton set. That is, it has only one behavior. An actor with no output ports (only input ports) is always functional.

A composition of actors and connectors is itself an actor. The input ports of such a composition can include any input port of a component actor that does not share a connection with an output port of a component actor. If the composition has no input ports, it is said to be **closed**. A composition is **determinate** if it is functional. A key question in many actor-oriented formalisms is, given a set of total functional actors and connectors, is the composition functional and total? This translates into the question of existence and uniqueness of behaviors of compositions. It determines whether a composition is determinate and whether it is receptive. Note that

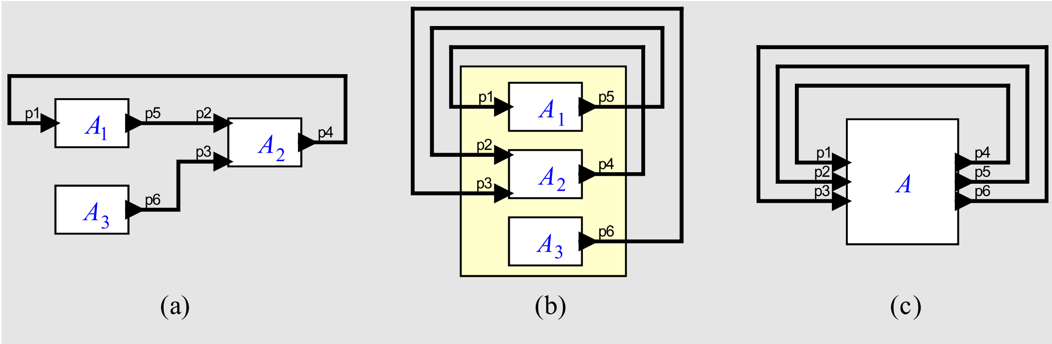

Fig. 1. A composition of three actors and its interpretation as a feedback system.

determinacy here is relative to the tag system. Anything not expressed in the tag system is irrelevant.

3.4 Syntax

Actor-oriented languages can be either self-contained programming languages (e.g. Esterel, Lustre) or coordination languages (e.g. Manifold [39], Simulink, Ptolemy II). In the former case, the "atomic actors" are the language primitives. In the latter case, the "atomic actors" are defined in a host language that is typically not actor oriented (but is often object oriented). Actor-oriented design is amenable to either textual syntaxes, which resemble those of more traditional computer programs, and visual syntaxes, with "boxes" representing actors and "wires" representing connections. The synchronous languages Esterel, Lustre, and Signal, for example, have principally textual syntaxes, although recently visual syntaxes for some of them have started to catch on. Ports and connectors are syntactically represented in these languages by variable names. Using the same variable name in two modules implicitly defines ports for those modules and a connection between those ports. Visual syntaxes are more explicit about this architecture. Examples with visual syntaxes include Simulink, LabVIEW, and Ptolemy II.

A visual syntax for a simple three-actor composition is shown in figure 1(a). Here, the actors are rendered as boxes, the ports as triangles, and the connectors as wires between ports. The ports pointing into the boxes are input ports and the ports pointing out of the boxes are output ports. A textual syntax for the same composition might associate a language primitive or a user-defined module with each of the boxes and a variable name with each of the wires.

3.5 Fixed Point Semantics

The composition in figure 1(a) can be redrawn as shown in figure 1(b), which suggests the abstraction shown in figure 1(c). It is easy to see that any block diagram of this type can be redrawn in this way and abstracted to a single actor with the same number of input and output ports, with each output port connected back to a corresponding input port.

It is also easy to see that if actors A_1, A_2, and A_3 in figure 1(b) are functional and receptive, then the composite actor A in figure 1(c) is functional and receptive. Let F_A denote the actor function for actor A. Assuming the component actors are functional and receptive, it has the form

$$F_A \colon (P_i \to \mathcal{S}_i) \to (P_o \to \mathcal{S}_o).$$

The feedback connections in figure 1(c) are an actor with function

$$C \colon (P_o \to \mathcal{S}_o) \to (P_i \to \mathcal{S}_i)$$

that requires the signals at ports P_i to be the same as the signals at ports P_o. The **feedback system function** is thus a composition of the actor function and the feedback connections,

$$(C \circ F_A) \colon (P_i \to \mathcal{S}_i) \to (P_i \to \mathcal{S}_i). \tag{2}$$

Then the behavior of the feedback composition in figure 1(c) is $\sigma \in (P_i \to \mathcal{S}_i)$ that is a fixed point of $C \circ F_A$. That is,

$$(C \circ F_A)(\sigma) = \sigma.$$

A key question, of course, is whether such a fixed point exists (does the composition have a behavior?) and whether it is unique (is the composition determinate?). This question has been addressed for dataflow process networks using fixed-point theorems on CPOs [26]. For discrete-event models, it is customary to define semantics somewhat differently, by defining a metric space on the set $\mathcal{S}$ of signals [48,28,19], and to make causality requirements on the components. We show here that the causality requirements are unnecessary for existence and uniqueness.

