Bases géométriques de l'informatique – Part deux

Jean Ponce
"Hauts du DI"
Email: ponce@di.ens.fr
Web: http://www.di.ens.fr/~ponce

Planches après les cours sur :
http://www.di.ens.fr/~ponce/geomvis/lect1.pptx
http://www.di.ens.fr/~ponce/geomvis/lect1.pdf

References

- R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision", Cambridge University Press, 2000.
- O.D. Faugeras, Q.-T. Luong, and T. Papadopoulo, "The Geometry of Multiple Images", MIT Press, 2001.
- D.A. Forsyth and J. Ponce, "Computer Vision: A Modern Approach", Prentice-Hall, 2002, 2011 (2nd edition).

- J.J. Koenderink, "Solid Shape", MIT Press, 1990.

- M. Berger, "Géométrie", Nathan, 1992.
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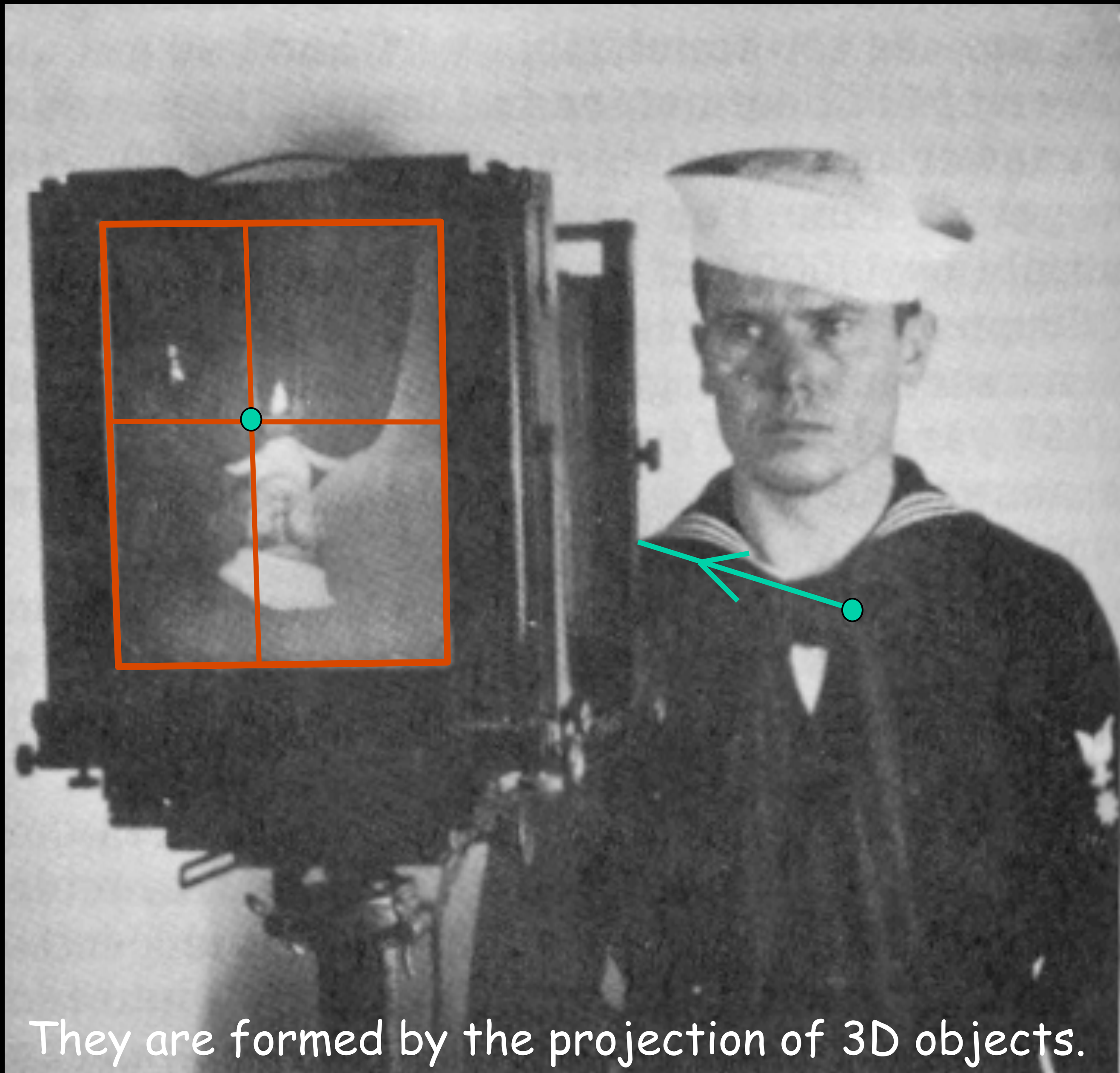
Images are two-dimensional patterns of brightness values.



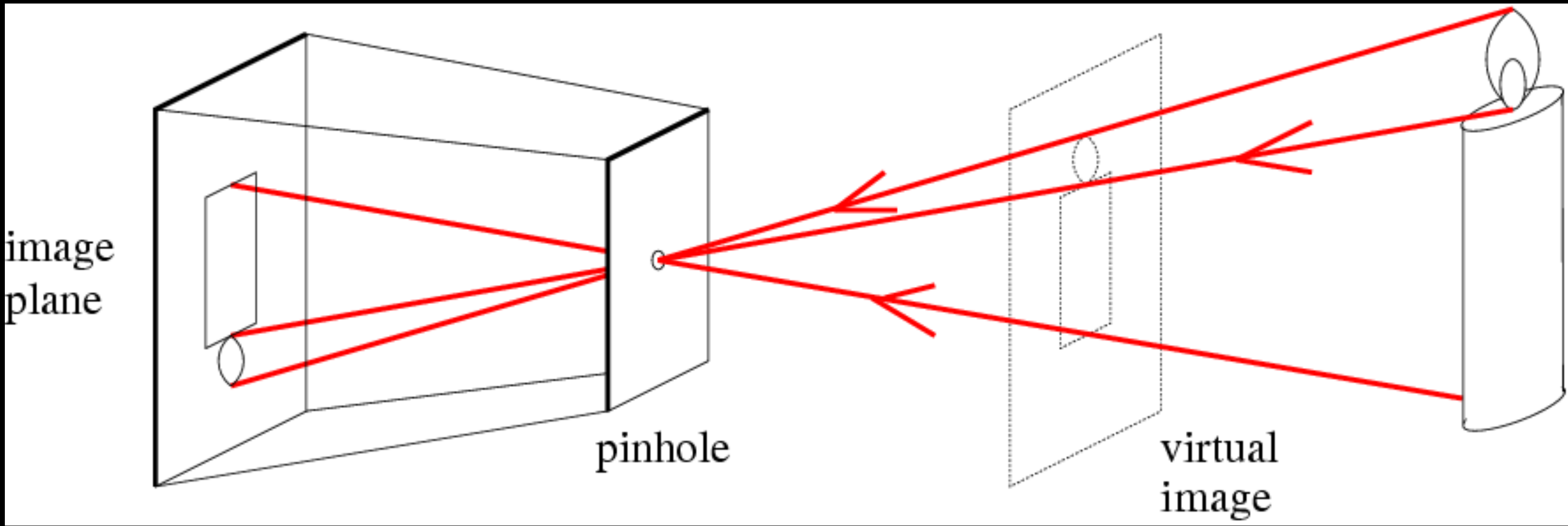
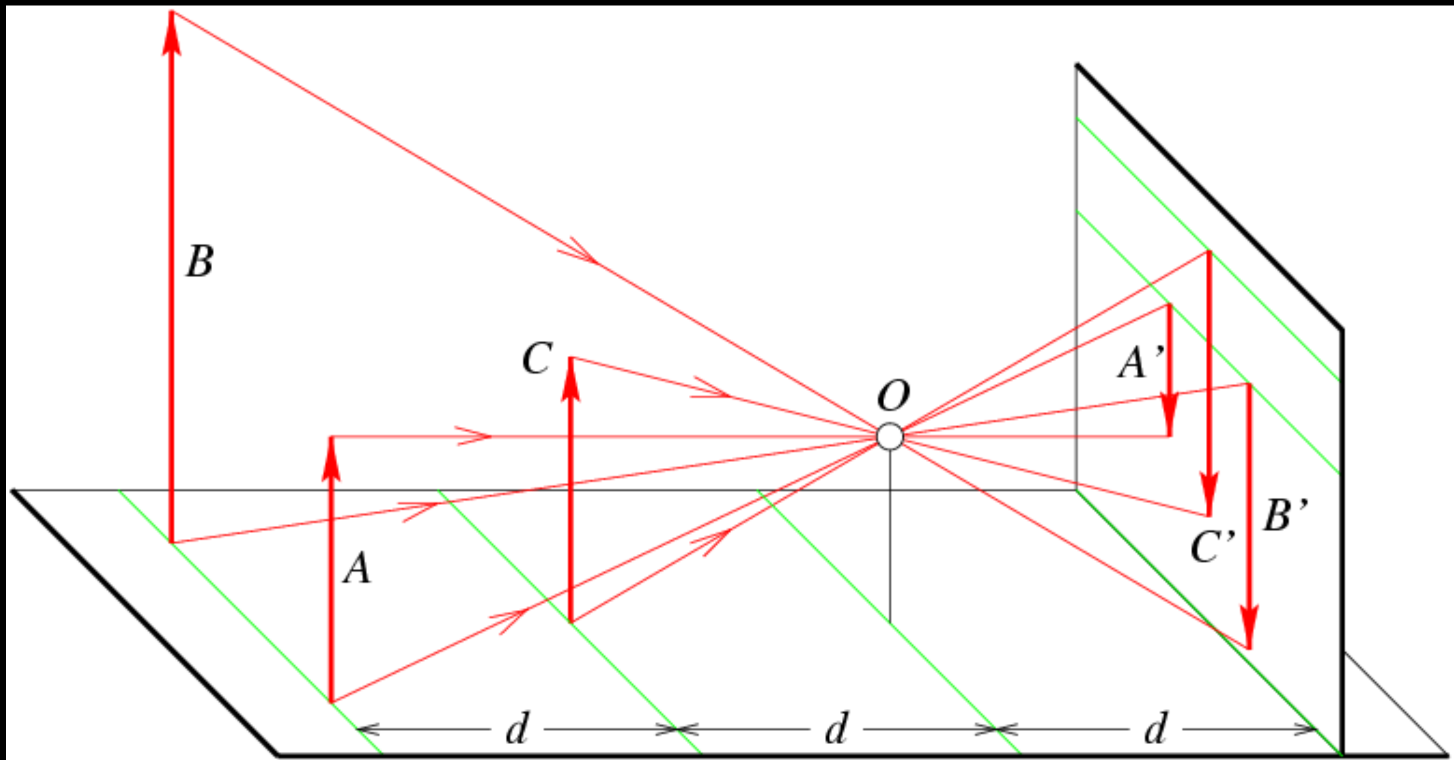


image plane
pinhole
virtual image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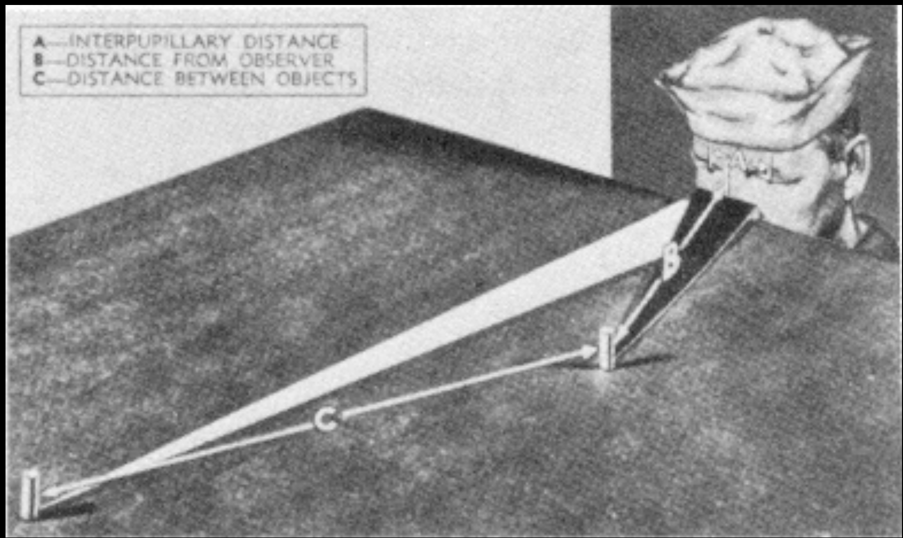
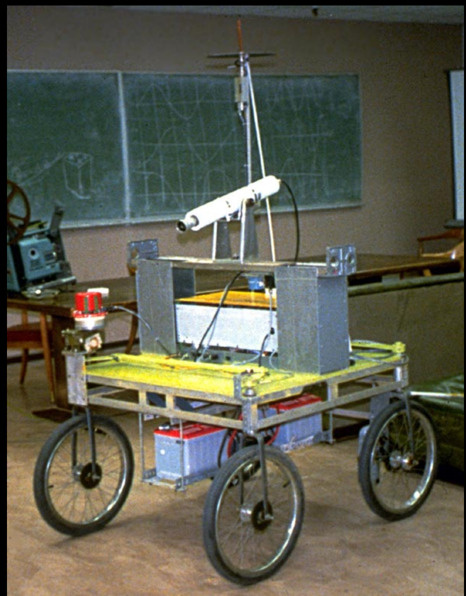
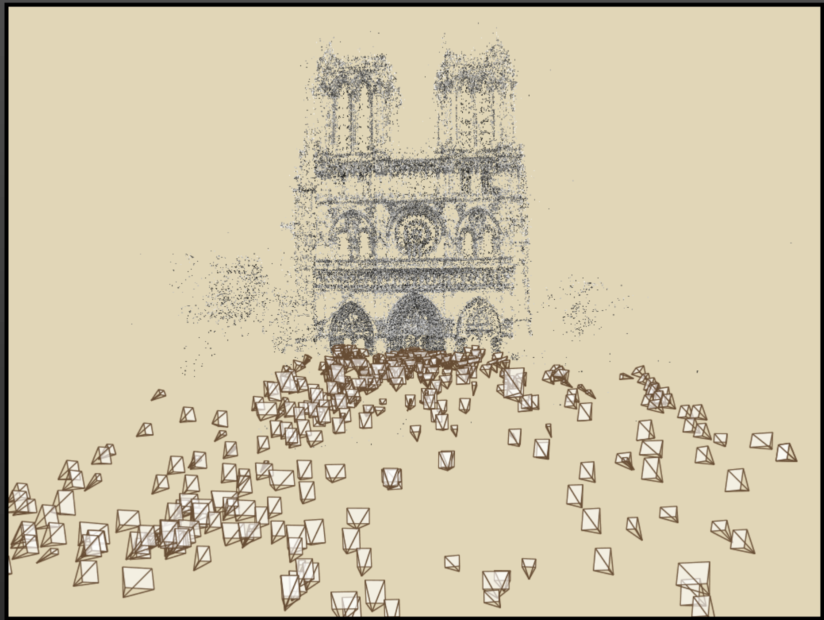
How do we perceive depth?





Building large-scale 3D models from photographs

Structure from motion



- Snavely, Seitz, Szeliski, 2007
- Vergauwen, Van Gool, 2006
- Brown, Lowe, 2005
- Schaffalitzky, Zisserman, 2002

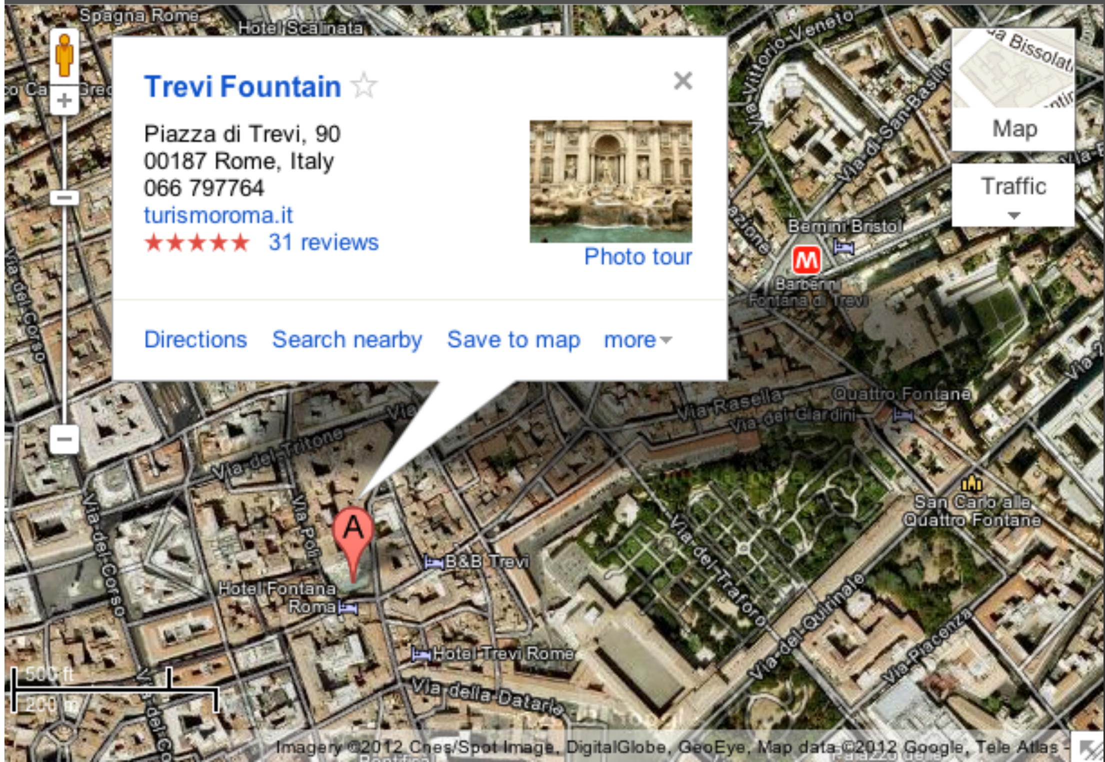
http://phototour.cs.washington.edu/bundler/

Dense multiview stereo

- Furukawa, Ponce, 2007
- Labatut, Pons, Keriven, 2009
- Goesele, et al., 2007

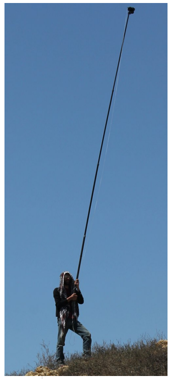
http://grail.cs.washington.edu/software/pmvs/

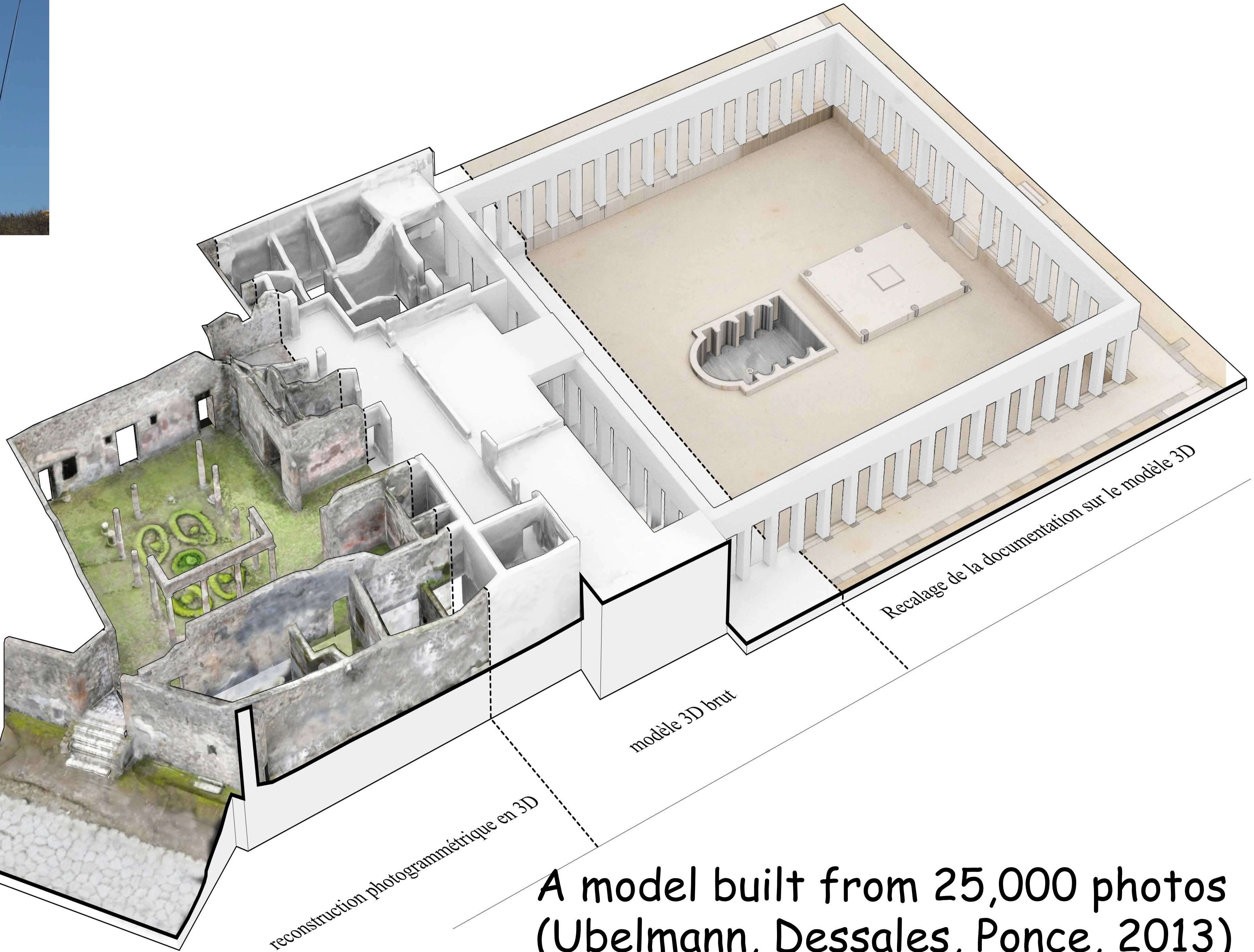
PMVS (http://www.di.ens.fr/pmvs)



(© Bath & Burke, Weta Digital, Siggraph'11)

- Google Maps Photo Tour
- Lucasfilm
- Weta Digital



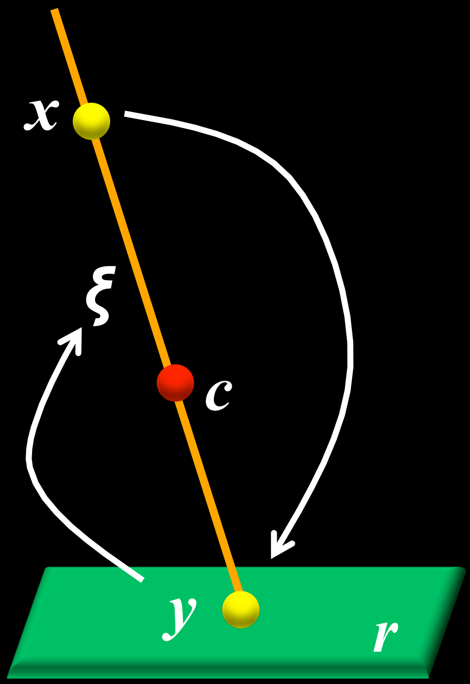


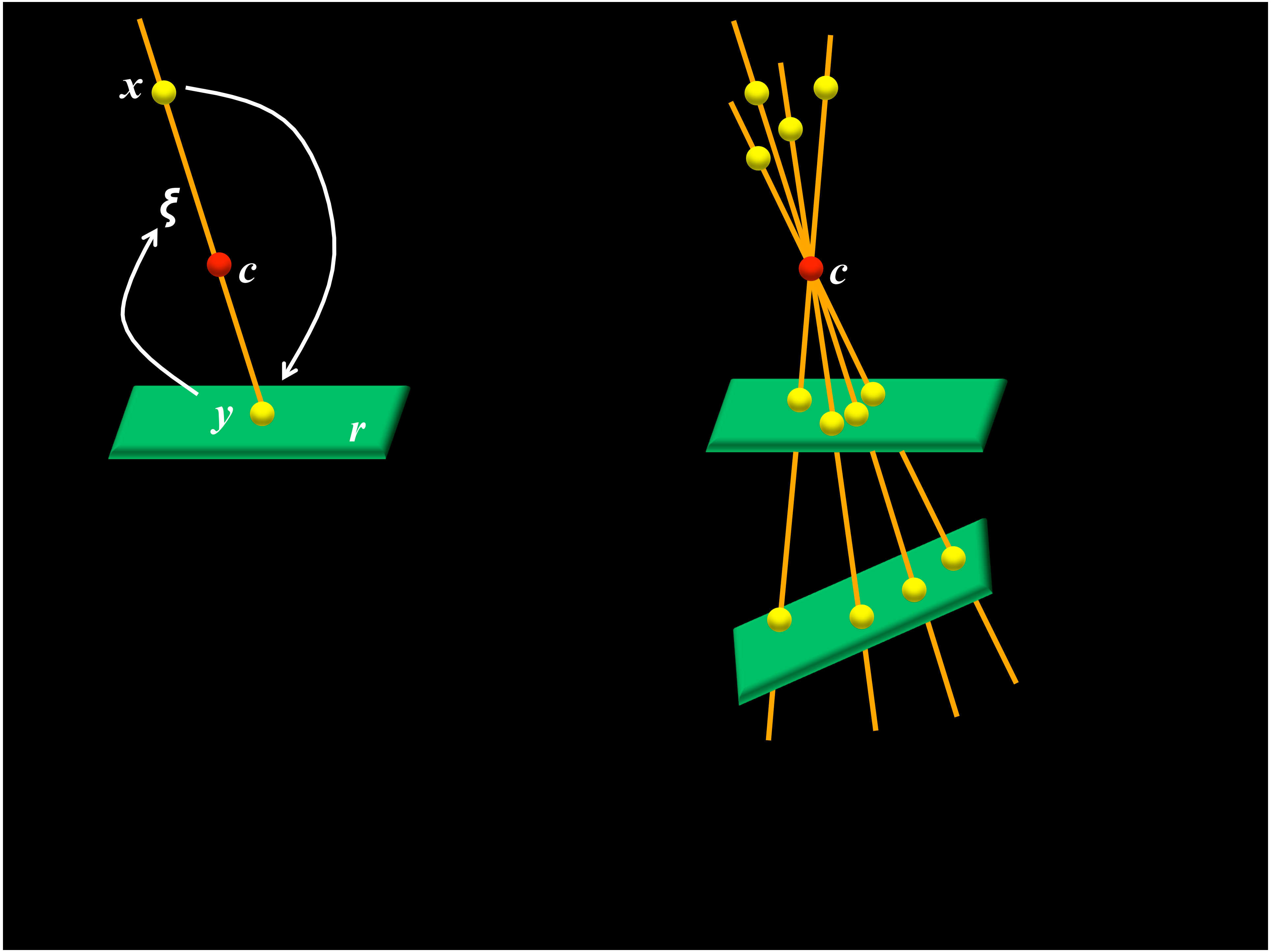
A model built from 25,000 photos
(Ubelmann, Dessales, Ponce, 2013)

What is a camera?

What is a camera?

What is a camera?

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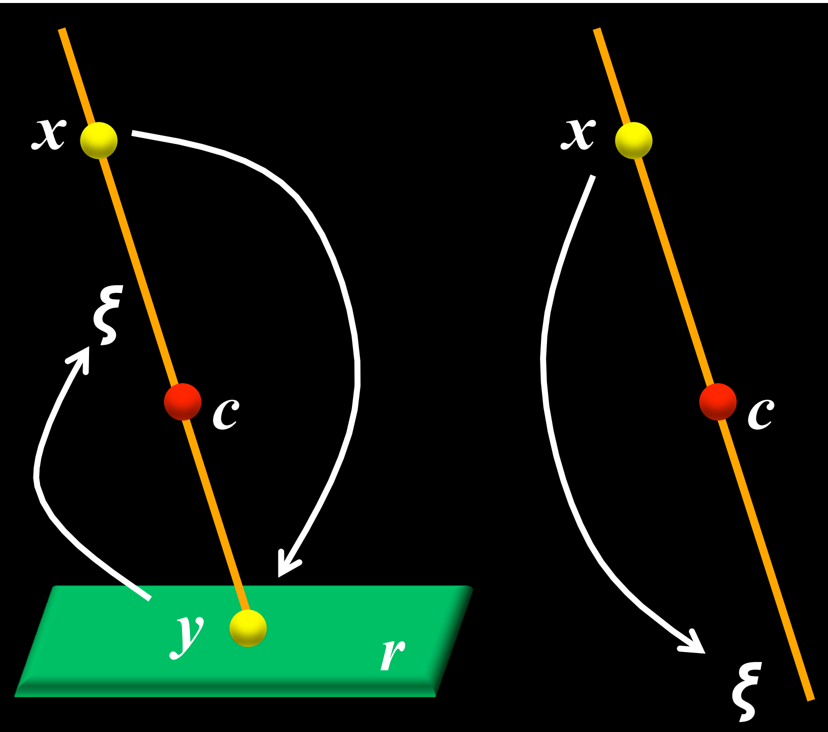
x
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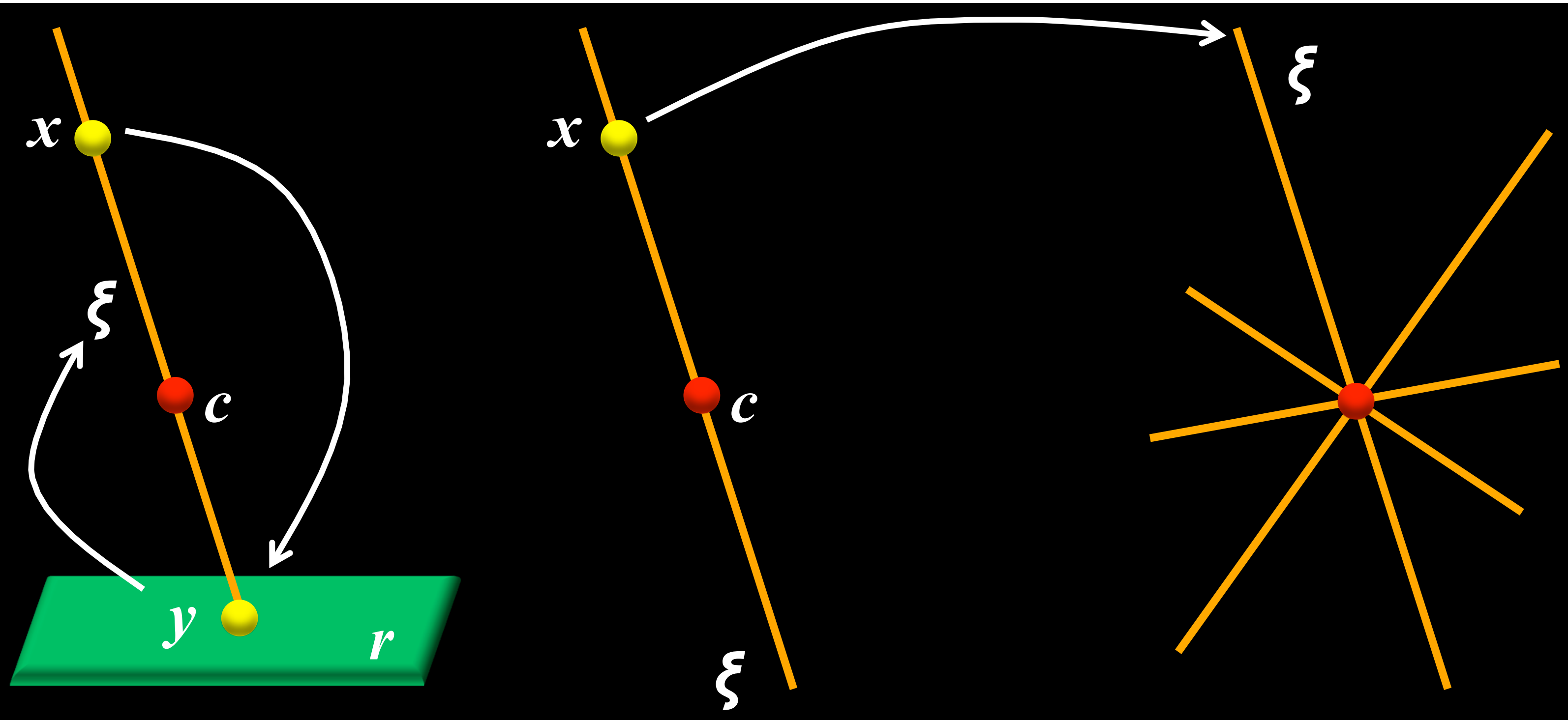