3.6 Open Systems

Note that the composition in figure 1 is closed (it has no inputs). We can generalize the formulation to allow open compositions like the example in figure 2. In such cases, we partition the input ports of the composition actor into two disjoint sets $P_i = P_i' \cup P_i''$, where P_i' is the set of input ports of actor A that are not connected to any output port of A, and $P_i'' = P_i \setminus P_i'$. Thus, in figure 2, $P_i' = \{p_2\}$ and

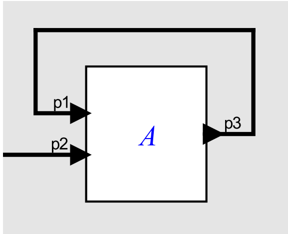

Fig. 2. A composition with feedback and input ports.

$P_i'' = \{p_1\}$. Let P_o denote the output ports of A. In figure 2, $P_o = \{p_3\}$. We assume without loss of generality that all output ports are connected back to input ports in P_i''. Then the actor function can be written

$$F_A' \colon (P_i' \to \mathcal{S}_i') \times (P_i'' \to \mathcal{S}_i'') \rightharpoonup (P_o \to \mathcal{S}_o),$$

where $\mathcal{S}_i'$ and $\mathcal{S}_i''$ are the signal sets of ports p_i' and p_i''. As before, we define a connector for the feedback path, which will be a function of the form

$$C \colon (P_o \to \mathcal{S}_o) \to (P_i'' \to \mathcal{S}_i'').$$

The feedback system function is then

$$(C \circ F_A') \colon (P_i' \to \mathcal{S}_i') \times (P_i'' \to \mathcal{S}_i'') \rightharpoonup (P_i'' \to \mathcal{S}_i''). \tag{3}$$

Given an input behavior $\sigma_i \in (P_i' \to \mathcal{S}_i')$, if the feedback composition of figure 2 has a feedback behavior $\sigma_o \in (P_i'' \to \mathcal{S}_i'')$, then it must be true that

$$(C \circ F_A')(\sigma_i, \sigma_o) = \sigma_o.$$

That is, the behavior on the output ports is a fixed point of a function that is parameterized by the input signal. If this fixed point exists and is unique for all input behaviors, then the **composition function** of figure 2 has the form

$$F \colon (P_i' \to \mathcal{S}_i') \to (P_o \to \mathcal{S}_o). \tag{4}$$

The ability to cleanly model open systems significantly reduces the incentive to model nondeterministic systems (where actors are not functional). If the source of nondeterminism in a system is external events, then a determinate model of an open system is probably better than a nondeterminate model of a closed system. Nonetheless, we conjecture that an adaptation of Plotkin's powerdomain construction would work to provide a generalization to nondeterminate systems [40]. (It needs to be adapted because it is based on transformations of global system state, which is not a well-defined concept in our model.)

We examine next the conditions for existence and uniqueness of the fixed point.

3.7 Existence and Uniqueness of Fixed Points

In this section, we review classical results [18] and apply them to our formulation of actor networks. Let $(D, \sqsubseteq)$ and $(E, \sqsubseteq)$ be CPOs. A function $G\colon D \to E$ is **monotonic** if it is order-preserving,

$$\forall d_1, d_2 \in D,\ \mathrm{d}_1 \sqsubseteq d_2 \implies G(d_1) \sqsubseteq G(d_2).$$

The same function is (Scott) **continuous** if for all directed sets $D' \subseteq D$, $G(D')$ is a directed set and

$$G(\bigvee D') = \bigvee G(D').$$

Here, $G(D')$ is defined in the natural way as $\{G(d) \mid d \in D'\}$, and $\vee X$ denotes the least upper bound of the set X.

It is easy to show that every continuous function is monotonic. A classic fixed point theorem [18] states that if $G\colon D \to D$ for CPO D is continuous, then it has a least fixed point, and that least fixed point is

$$\bigvee\{G^n(\perp_D) \mid n \in \mathbb{N}\}, \tag{5}$$

where $\perp_D$ is the least element of D and $\mathbb{N}$ is the natural numbers.

These results can be immediately applied to closed actor systems like those in figure 1. If each component actor is receptive and continuous, then the system function $C \circ F_A$ of (2) is a continuous function on a CPO. Thus, it has a least fixed point, and that fixed point is given by (5). Following [26], we can define the semantics of the feedback system to be the single unique behavior that is the least fixed point. As we will see, however, this result applies much more broadly than to the process networks of [26].

To handle open systems like those in figure 2, we have a bit more work to do, but again, classic results can be applied almost immediately. As before, let $(D, \sqsubseteq)$ and $(E, \sqsubseteq)$ be CPOs, but now we consider a function of the form

$$G\colon D \times E \to E. \tag{6}$$

For a given $d \in D$, let $G(d)\colon E \to E$ be the function such that

$$\forall\, e \in E, \quad (G(d))(e) = G(d, e).$$

If G is continuous, then for all $d \in D$, $G(d)$ is continuous (lemma 8.10 in [47]). Hence, $G(d)$ has a unique least fixed point, and that fixed point is

$$\bigvee\{(G(d))^n(\perp_E) \mid n \in \mathbb{N}\},$$

where $\perp_E$ is the least element of E.

We recognize immediately that the feedback system function of (3) is a function of form (6). Moreover, if the component actors are receptive and continuous, then the feedback system function will be receptive and continuous, and given an input behavior $\sigma_i \in (P'_i \to \mathcal{S}')$,

$$(C \circ F'_A)(\sigma_i) \colon (P''_i \to \mathcal{S}''_i) \to (P''_i \to \mathcal{S}''_i)$$

is continuous and hence has a least fixed point. We take that least fixed point to be the semantics of the system. Thus, for any input behavior σ_i, the feedback composition has a unique semantics, and that semantics is a function of the form (4). We now show that that function is receptive and continuous.

Since for any input behavior the system in figure 2 has a unique semantics, the composition function F of (4) is well defined. More interestingly, we can show that if each of the component actors is receptive and continuous, then the composition function F is receptive and continuous. This follows first from the (trivial) observation that F'_A, C, and $(C \circ F'_A)$ are receptive and continuous, and second from the following theorem.

Theorem 4 *Let $(D, \sqsubseteq), (E, \sqsubseteq)$ be CPOs, and let $G \colon D \times E \to E$ be a continuous function. Define a function $F \colon D \to E$ such that*

$$\forall\, d \in D, \quad F(d) = \bigvee \{(G(d))^n(\bot_E) \mid n \in \mathbb{N}\}.$$

That is, $F(d)$ yields the least fixed point of the function $G(d) \colon E \to E$ (which exists and is unique). F is continuous.

Proof. Let $[E \to E]$ be the set of all continuous functions from E to E. We can define a partial order on this set by $\forall\, p, q \in [E \to E]$,

$$p \sqsubseteq q \iff \forall\, y \in E,\ p(y) \sqsubseteq q(y).$$

With this partial order, $[E \to E]$ is a CPO. For any directed set $D' \subseteq D$, $\{G(d) \mid d \in D'\} \subseteq [E \to E]$ is a directed set, and

$$\bigvee \{G(d) \mid d \in D'\} = G(\bigvee D'),$$

so the function $G \colon D \to [E \to E]$ is continuous. Let $\mathrm{fix} \colon [E \to E] \to E$ denote the function that yields the unique least fixed point of any continuous function in $[E \to E]$. By Theorem 2.1.19 in [2], fix is continuous. Note that

$$F = \mathrm{fix} \circ G.$$

Since this is the composition of two continuous functions, F is continuous. □

4 Timed Interactive Networks

Our framework so far can easily subsume some classical results. For example, if the tag set for all signals is $T = \mathbb{N}$, the natural numbers, then our networks are Kahn process networks [26]. The constraint that signals be defined on a down set of T is natural in this case. However, our framework is more general, and in this paper, we focus on its use for timed interactive networks.

4.1 Models of Time

Our framework admits several models of time. In all cases, the tag set T will be totally ordered. Perhaps the most natural choice, where $T = \mathbb{R}_+$, the non-negative reals, reflects a Newtonian physical view of time. The fact that we include only the non-negative reals implies that our timed interactive networks have a starting point.

A more interesting model of time is super dense time (SDT) [35], where $T = \mathbb{R}_+ \times \mathbb{N}$ with lexical ordering,

$$(r_1, n_1) \leq (r_2, n_2) \iff r_1 < r_2, \text{ or } r_1 = r_2 \text{ and } n_1 \leq n_2 \,. \tag{7}$$

This is a total order. SDT can be similarly defined as $T = I \times \mathbb{N}$, where I is any interval of real numbers. SDT has been used in studying the semantics of hybrid systems [27,32,34]. A subset $T = \mathbb{N} \times \mathbb{N}$, is used as the model of time in some hardware description languages (notably VHDL). SDT is in a sense “strictly richer” than $\mathbb{R}_+$ as a model of time, in that one can show that there is no order-embedding of $T = \mathbb{R}_+ \times \mathbb{N}$ in $\mathbb{R}_+$.

We make few constraints on the value sets, but for most models, it is useful to assume that every value set V contains a special element $\varepsilon \in V$ that represents absence of a value. Without this choice, only signals defined on a connected interval of $\mathbb{R}_+$ including 0 would meet our requirement that signals be defined on a down set. This would unnecessarily constrain us to continuous-time signals.