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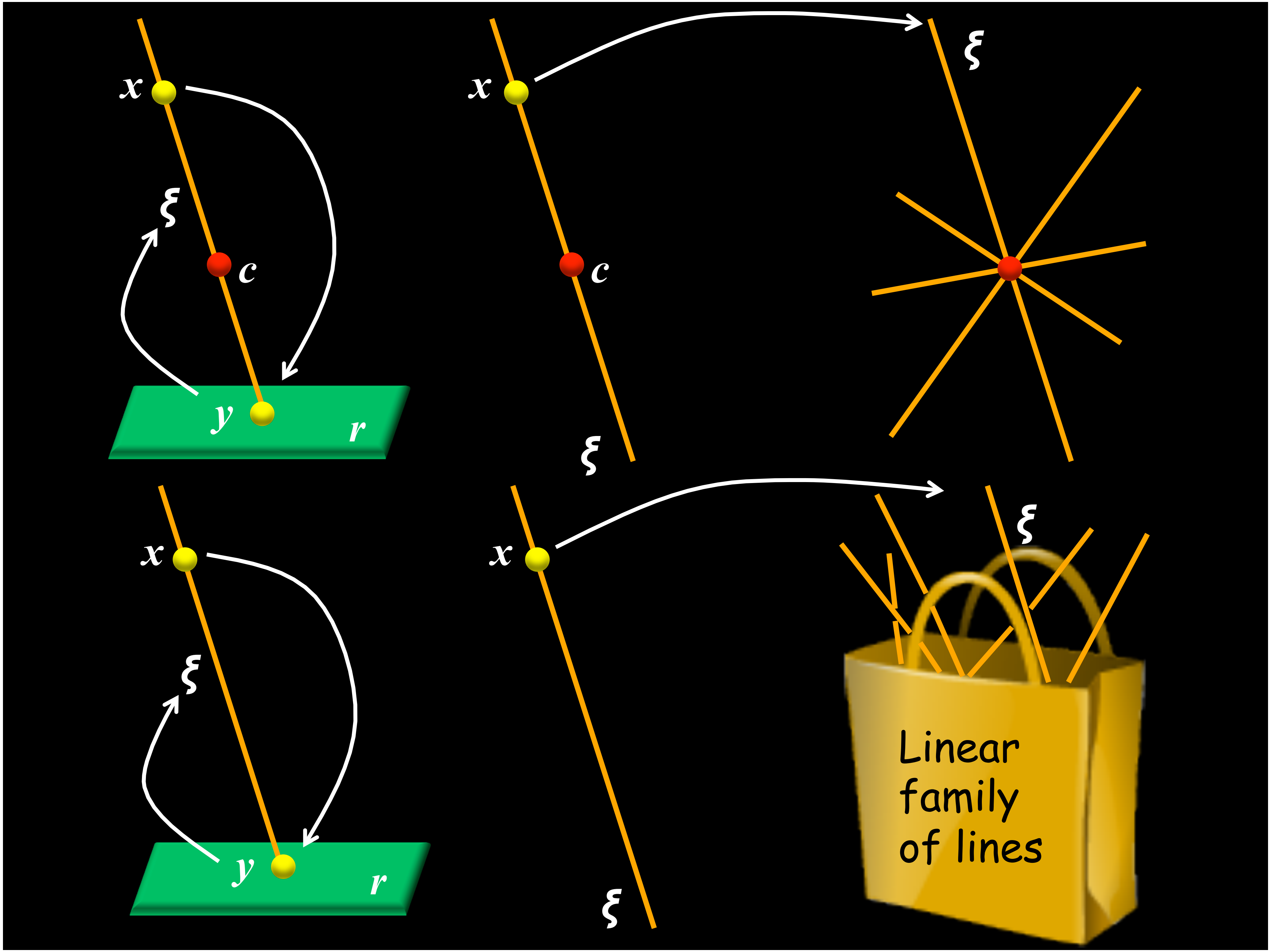
x
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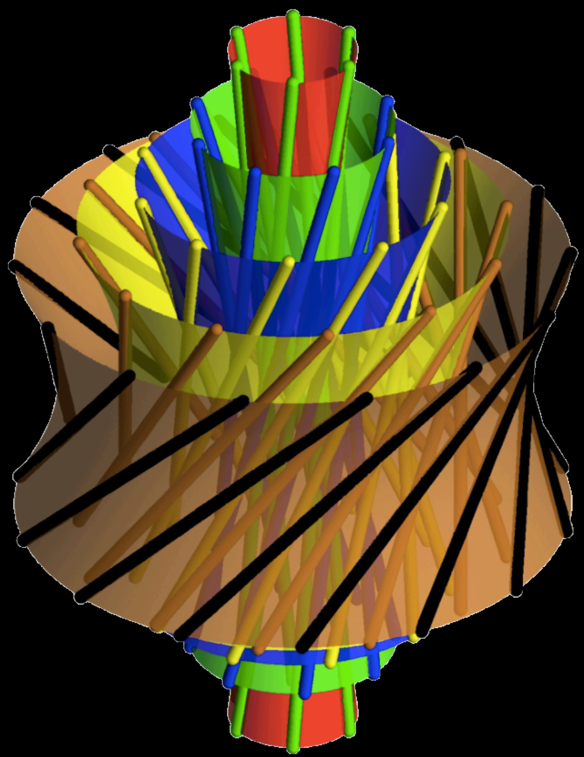
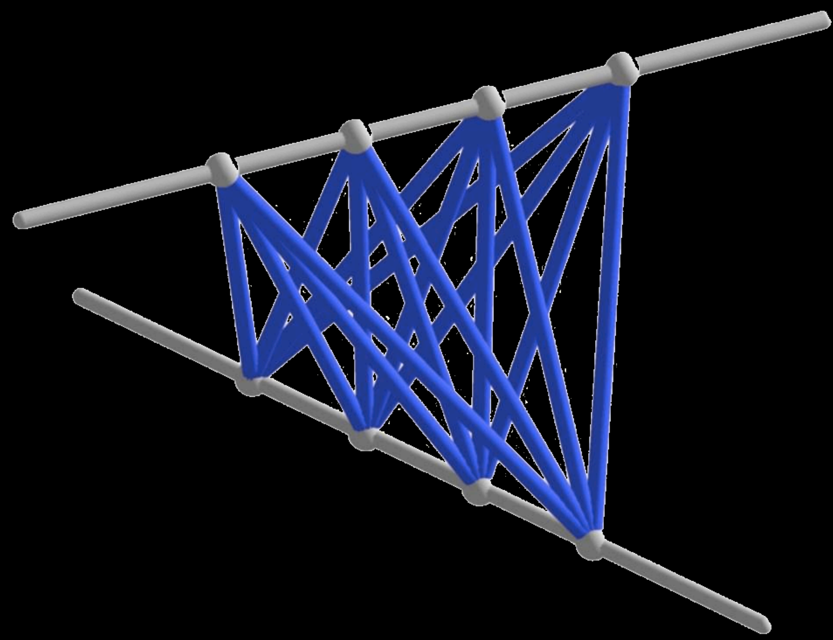
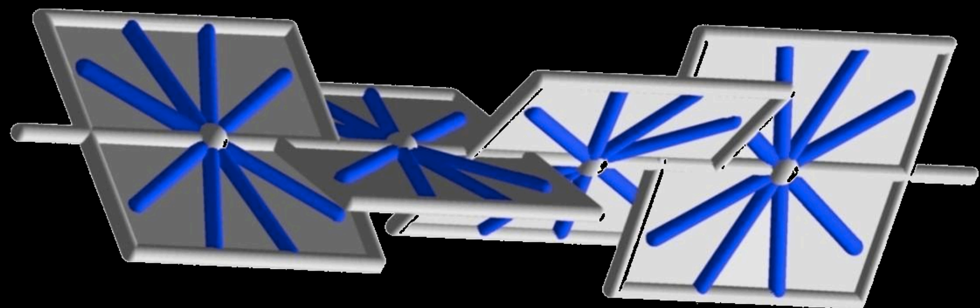
Linear
family
of lines

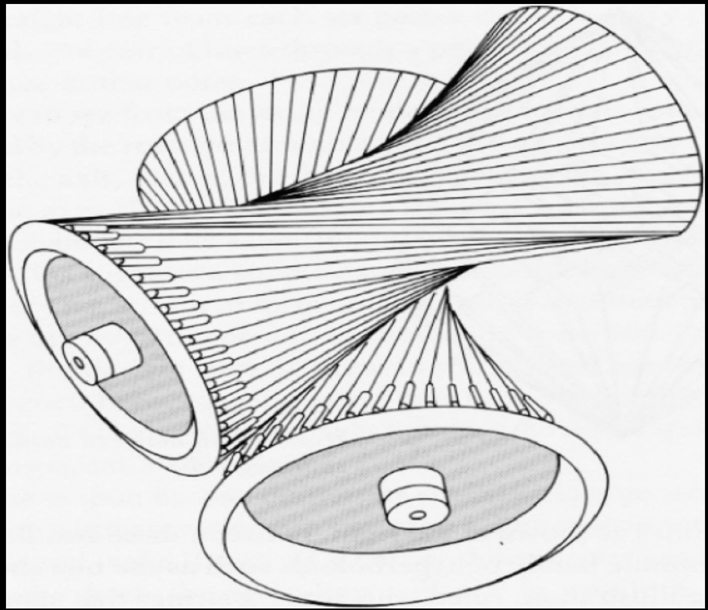
Rank-4 (nondegenerate) families:
Linear congruences

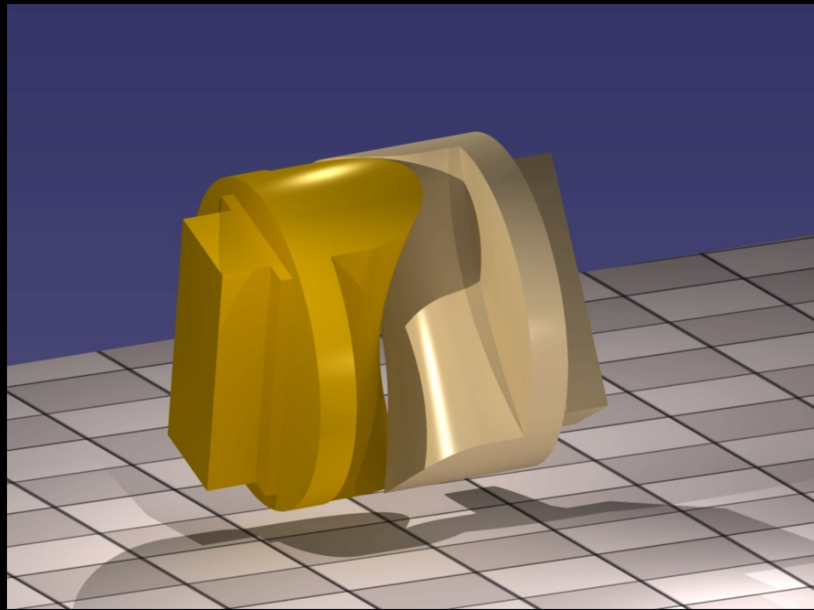






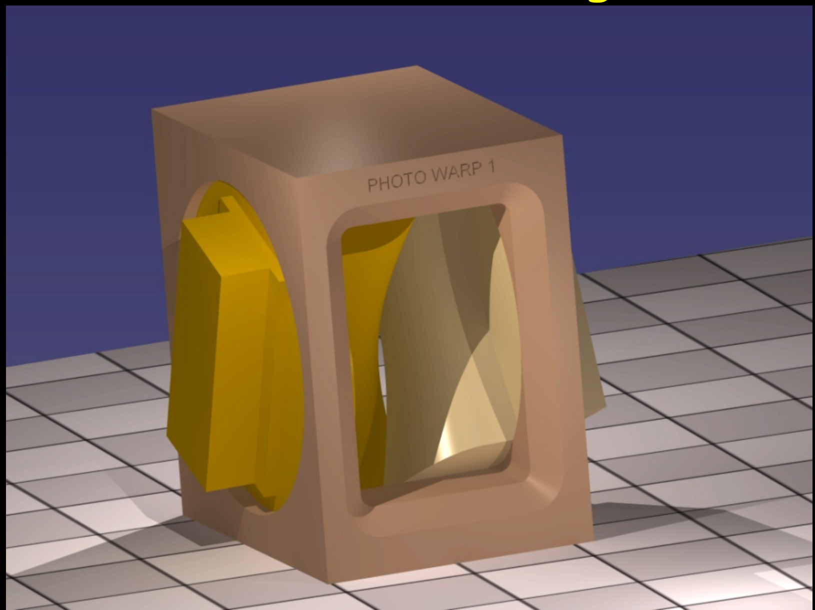
Figures © H. Havlicek, VUT



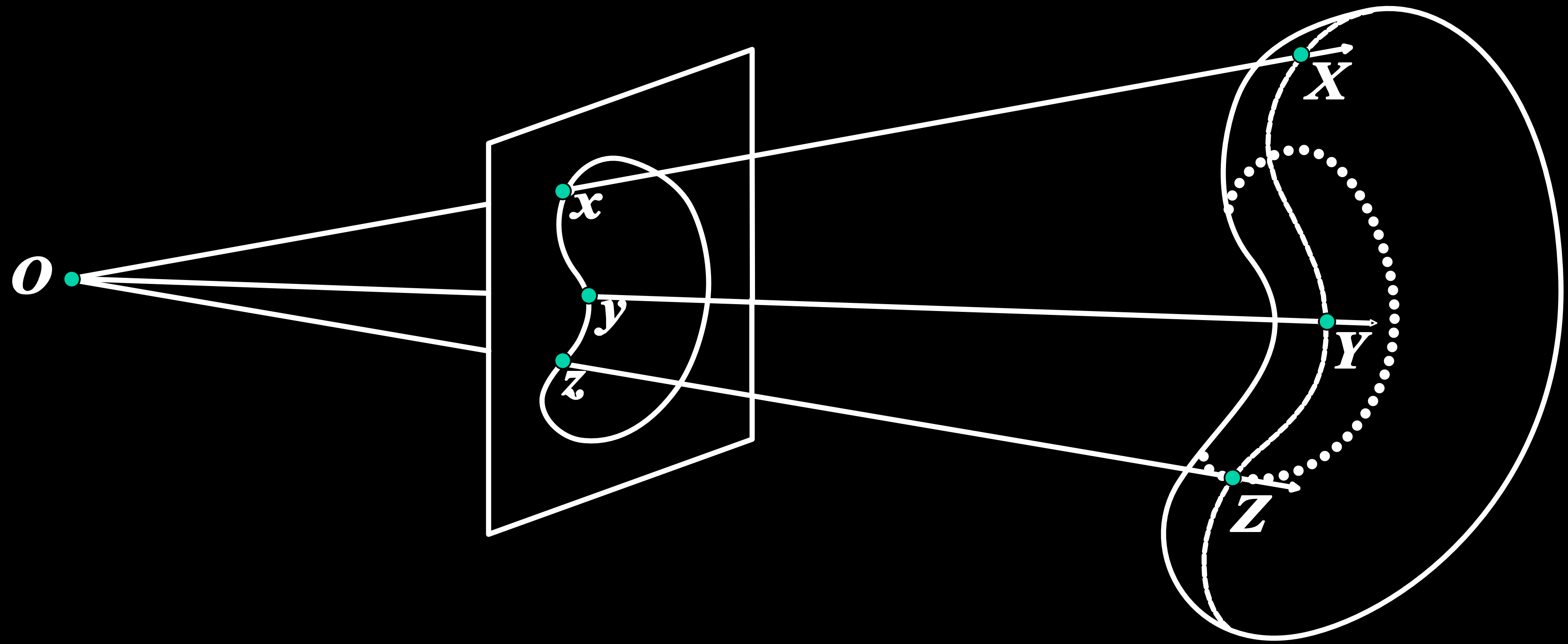


Building a parabolic camera

(Batog, Goaoc, Lavandier, Ponce, 2010)

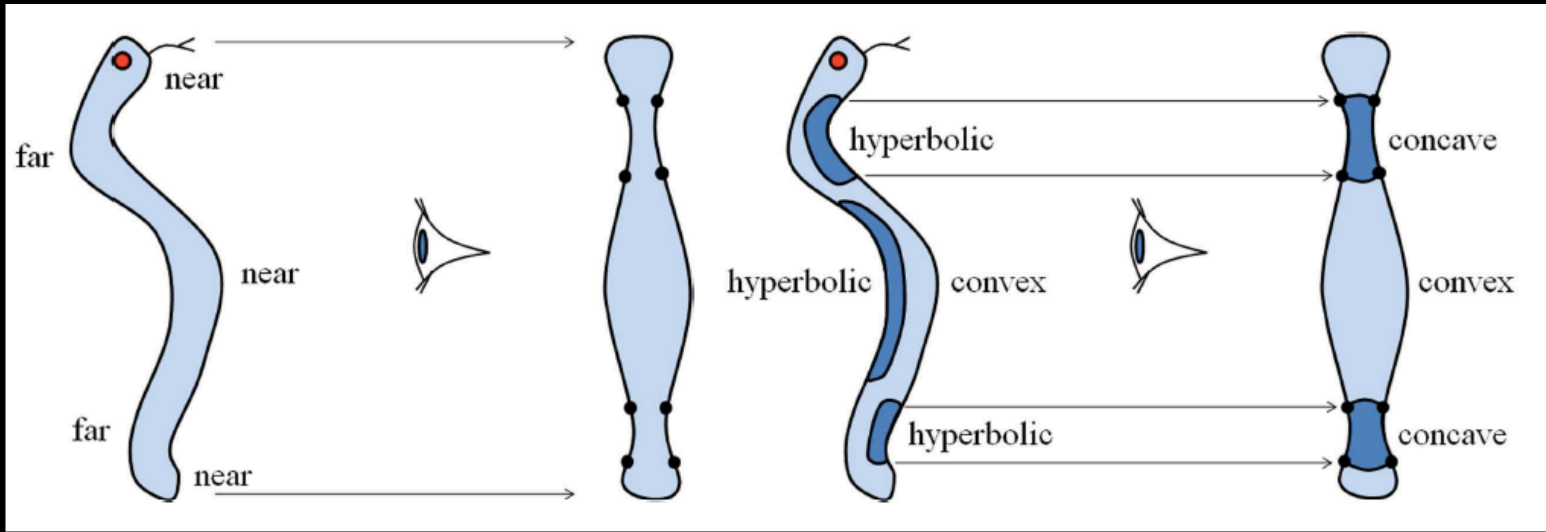


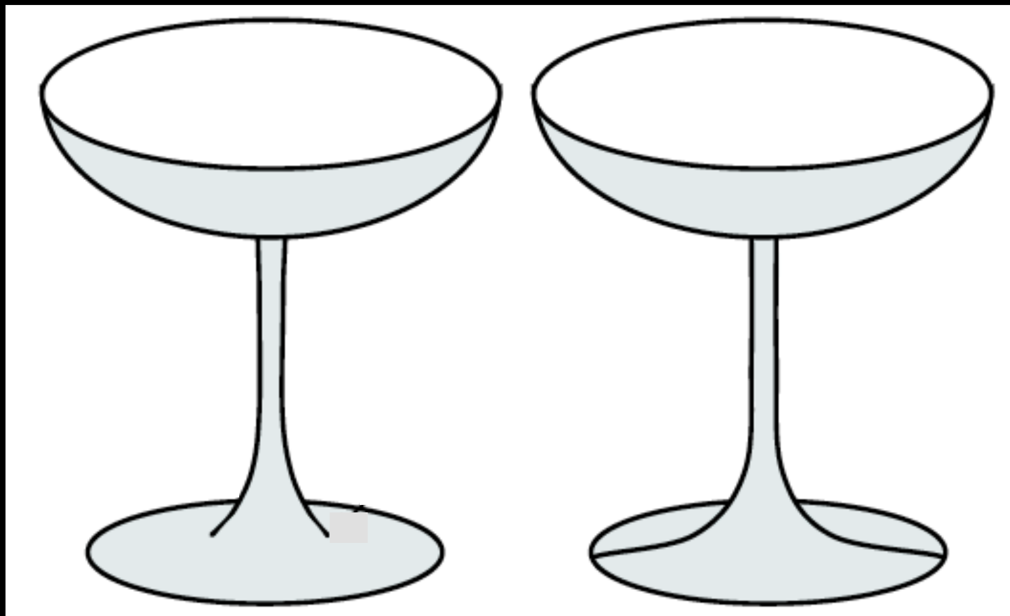
Koenderink (1984)



Solid shapes and their outlines

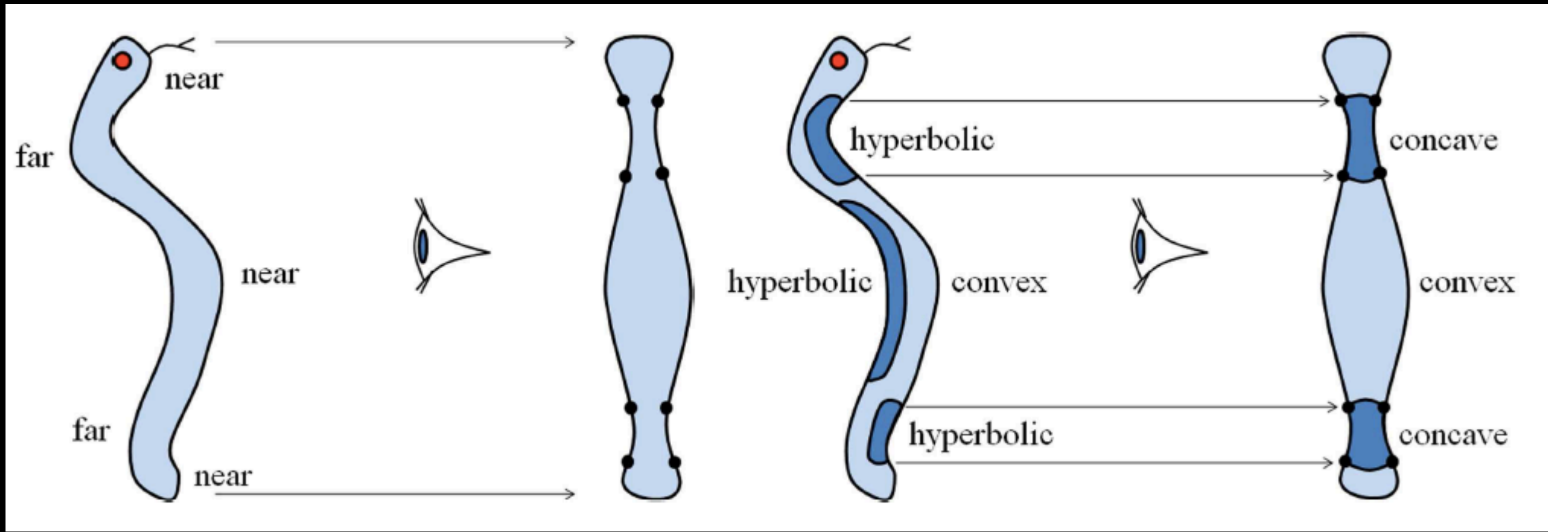
What does the occluding contour tell us about shape?

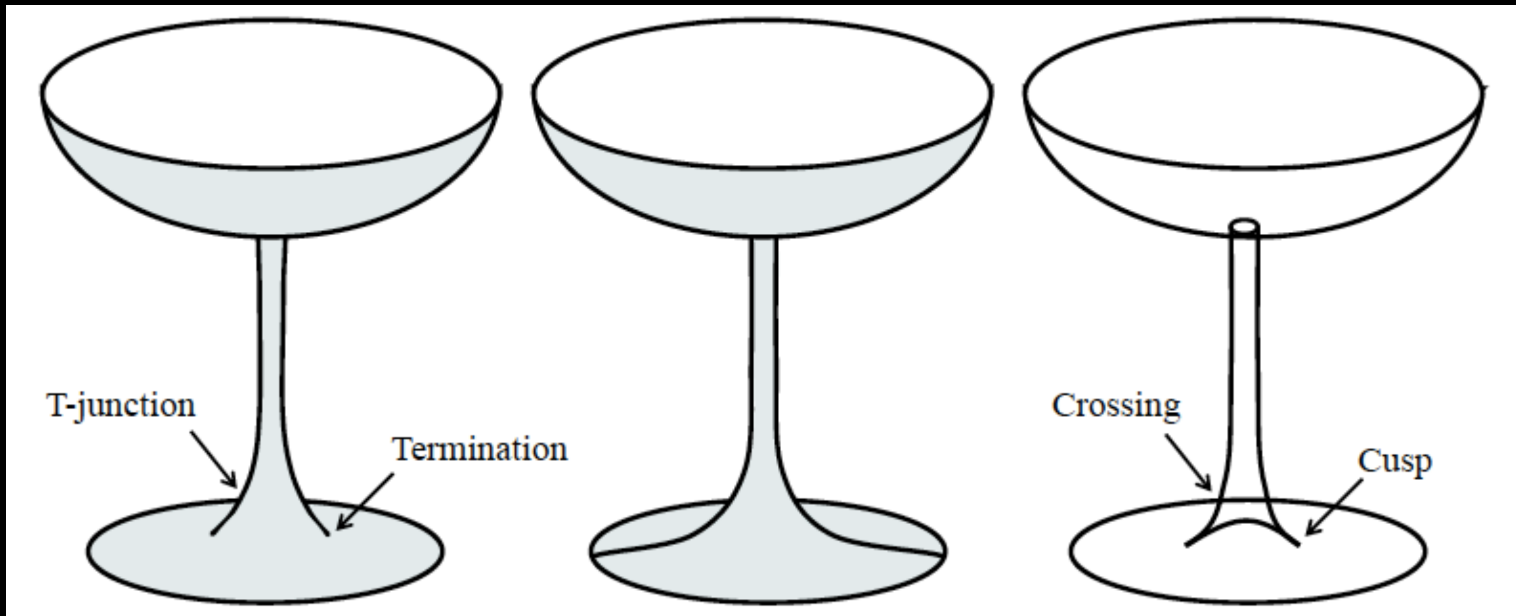




(Marr & Nishihara, 1978; Koenderink, 1984)

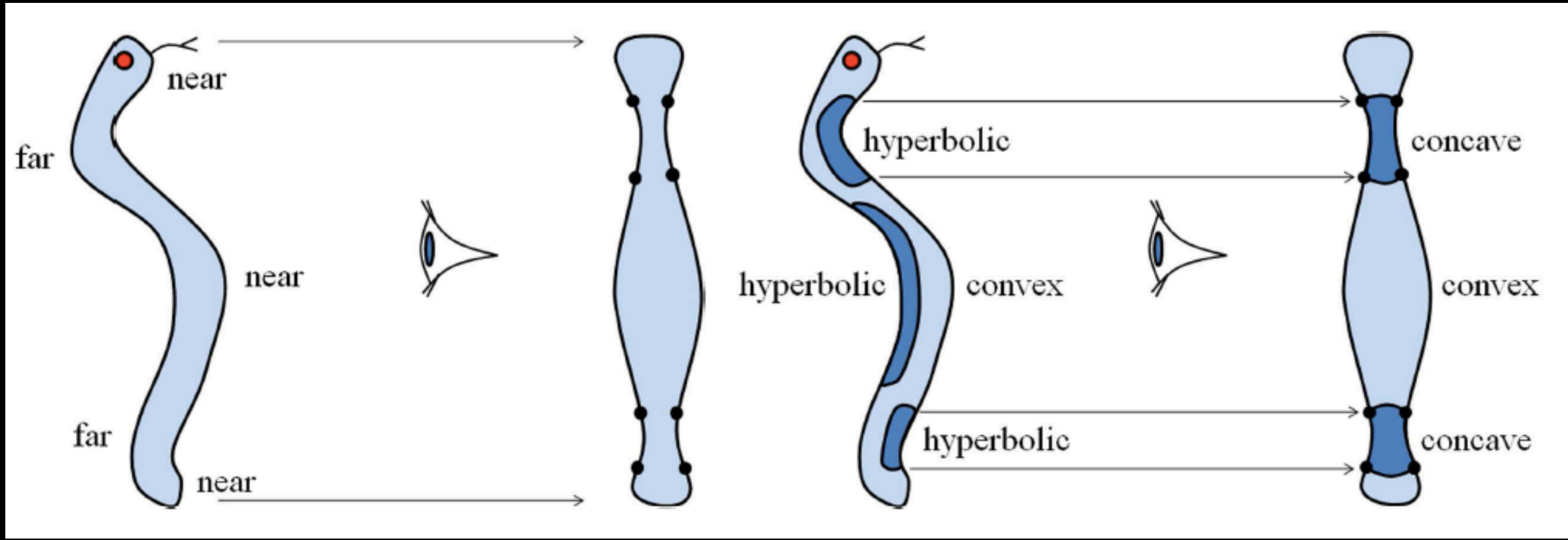
What does the occluding contour tell us about shape?

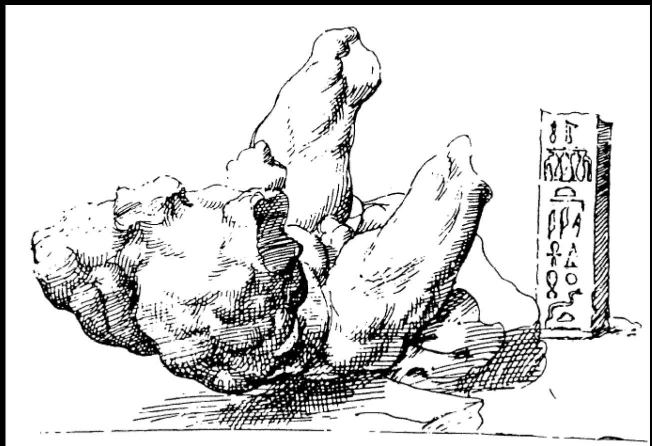




(Marr & Nishihara, 1978; Koenderink, 1984)

What does the occluding contour tell us about shape?



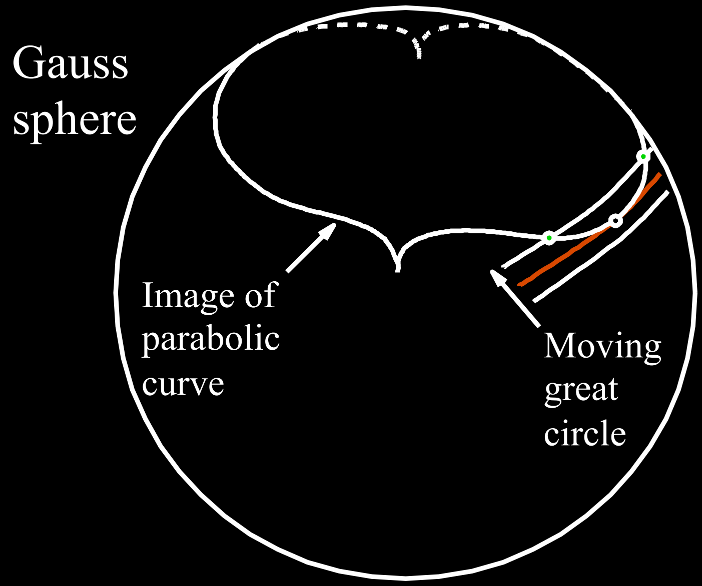


M. Van Hemskeerk

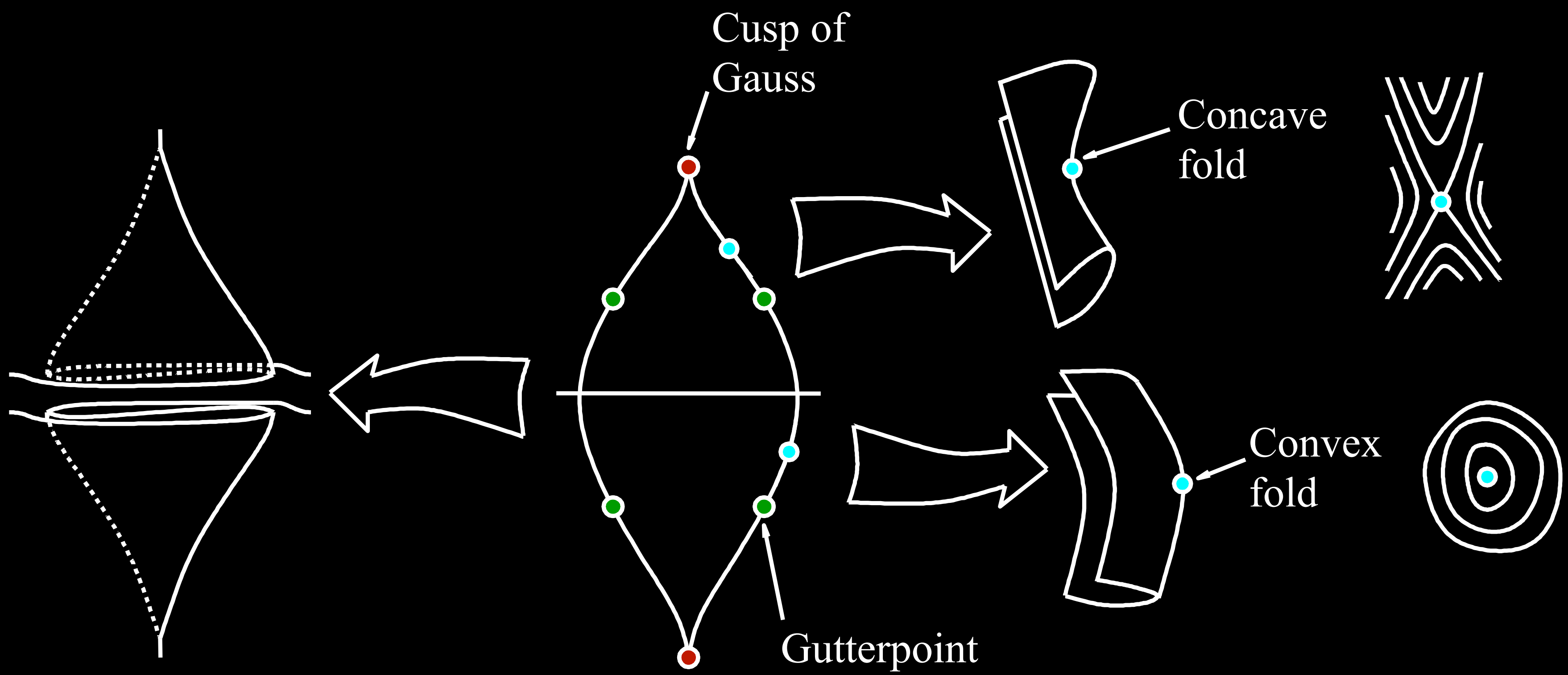
Picasso

Dürer

The Geometry of the Gauss Map



Reprinted from "On Computing Structural Changes in Evolving Surfaces and their Appearance," By S. Pae and J. Ponce, the International Journal of Computer Vision, 43(2):113-131 (2001).