4.2 Defining Signals

For convenience in giving examples, we will give signals as a tuple, $(\mathrm{dom}(s), E)$ where $\mathrm{dom}(s)$ is the domain of the signal (a down set of T), and E is the set of events that are not absent,

$$E = \{(t, s(t)) \,|\, t \in \mathrm{dom}(s),\ s(t) \neq \varepsilon\}.$$

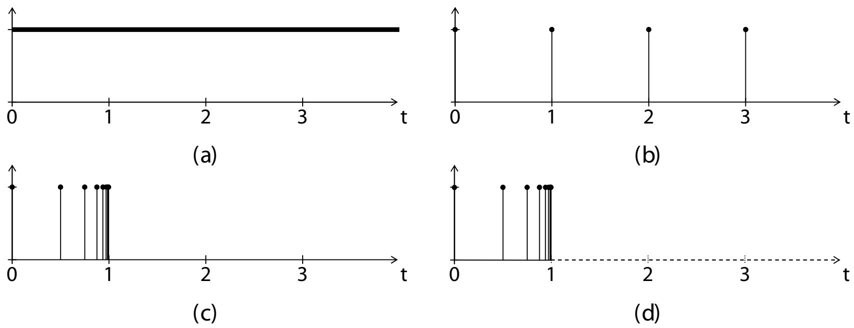

Fig. 3. Examples of timed signals: (a) $const_1$, (b) $clock_1$, (c) $zeno$, (d) $dzeno$.

By implication, all other events with a tag in the domain are absent. If E is a finite set, signal s is called a **finite signal**. For example,

$$s_\perp = (\emptyset, \emptyset),$$
$$s_\varepsilon = (T, \emptyset).$$

The empty signal $s_\perp$ has no events, whereas the absent signal s_ε has absent events (t, ε) for all $t \in T$.

The following examples, with $T = \mathbb{R}_+$ and $V = \{0, 1, \varepsilon\}$, are sketched in figure 3:

$$\begin{aligned} const_1 &= (\mathbb{R}_+, \{(t, 1) \mid t \in \mathbb{R}_+\}), \\ clock_1 &= (\mathbb{R}_+, \{(k, 1) \mid k \in \mathbb{N}\}), \\ zeno &= (\mathbb{R}_+, \{(1 - 1/2^k, 1) \mid k \in \mathbb{N}\}), \\ dzeno &= ([0, 1), \{(1 - 1/2^k, 1) \mid k \in \mathbb{N}\}). \end{aligned}$$

4.3 *Examples of Actors*

We now consider two example actors, $Delay_d$ and $Merge$. Let d be any positive real number. The $Delay_d \colon S \to S$ actor shifts every event in its input signal by d into the future such that if $r = Delay_d(s)$, then

$$\begin{aligned} \operatorname{dom}(r) &= \{t \in T \mid t - d \in \operatorname{dom}(s) \text{ or } t - d \notin T\}, \\ r(t) &= \begin{cases} s(t-d) & t - d \in \operatorname{dom}(s), \\ \varepsilon & \text{otherwise.} \end{cases} \end{aligned} \tag{8}$$

The $Merge \colon S^2 \to S$ actor combines the present events in its input signals into its output signal, giving precedence to its first input when both input signals are

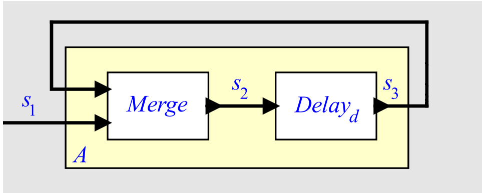

Fig. 4. A composition that can be shown to be live.

present at the same time. Specifically, if $s = Merge(s_1, s_2)$, then

$$\begin{aligned} \mathrm{dom}(s) &= \mathrm{dom}(s_1) \cap \mathrm{dom}(s_2), \\ s(t) &= \begin{cases} s_1(t) & s_1(t) \neq \varepsilon, \\ s_2(t) & \text{otherwise.} \end{cases} \end{aligned} \tag{9}$$

It is easy to prove that actors $Delay_d$ and $Merge$ are both continuous [33]. They are also obviously both receptive.

4.4 Live Systems

A closed composition of actors is said to be **live** if all its behaviors are total (if it is determinate, then there is only one behavior). An open composition of actors is live if, given input signals that are total, all behaviors are total. This broadly captures the notions of freedom from deadlock, livelock, and causality loops.

Consider the composition shown in figure 4, which has the form of that in figure 2 when the $Merge$ and $Delay_d$ are aggregated. Since both $Merge$ and $Delay_d$ are receptive and continuous, the composite actor A is receptive and continuous, and hence the feedback composition is itself a continuous function, using results from section 3.7 above. We can also show that it is live.

To show that the composition in figure 4 is live, we use abstract interpretation [17], considering the actors only to be relations on the domains of the signals,

$$\begin{aligned} \mathrm{dom}(s_2) &= \mathrm{dom}(s_3) \cap \mathrm{dom}(s_1), \\ \mathrm{dom}(s_3) &= [0, d) \cup \{t + d \,|\, t \in \mathrm{dom}(s_2)\}. \end{aligned}$$

If the input s_1 is total, then $\mathrm{dom}(s_1) = T$ and $\mathrm{dom}(s_2) = \mathrm{dom}(s_3)$. This implies that

$$\mathrm{dom}(s_3) = [0, d) \cup \{t + d \,|\, t \in \mathrm{dom}(s_3)\}.$$

The only subset of $\mathbb{R}_+$ that satisfies the last equation is $\mathbb{R}_+$, so both s_2 and s_3 are total signals.