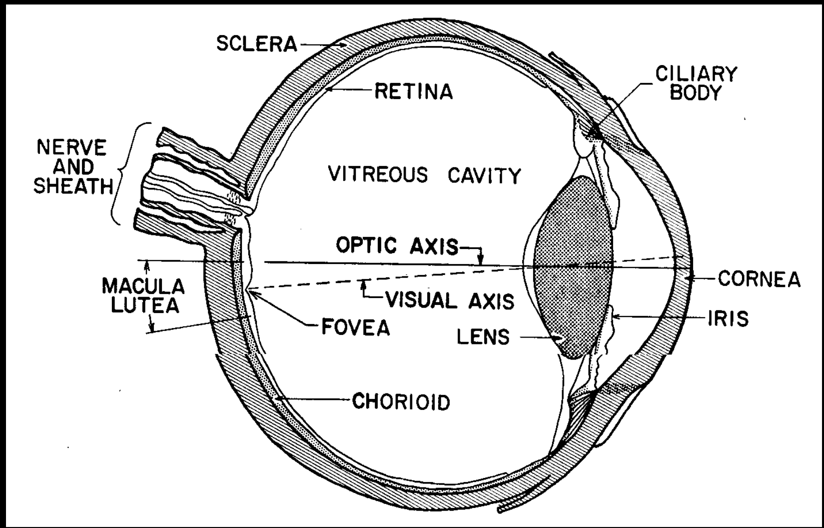


PLAN DU COURS:

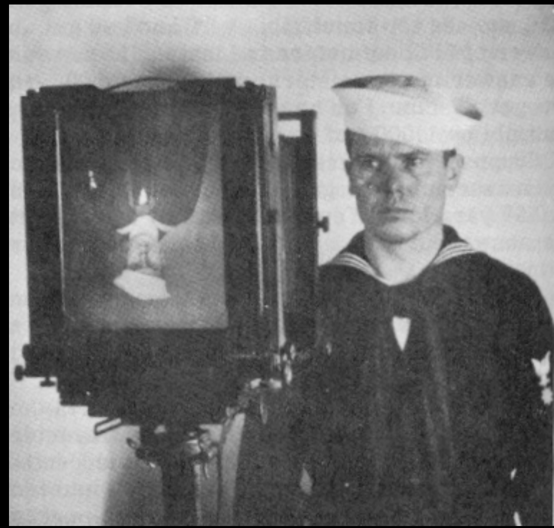
1. Introduction générale
2. Caméras Euclidiennes : perspective centrale, projection parallèle; paramètres intrinsèques et extrinsèques; mires et étalonnage Euclidien.
3. Caméras affines : éléments de géométrie affine; géométrie multi vues, analyse du mouvement.
4. Caméras projectives : éléments de géométrie projective; géométrie multi vues, analyse du mouvement.
5. Étalonnage Euclidien sans mire : la conique absolue de Chasles et ses cousines; analyse du mouvement Euclidien.
6. Caméras purement projectives : éléments de géométrie des droites; caméras linéaires « générales ».
7. Les surfaces Euclidiennes lisses et leurs silhouettes : géométrie différentielle descriptive; le théorème de Koenderink et les graphes d'aspects.

Euclidean Cameras

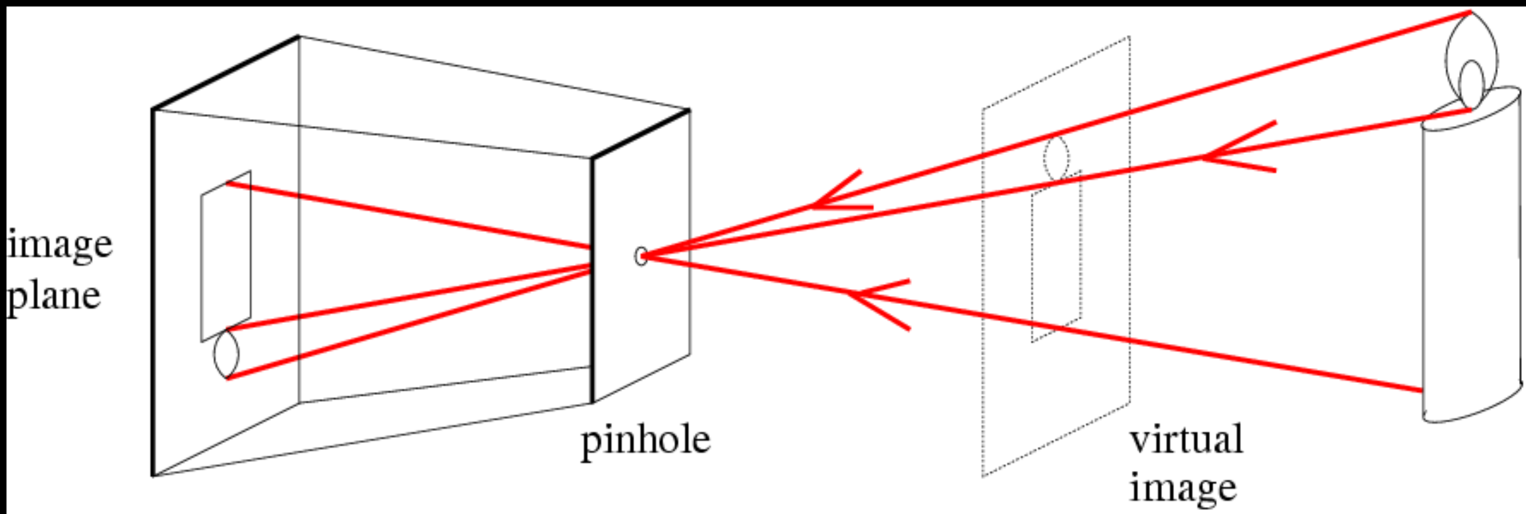
- Pinhole perspective projection
- Orthographic and weak-perspective models
- Non-standard models
- A detour through sensing country
- Intrinsic and extrinsic parameters



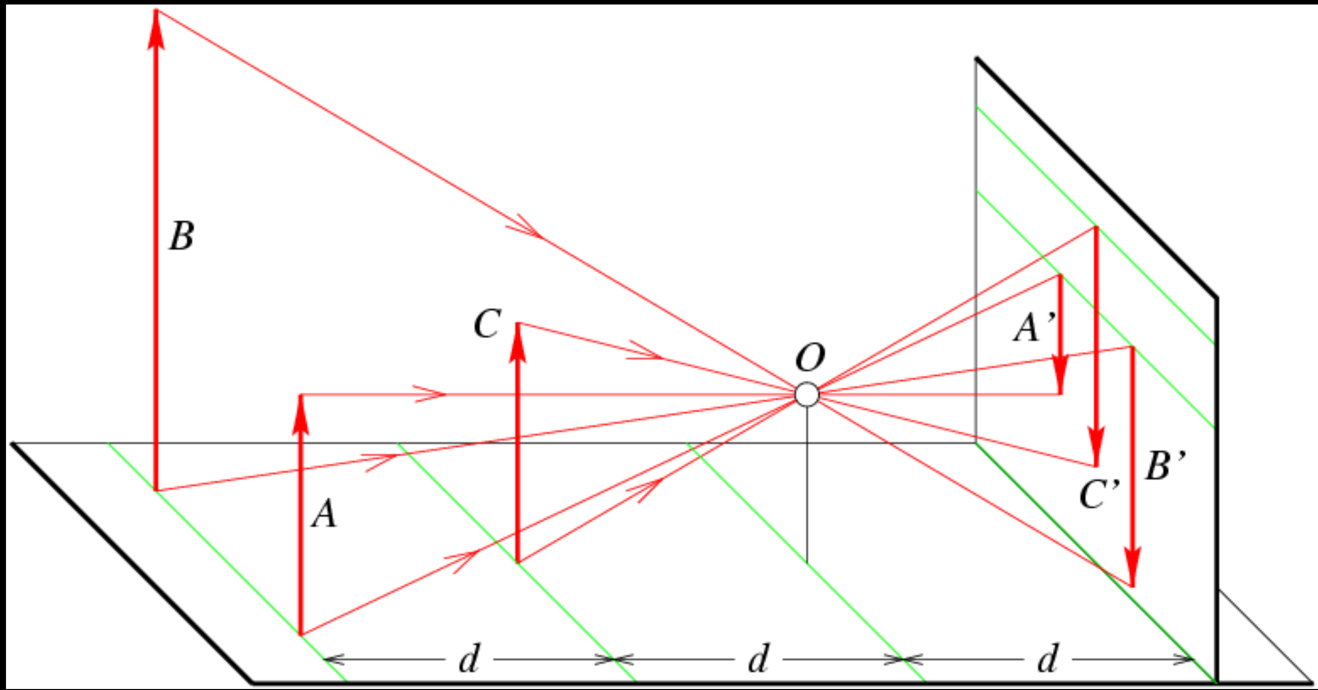
Animal eye: a looonnng time ago.



Photographic camera: Niepce, 1816.



Pinhole perspective projection: Brunelleschi, XVth Century.
Camera obscura: XVIth Century.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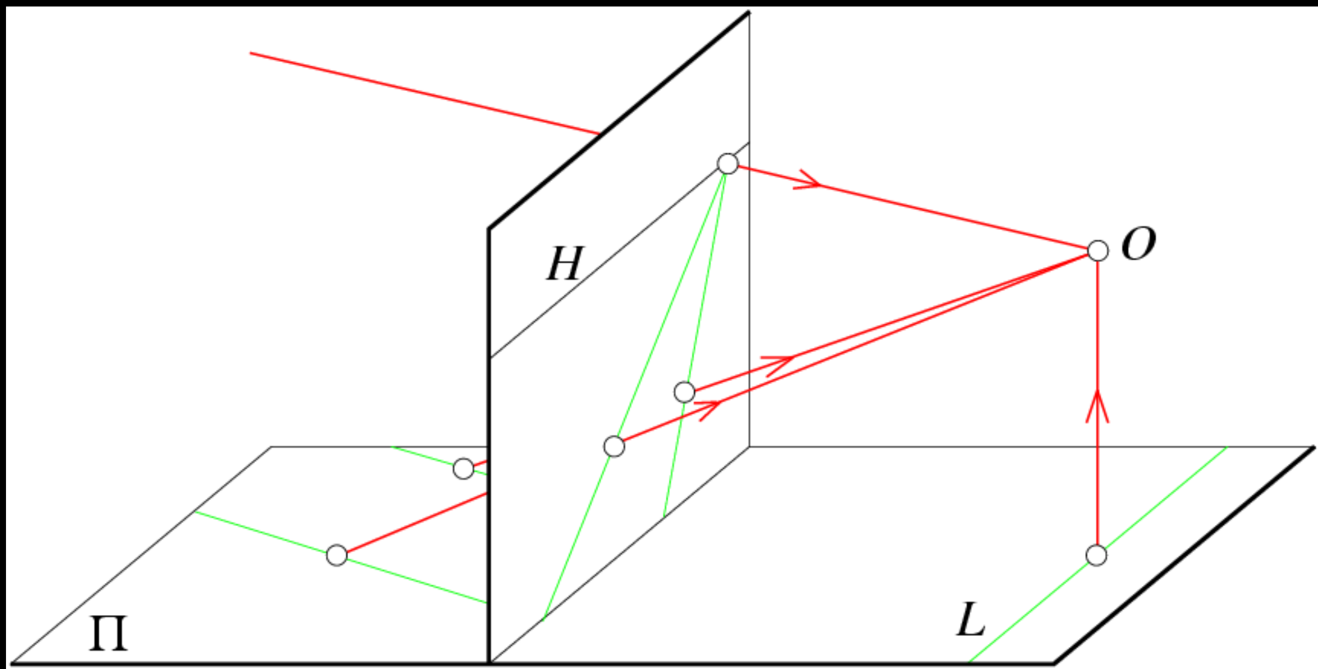
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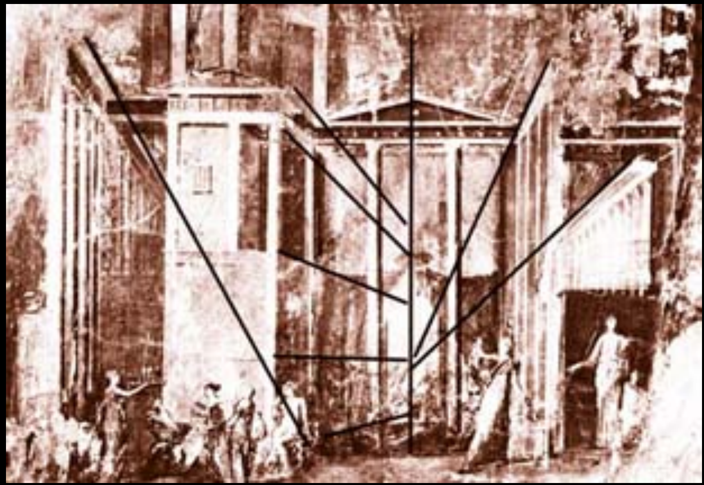
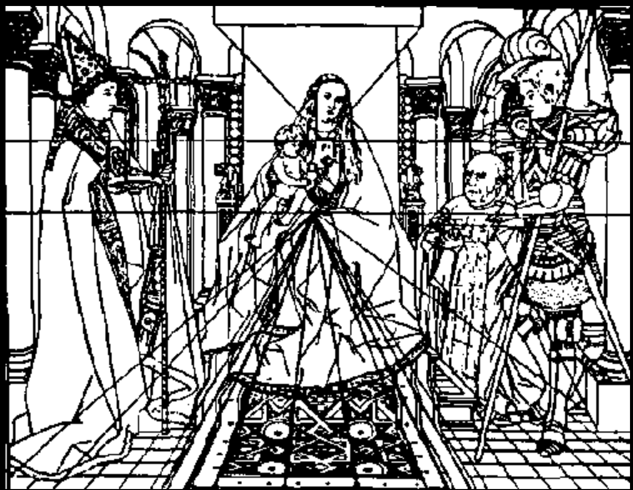
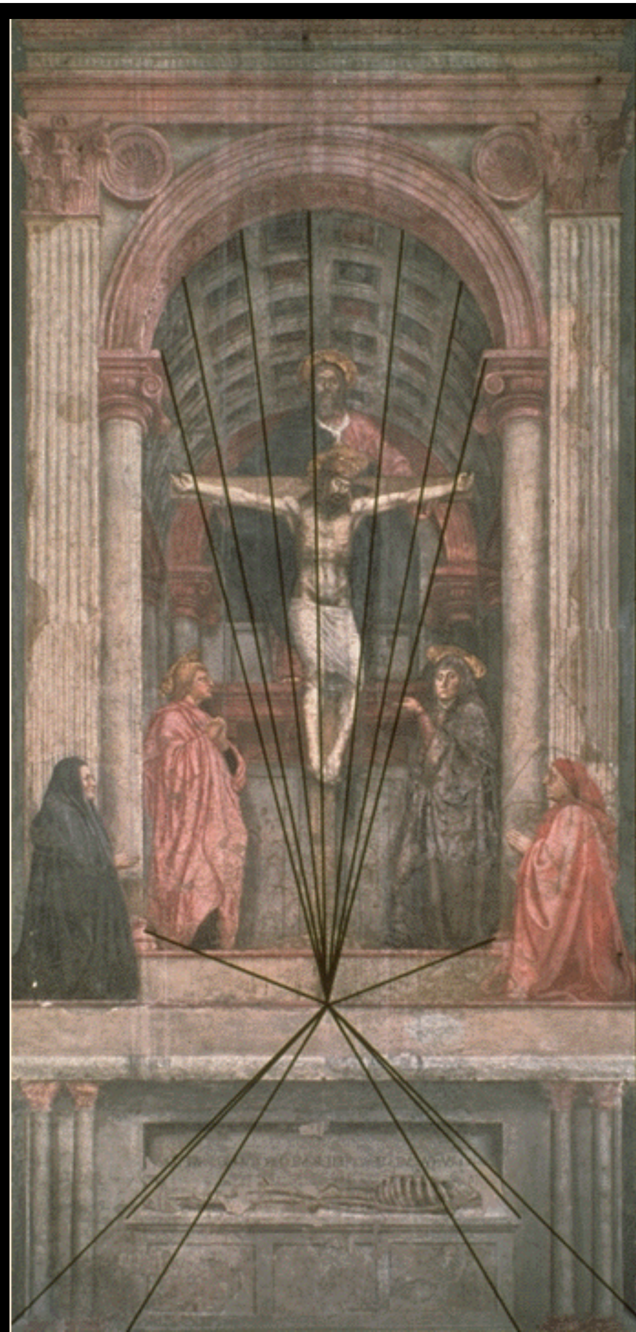