Not all timed process networks have this property. Suppose we replace the $Delay_d$ actor in figure 4 with an actor $LookAhead_a \colon S \to S$, where a is a positive real number. Given a signal s, the output $r = LookAhead_a(s)$ is defined by

$$\begin{aligned} \mathrm{dom}(r) &= \{t \in T \mid t + a \in \mathrm{dom}(s)\}, \\ r(t) &= s(t + a), \end{aligned}$$

It is easy to show that $LookAhead_a$ is continuous.

If we replace $Delay_d$ in figure 4 with $LookAhead_a$, the composition still yields a receptive and continuous function from inputs to outputs, because like $Delay_d$, $LookAhead_a$ is continuous. However, the composition is not live. Given any input s_1, the least fixed point is $s_2 = s_3 = s_\perp$, the empty signal. Thus, the feedback composition gives a function that maps all inputs to the empty signal. This function is certainly receptive and continuous, but it's not very useful. This situation is analogous to deadlock in Kahn process networks.

It is well known that, in general, whether a network of actors is live is undecidable (this is known for Kahn process networks, which are a special case of our framework, so we must assume that in general liveness is undecidable). We have two alternatives. We can specialize the semantics of actors and tag systems to decidable subsets (such as synchronous dataflow [29] and the synchronous/reactive languages [9]), or we can find sufficient conditions for a network to be live, where the sufficient conditions are checkable and not overly restrictive.

The latter approach is commonly used in timed systems such as discrete-event languages [48,28], where a metric space of signals is constructed and contraction maps combined with the Banach fixed point theorem yield live systems. For systems of the types represented by figures 1 and 2, a sufficient condition for a system to be live is that the composite actor A be a contraction map. This corresponds to the more intuitive requirement that every directed loop in a timed actor network include a time delay greater than some $\alpha > 0$. In practice, even though this condition is only sufficient and not necessary, this constraint is not onerous. Designers using discrete-event languages, such as hardware description languages, have no difficulty complying, and no difficulty understanding why the requirement is needed. Indeed, they would consider systems that do not comply but are still live to be pathological. However, in the context of hybrid systems [32], contraction maps are, in fact, overly restrictive.

The metric space approach has been adapted to untimed systems (specifically Kahn process networks) by Mathews [36], who uses a partial metric where the distance of a sequence to itself is greater than zero if the sequence is finite, and is zero only if the sequence is infinite. Mathews develops a generalization of the Banach fixed point theorem to partial metrics and shows that if you have a contraction, then the system is live (he calls the system “complete” rather than “live”).

In this paper, we give a sufficient condition for a system to be live that does not require the machinery of a metric or a partial metric, and yet subsumes these mechanisms as special cases. Our approach is based on a simple and intuitive definition of causality.

4.5 Causality

Causality is the relationship between causes and effects. If a timed process models a physical or computational process, the time of an effect cannot be earlier than the time of the corresponding cause. This intuition is captured by the following definition.

Definition 5 (Causality) *An actor A with input ports P_i and output ports P_o is causal if it is monotonic, and for all behaviors $\sigma \in A$,*

$$\bigcap_{p \in P_i} \mathrm{dom}(\sigma(p)) \subseteq \bigcap_{p \in P_o} \mathrm{dom}(\sigma(p))\,. \tag{10}$$

An immediate consequence of this definition is that a causal actor is live. Thus, whether a composition of actors is causal will tell us whether it is live.

To understand this definition intuitively, consider the case where the tag set T is totally ordered. Then this definition says that if the inputs to a causal actor are known up to some tag $t \in T$, then the outputs are known at least up to that same tag t.

Also, a consequence of this definition is that if the input signals in one behavior $\sigma \in A$ are the same as the input signals in another behavior $\sigma' \in A$ up to some tag t, then the corresponding output signals will be the same up to the same tag t.

We can make this precise. Let $D(t) = \{\tau \in T \mid \tau \leq t\}$ for some $t \in T$ denote the smallest down set including t. If an actor A is causal, then for any two behaviors $\sigma, \sigma' \in A$ and time t such that

$$t \in \bigcap_{p \in P_i} \mathrm{dom}(\sigma(p)) \cap \mathrm{dom}(\sigma'(p)),$$

$$\begin{aligned} &\forall\, p \in P_i,\ \sigma(p) \restriction D(t) = \sigma'(p) \restriction D(t) \implies \\ &\forall\, p \in P_o,\ \sigma(p) \restriction D(t) = \sigma'(p) \restriction D(t). \end{aligned}$$

This follows immediately from the definition of causality and the fact that the actor is monotonic.

Among the actors discussed so far, $Delay_d$ and $Merge$ are causal, whereas $LookAhead_a$ is not.