Pompei painting, 2000 years ago.



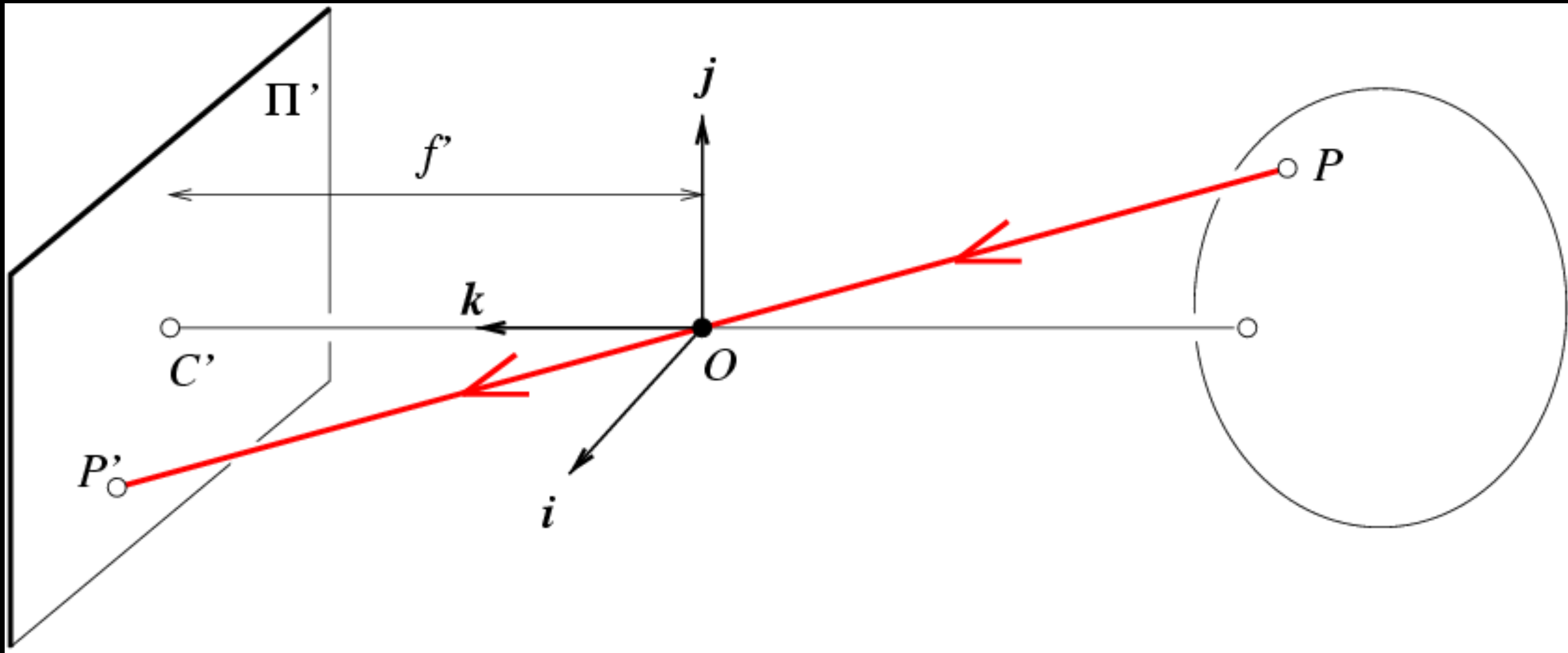
Van Eyk, XIV^th^ Century

Brunelleschi, 1415



Massaccio's Trinity, 1425

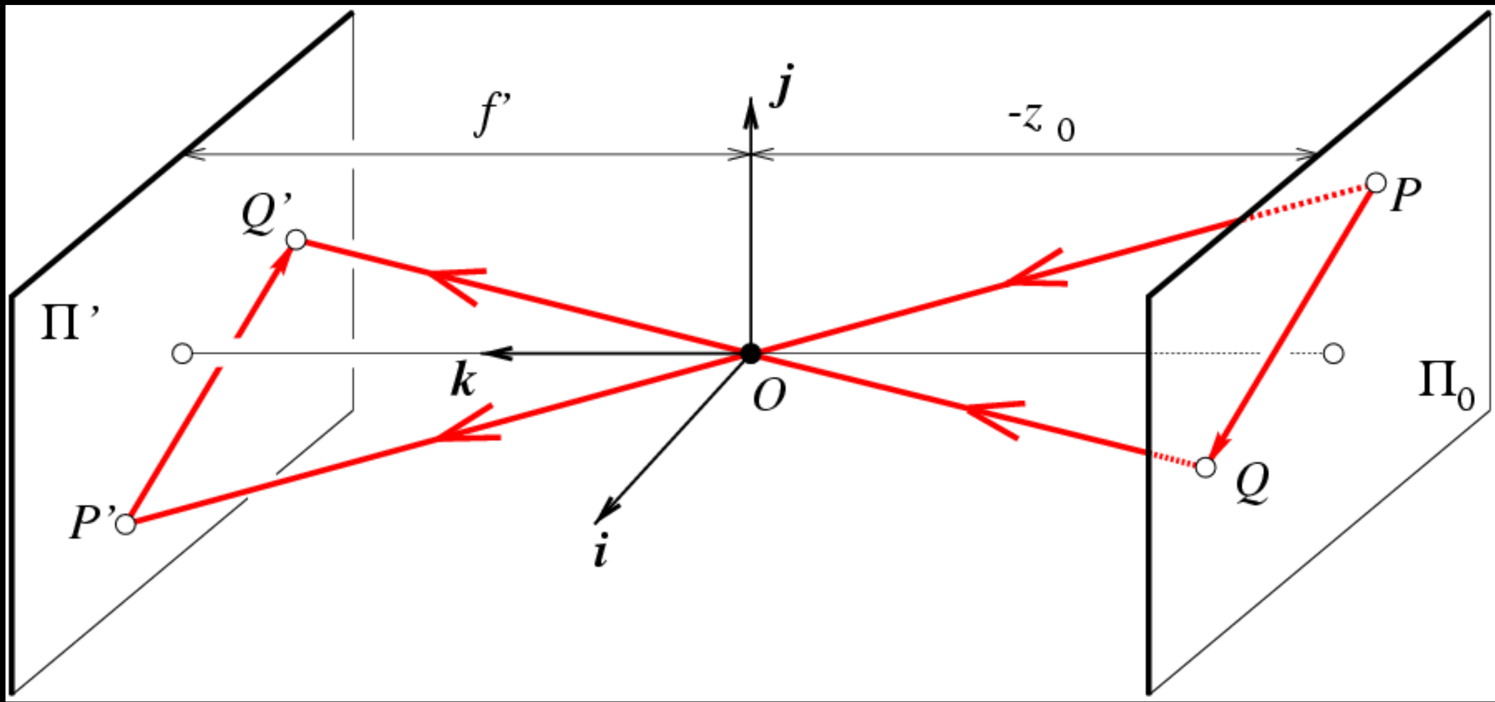
Pinhole Perspective Equation



$$\begin{cases} x' = f' \dfrac{x}{z} \\ y' = f' \dfrac{y}{z} \end{cases}$$

NOTE: z is always negative..

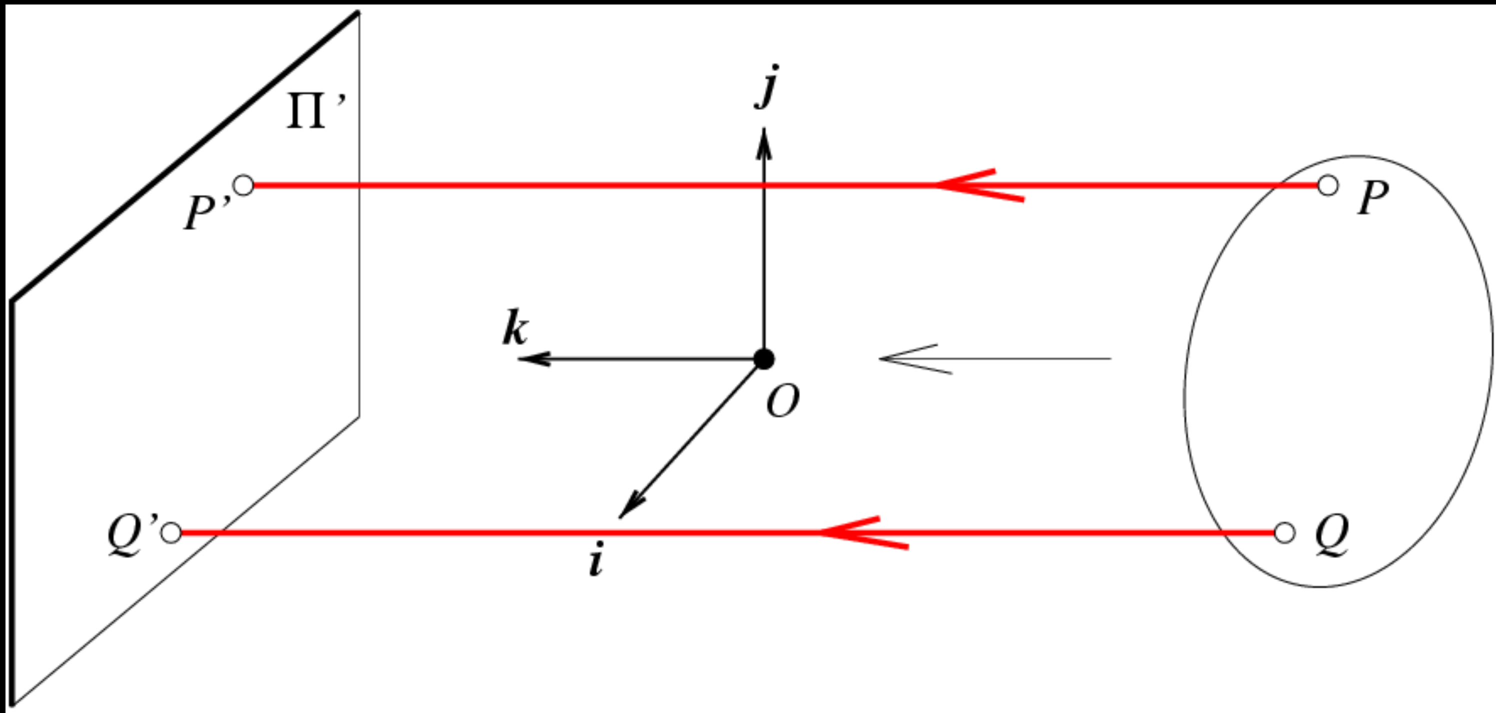
Affine projection models: Weak perspective projection



$$\begin{cases} x' = -mx \\ y' = -my \end{cases} \quad \text{where} \quad m = -\frac{f'}{z_0}$$ is the magnification.

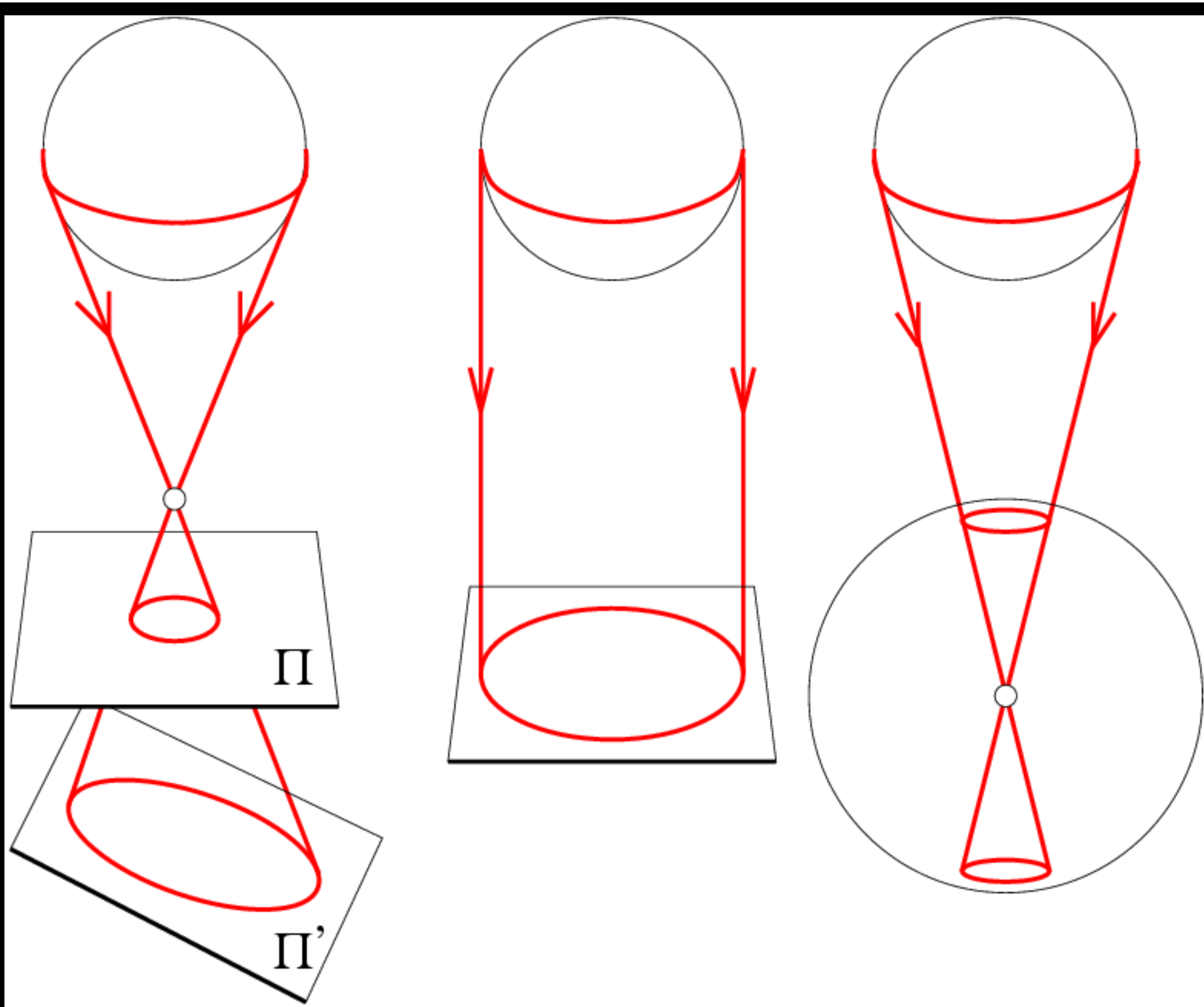
When the scene relief is small compared its distance from the Camera, m can be taken constant: weak perspective projection.