Neither causality nor continuity implies the other. The $\mathit{LookAhead}_a$ process is continuous but not causal. A minor variant of the Merge actor that we call $\mathit{MaxMerge}$ is causal but not continuous. The $\mathit{MaxMerge}\colon S^2 \to S$ actor is such that $s = \mathit{MaxMerge}(s_1, s_2)$ is given by

$$\mathrm{dom}(s) = \{t \in \mathrm{dom}(s_1) \mid \forall \tau \in D(t) \setminus \mathrm{dom}(s_2),\ s_1(\tau) \neq \varepsilon\}, \tag{11}$$

$$s(t) = \begin{cases} s_1(t) & s_1(t) \neq \varepsilon, \\ s_2(t) & \text{otherwise.} \end{cases} \tag{12}$$

Intuitively, if the input signal s_1 is continuously present over a time interval beyond $\mathrm{dom}(s_2)$, then those present events are in the output of $\mathit{MaxMerge}$. The "Max" in the name is suggestive that this actor, unlike Merge, produces the maximal output for a given pair of inputs.

Lemma 6 *$\mathit{MaxMerge}$ is not continuous.*

Proof. Assume $T = \mathbb{R}_+$ and consider two signals

$$u_1 = ([0,1], \{(1,1)\}), \tag{13}$$

$$u_2 = ([0,1), \emptyset), \tag{14}$$

$$\mathit{MaxMerge}(u_1, u_2) = u_1. \tag{15}$$

Let

$$r_k = ([0, 1 - \frac{1}{2^k}), \emptyset),\ k \in \mathbb{N},$$
$$D = \{(u_1, r_k),\ k \in \mathbb{N}\}.$$

D is a directed set, and

$$\mathit{MaxMerge}(u_1, r_k) = r_k, \qquad \bigvee \mathit{MaxMerge}(D) = u_2.$$
$$\bigvee D = (u_1, u_2), \qquad \mathit{MaxMerge}(\bigvee D) = u_1.$$

$$\bigvee \mathit{MaxMerge}(D) \neq \mathit{MaxMerge}(\bigvee D)\,.$$

Hence, the actor is not continuous. □

It is easy to see that any composition of causal actors without directed cycles is itself a causal actor. This is not in general true when there are directed cycles. In this case, we will require that at least one actor in the loop be strictly causal, as defined next.

Definition 7 (Strict Causality) *An actor A with input ports P_i and output ports P_o is strictly causal if it is monotonic, and for all behaviors $\sigma \in A$, either $\sigma(p)$ is* *total for all* $p \in P_o$ *or*

$$\bigcap_{p \in P_i} \mathrm{dom}(\sigma(p)) \subset \bigcap_{p \in P_o} \mathrm{dom}(\sigma(p))\,. \tag{16}$$

Here $\subset$ denotes a strict subset relation. Note that if A is a strictly causal actor with one input and one output, then $A(s_\perp) \neq s_\perp$. A must "come up with something from nothing." This is, of course, why strictly causal actors are useful in directed cycles. Strict causality in our sense serves a similar role to "delta causality" in metric space formulations, but ours does not require a metric.

We might assume that $Delay_d$ is strictly causal, but this is not always the case. If the tag set is $T = \mathbb{R}_+$, then $Delay_d$ is strictly causal for any $d > 0$. The same holds if T is any interval in the reals that is not a down set of $\mathbb{R}$. If T is a down set of $\mathbb{R}$, such as $(-\infty, 0]$ or $\mathbb{R}$ itself, then $Delay_d$ is not strictly causal, as evidenced by the fact that $Delay_d(s_\perp) = s_\perp$.

We finally come to the main result of this section. The following theorem effectively gives us a sufficient condition for networks to be live, since causal actors are live.

Theorem 8 (Causality of Feedback Compositions) *Given a totally ordered tag set and a network of causal, receptive, and continuous actors where in every dependency loop in the network there is at least one strictly causal actor, then the network is a causal, receptive, and continuous actor.*

Proof. (Sketch) We will prove the theorem for networks of the form of figure 2. The composition actor has input port p_2 and output port p_3. Note that since actor A is continuous, the composite actor is receptive and continuous by the results of section 3.7. So we only have to show causality. The generalization to arbitrary networks is notationally more tedious, but conceptually identical, and is given in [33].

We proceed by contradiction. Suppose the composite actor is not causal. Then there exists an input signal s_2 at port p_2 and output signal s_3 at p_3 where $\mathrm{dom}(s_2) \not\subseteq \mathrm{dom}(s_3)$. Since the tag set is totally ordered, the set of down sets of the tag set is totally ordered by set inclusion. Thus, if $\mathrm{dom}(s_2) \not\subseteq \mathrm{dom}(s_3)$, then it must be true that $\mathrm{dom}(s_3) \subset \mathrm{dom}(s_2)$ (a strict subset). The signal s_1 at port p_1 is the same as s_3, so $\mathrm{dom}(s_1) \subset \mathrm{dom}(s_2)$ and $\mathrm{dom}(s_1) \cap \mathrm{dom}(s_2) = \mathrm{dom}(s_1)$. Hence, strict causality requires $\mathrm{dom}(s_1) \subset \mathrm{dom}(s_3)$, but we have $\mathrm{dom}(s_1) = \mathrm{dom}(s_3)$, a contradiction. □

Note that this proof is not constructive. It does not tell us how to find the behavior of the actor network, it just tells us that there is a well-defined behavior, and it implies that if the input is total then the output is total. Since we assume the actors are receptive and continuous, the behavior of the network is the same as obtained constructively by theorem 4. However, although theorem 4 is constructive, behaviors of the system may not be computable in practice. We examine this issue next.