Affine projection models: Orthographic projection



$$\begin{cases} x' = x \\ y' = y \end{cases}$$

When the camera is at a (roughly constant) distance from the scene, take m=-1.



Planar pinhole perspective

Orthographic projection

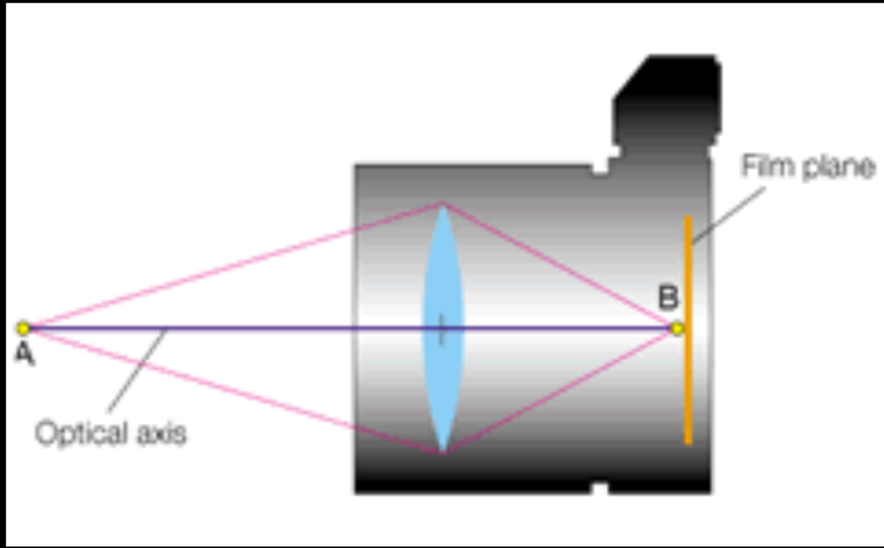
Spherical pinhole perspective

Diffraction effects in pinhole cameras.

Shrinking pinhole size

Use a lens!