5 Discrete-Event Systems

An important subclass of timed systems are discrete event (DE) systems [14,22,28]. Here, we give a strong definition of such systems, showing that they provide a subset of timed systems that can be computed one event at a time. In particular, appropriately constrained DE systems yield a countable set of events and avoid Zeno conditions, which in practice can be as big an obstacle to practical utility as lack of liveness. We begin with the definition of DE signals and their properties.

5.1 DE Signals

Definition 9 (Discrete Event Signal) *A timed signal $s \in \mathcal{S}$ is a discrete event (DE) signal if there exists a directed set $D \subseteq \mathcal{S}$ of finite timed signals such that*

$$s = \bigvee D\,.$$

Let $\mathcal{S}_{\mathrm{d}} \subseteq \mathcal{S}$ denote the set of all DE signals with the same tag and value sets as $\mathcal{S}$. Among the signals in figure 3, $clock_1$ and $dzeno$ are DE signals, but not $const_1$ and $zeno$. Both the empty signal $s_\perp$ and the absent signal s_ε are DE signals.

There are several equivalent definitions of DE signals, as established by the following lemmas.

Lemma 10 *A timed signal s is a DE signal if and only if for all $t \in \mathrm{dom}(s)$, $s \restriction D(t)$ is a finite signal.*

Proof. Let s be a DE signal and D a directed set of finite signals such that $s = \bigvee D$. For all $t \in \mathrm{dom}(s)$, there exists $r \in D$ such that $t \in \mathrm{dom}(r)$.

$$r \sqsubseteq s \implies s \restriction D(t) \ = r \restriction D(t).$$

r is a finite signal implies $r \restriction D(t)$ is a finite signal, so is $s \restriction D(t)$.

For any timed signal s, let

$$D_s = \{s \restriction D(t) \,|\, t \in \mathrm{dom}(s)\} \cup \{s_\perp\}\,.$$

D_s is a directed set and $s = \bigvee D_s$. If for all $t \in \mathrm{dom}(s)$, $s \restriction D(t)$ is finite, then s is a DE signal. $\Box$

Lemma 11 *A timed signal $s \in \mathcal{S}$ is a DE signal if and only if $s^{-1}(V \setminus \{\varepsilon\})$ is order-isomorphic to a down set of $\mathbb{N}$, and if $s^{-1}(V \setminus \{\varepsilon\})$ is an infinite set, then*

$$\mathrm{dom}(s) = \bigcup_{t \in s^{-1}(V \setminus \{\varepsilon\})} D(t)\,. \tag{17}$$

This definition is used in [28]. If $s^{-1}(V \setminus \{\varepsilon\})$ is order-isomorphic to a down set of $\mathbb{N}$, then the present events of s can be enumerated in the order of their time. If s is present at an infinite number of times, then equation 17 guarantees that for any $t \in \mathrm{dom}(s)$, s is present at a time later than t.

With these lemmas, we have three equivalent definitions of DE signals. Definition 9 states that DE signals can be approximated by "simple" elements of $\mathcal{S}$, the finite signals. Lemma 10 is very useful in proving properties of DE signals. By lemma 11, the present events in a DE signal can be treated as a sequence with increasing time tags.

The following lemma summarizes the properties of $\mathcal{S}_\mathrm{d}$, the set of DE signals.

Lemma 12 *For any totally ordered tag set T,*

(a) $\mathcal{S}_\mathrm{d}$ is a down set of $\mathcal{S}$.
(b) $\mathcal{S}_\mathrm{d}$ with the prefix order is a CPO.
(c) $\mathcal{S}_\mathrm{d}$ is a complete semilattice.

Proof. Part (a) is straightforward, as any prefix of a DE signal is also a DE signal.

Part (b). Let D be a directed set of DE signals from $\mathcal{S}_\mathrm{d}$. As a subset of $\mathcal{S}$, D is also a directed set. Since $\mathcal{S}$ is a CPO, there exists $u \in \mathcal{S}$ such that $u = \bigvee D$ in the CPO $\mathcal{S}$. For all $t \in \mathrm{dom}(u)$, there exists $s \in D$ such that $t \in \mathrm{dom}(s)$.

$$s \sqsubseteq u,\ t \in \mathrm{dom}(s) \implies u \restriction D(t) = s \restriction D(t).$$

$s \restriction D(t)$ is a finite signal, so is $u \restriction D(t)$. u is a DE signal, so D has a least upper bound in $\mathcal{S}_\mathrm{d}$. $\mathcal{S}_\mathrm{d}$ is a CPO.

Part (c). The proof follows directly from the fact that $\mathcal{S}$ is a complete semilattice and part (a) of this lemma. $\Box$

Definition 13 (Non-Zeno Signal) *A DE signal $s \in \mathcal{S}_\mathrm{d}$ is non-Zeno if either s is a finite signal, or s is a total signal, $\mathrm{dom}(s) = T$.*

Of the signals in figure 3, $clock_1$ is the only non-Zeno DE signal. The only other DE signal, $dzeno$, is a Zeno signal—it is present at an infinite number of times in a strict subset of its tag set. The significance of this is that if the signal is computed by enumerating its present events ordered by time, then any $t \in T \setminus \mathrm{dom}(dzeno)$ cannot be covered in any finite number of computational steps.

Note the role of the tag set T in definition 13. If we change the tag set to $T = [0, 1)$, then the signal

$$([0,1), \{(1 - \frac{1}{2^k}, 1) \mid k \in \mathbb{N}\})$$

is present at the same set of times as $dzeno$, but it is a non-Zeno signal because its tag set T is $[0, 1)$ and it is a total signal.

A key property of non-zeno DE signals is that all approximations defined over a subset of T have a finite number of (non-absent) events. This property is extremely helpful when computing the signals in a composition. It means that a computation can successively approximate signals over downsets of T, iteratively increasing these downsets towards the limit of T, and the computation will never have to represent more than a finite number of events. Discrete-event simulators, for example, execute a composition in precisely this manner, by advancing time and representing signals up to the advancing time. This observation motivates the following definitions.

Definition 14 (Computable closed compositions) *A closed composition is computable if it has a finite number of behaviors and every signal in every behavior is a non-Zeno DE signal.*

Definition 15 (Computable open compositions) *An open composition is computable if given non-Zeno DE signals for inputs it has a finite number of behaviors and every signal in every behavior is a non-Zeno DE signal.*

5.2 Discrete-Event Actors

Definition 16 (Discrete-Event Actor) *A* ***discrete event actor*** *is a function from DE signals to DE signals.*

All input and output signals of a DE actor have the same tag set. Among the actors discussed above, $Delay_d$, $Merge$, and $LookAhead_a$ are DE actors. $MaxMerge$ is not a DE actor, as it has the following behavior,

$$\begin{aligned} s_1 &= ([0,1], \{(1,1)\}), \\ s_2 &= dzeno, \\ MaxMerge(s_1, s_2) &= ([0,1], \{(1 - \frac{1}{2^k}, 1) \mid k \in \mathbb{N}\} \cup \{(1,1)\}). \end{aligned}$$

$MaxMerge(s_1, s_2)$ is not a DE signal.

Definition 17 (Non-Zeno Actor) *A DE actor $P\colon \mathcal{S}_\mathrm{d} \to \mathcal{S}_\mathrm{d}$ is a* ***non-Zeno actor*** *if for any non-Zeno signal $s \in \mathcal{S}_\mathrm{d}$, $P(s)$ is a non-Zeno signal.*

Such actors are called *simple processes* in [15].

Theorem 18 *A causal DE actor is non-Zeno.*

Proof. Let $P\colon \mathcal{S}_\mathrm{d} \to \mathcal{S}_\mathrm{d}$ be a causal DE process. Let $s \in \mathcal{S}_\mathrm{d}$ be any non-Zeno signal. If s is a total signal, P is causal implies $P(s)$ is a total DE signal. $P(s)$ is non-Zeno.

If s is not a total signal, then s is finite. Let $s' \in \mathcal{S}_\mathrm{d}$ be a total signal such that

$$s'(t) = \begin{cases} s(t) & \text{if } t \in \mathrm{dom}(s), \\ \varepsilon & \text{otherwise.} \end{cases}$$

s' is a total non-Zeno signal, and $s \sqsubseteq s'$. $P(s')$ is a non-Zeno signal. P is causal, so it is monotonic by definition. $P(s) \sqsubseteq P(s')$, so $P(s)$ is non-Zeno. $\square$

5.3 Composition of Discrete-Event Actors

Combining the previous results, from section 3, we know that if all actors in a network of DE actors are continuous, then the network, as a functional actor that maps input signals to the least solution of the network equations, is continuous. The following theorem is proved essentially identically to theorem 8.

Theorem 19 (Causal DE Process Network) *If all actors in a DE actor network are causal and continuous, and in every dependency loop in the network there is at least one strictly causal actor, then the network is causal and continuous.*

Corollary 20 *A DE actor network that satisfies the assumptions of theorem 19 is non-Zeno, and hence computable.*

This corollary follows directly from theorems 19 and 18. Note that unlike [49,48,28] and most other treatments of DE systems, we do not require that two present events in a signal be separated by a minimum time interval, nor that actors be required to introduce a minimum time delay.

6 Conclusion

We have given a domain-theoretic denotational framework for timed interactive systems. We have shown that classical CPO-based techniques determine existence and uniqueness of (least fixed point) solutions, while causality determines liveness. In particular, strict causality, the definition of which does not require a metric space, ensures live feedback loops, which in turn ensures freedom from Zeno conditions. Our approach contrasts with metric space approaches, where contractions and the Banach fixed point theorem ensure existence and uniqueness of fixed points together with liveness. Separating liveness concerns from existence/uniqueness concerns allows us to admit non-causal components. Moreover, our approach does not require a metric space and consequently embraces easily a wide variety of models of time, including super-dense time.

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