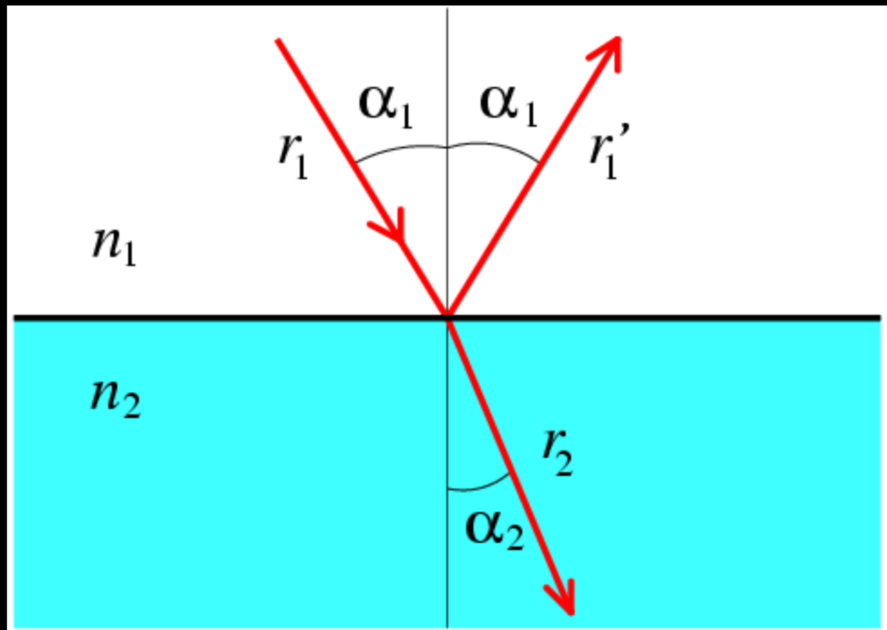
Lenses



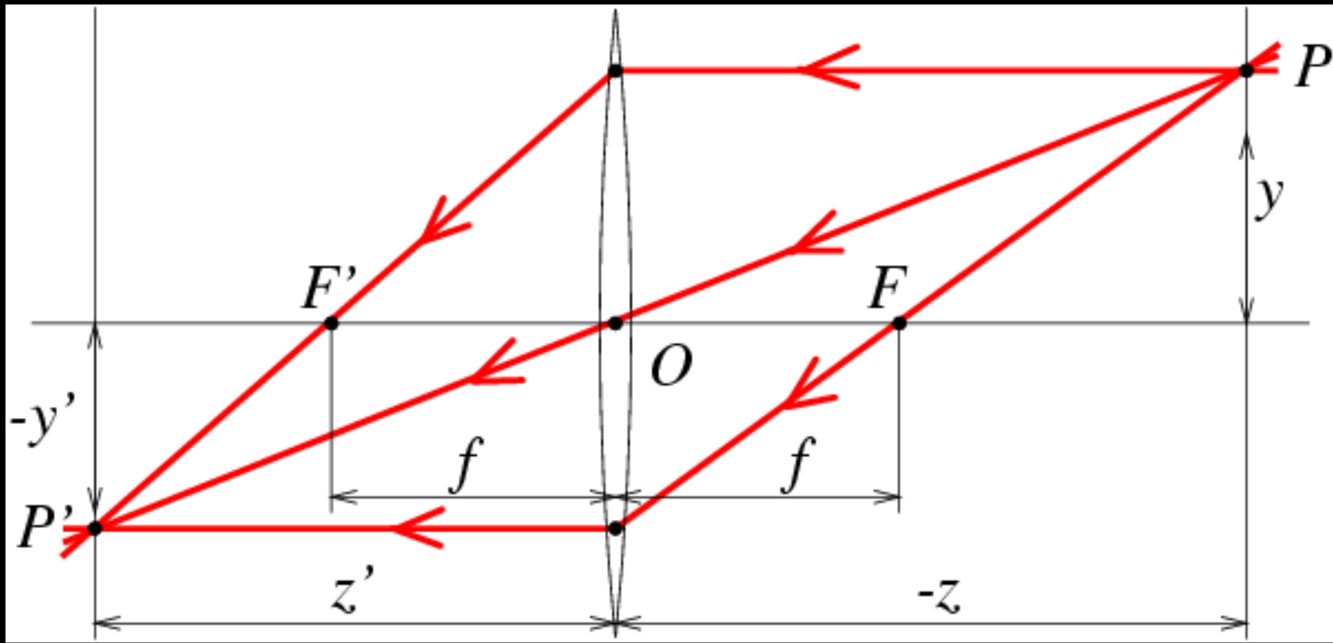
Snell's law

$$n_1 \sin \alpha_1 = n_2 \sin \alpha_2$$

Descartes' law



Thin Lenses



$$\begin{cases} x' = z'\dfrac{x}{z} \\ y' = z'\dfrac{y}{z} \end{cases}$$

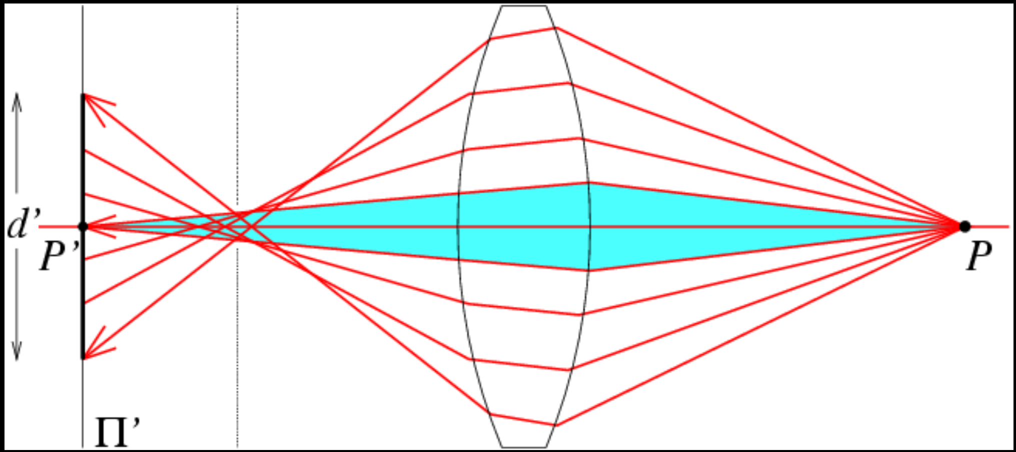
where

$$\frac{1}{z'} - \frac{1}{z} = \frac{1}{f}$$

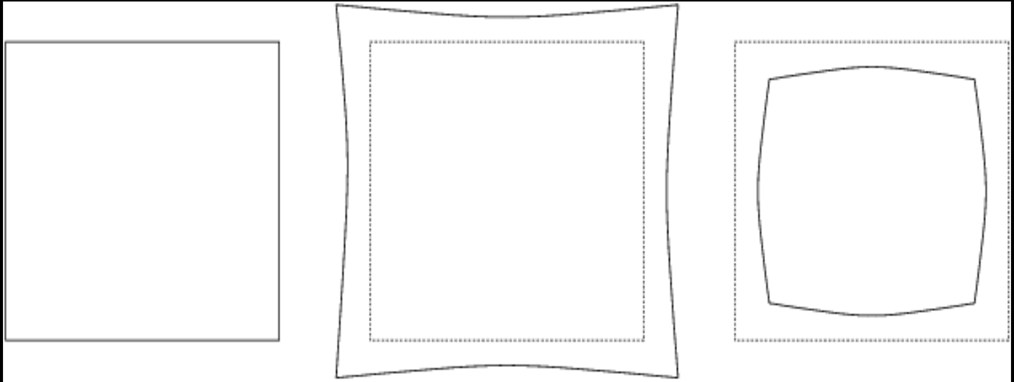
and

$$f = \frac{R}{2(n-1)}$$

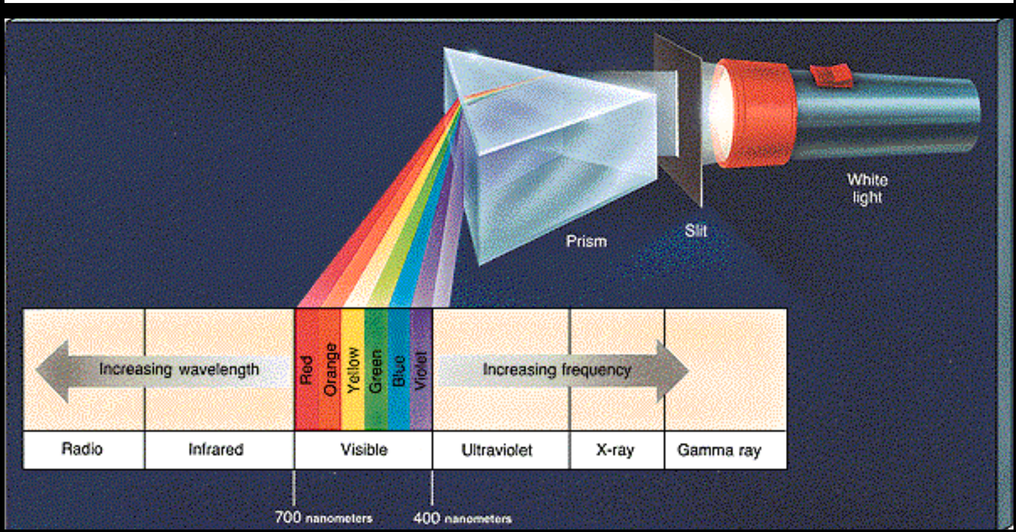
Spherical Aberration



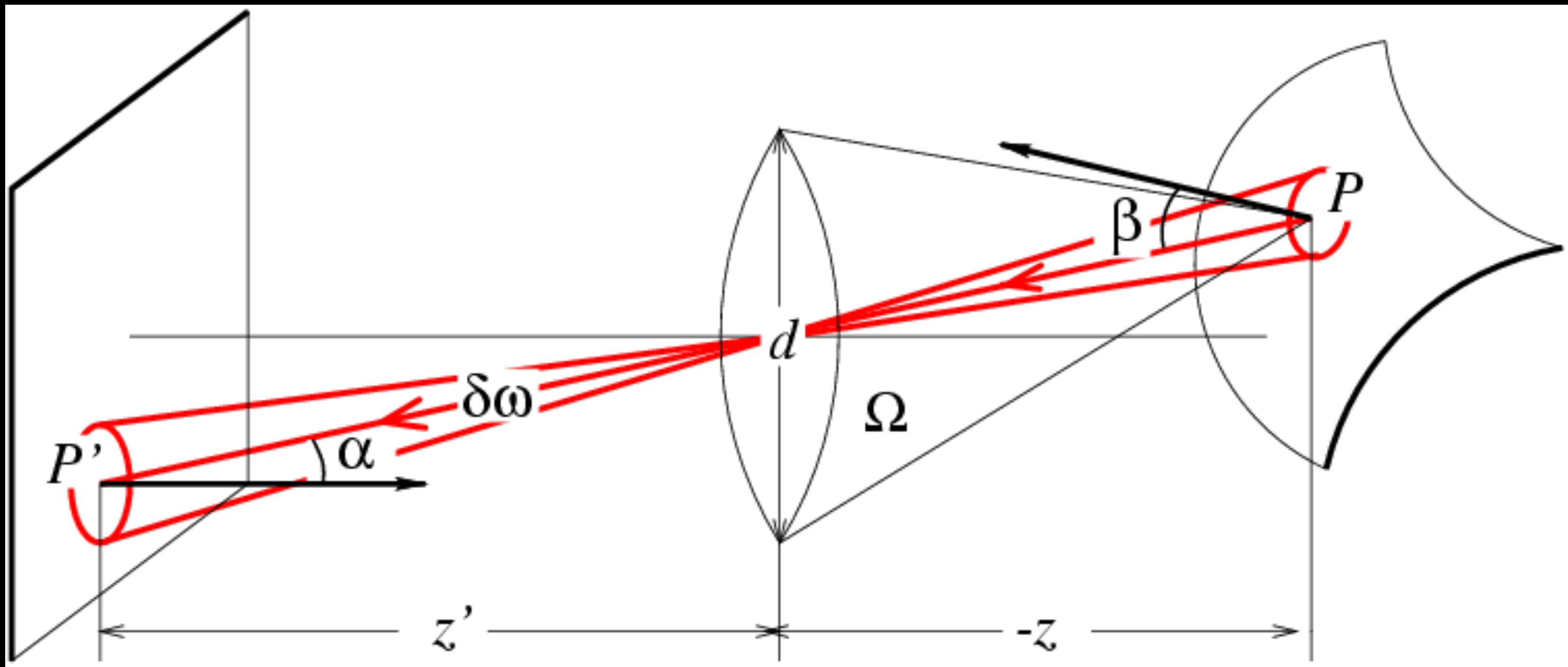
Distortion



Chromatic Aberration

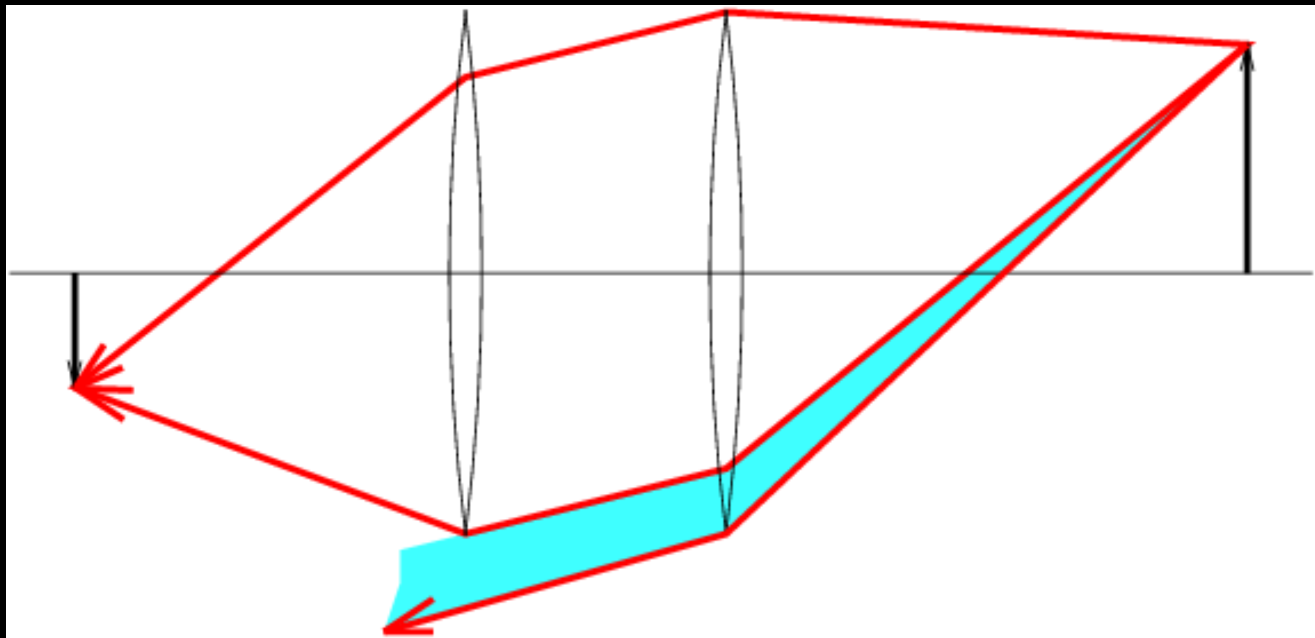


A compound lens



$$E = (\Pi/4)\left[\,(d/z')^2 \cos^4\alpha\,\right] L$$

Vignetting

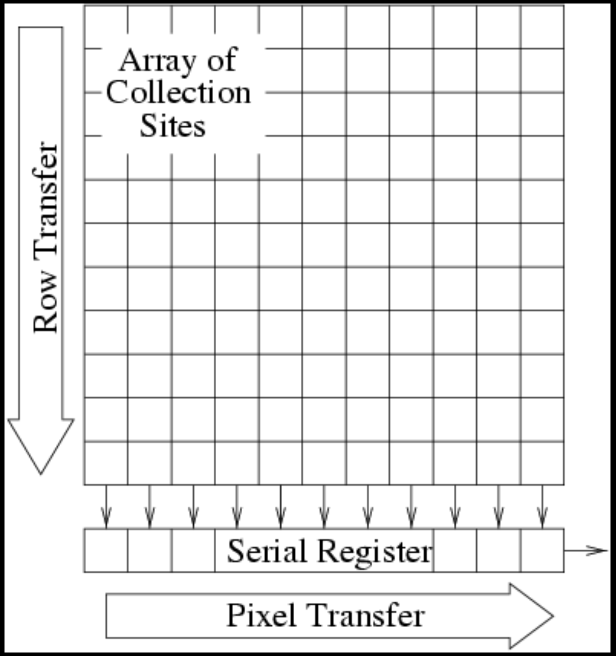


Challenge: Illumination – What is wrong with these pictures?

Photography (Niepce,
"La Table Servie," 1822)

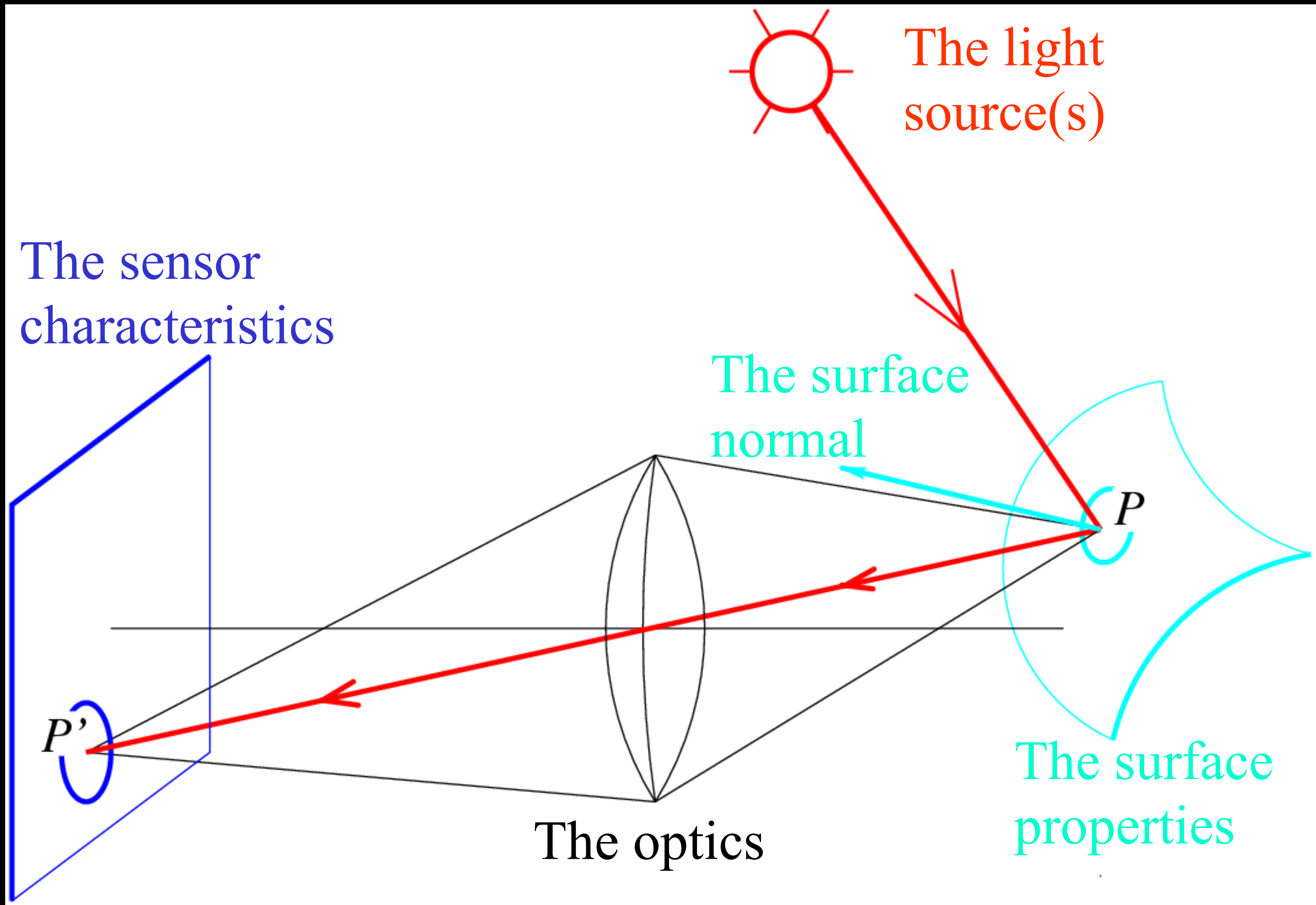
Milestones:

- Daguerréotypes (1839)
- Photographic Film (Eastman, 1889)
- Cinema (Lumière Brothers, 1895)
- Color Photography (Lumière Brothers, 1908)
- Television (Baird, Farnsworth, Zworykin, 1920s)



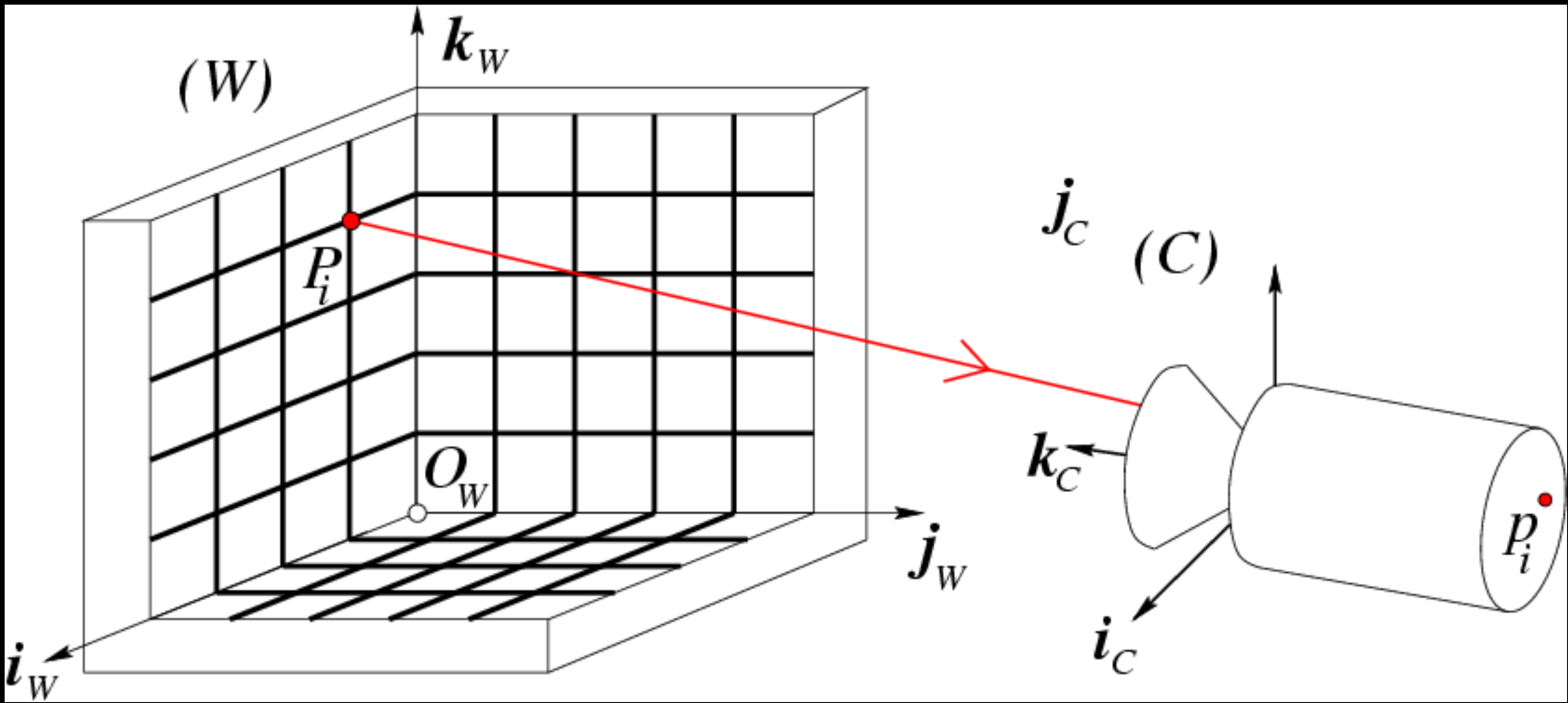
CCD Devices (1970)

Image Formation: Radiometry



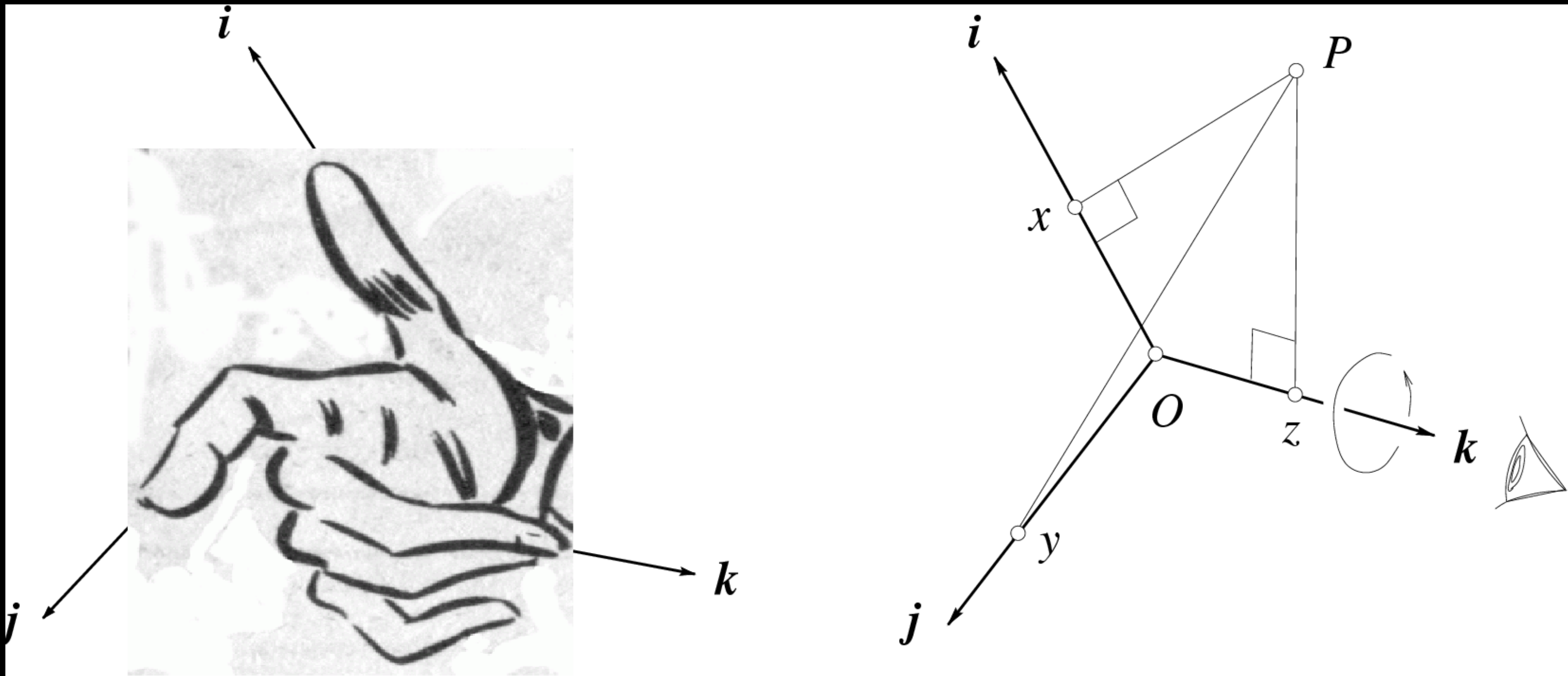
What determines the brightness of an image pixel?

Quantitative Measurements and Calibration



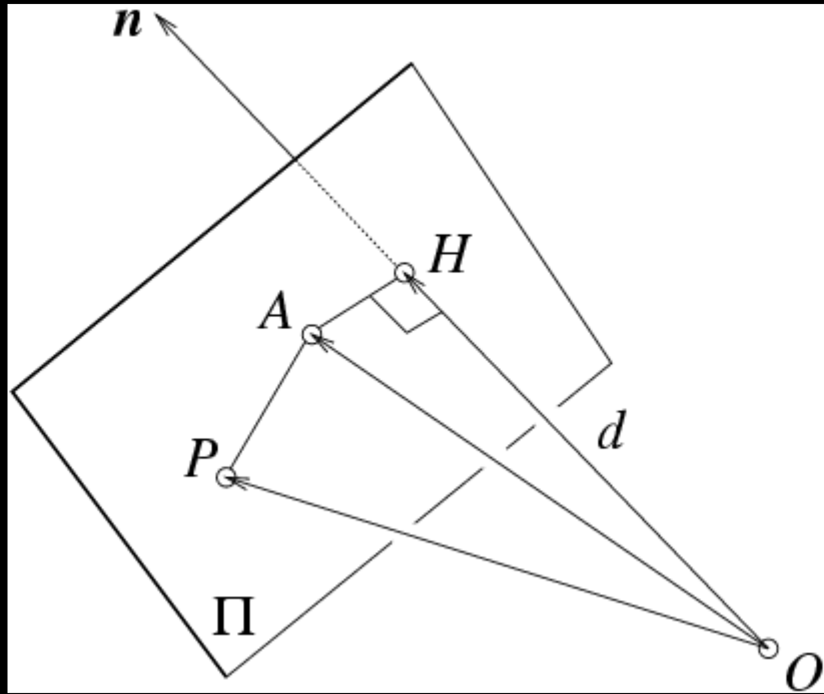
Euclidean Geometry

Euclidean Coordinate Systems



$$\begin{cases} x = \overrightarrow{OP}.\mathbf{i} \\ y = \overrightarrow{OP}.\mathbf{j} \\ z = \overrightarrow{OP}.\mathbf{k} \end{cases} \Leftrightarrow \quad \overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad \Leftrightarrow \quad \mathbf{P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

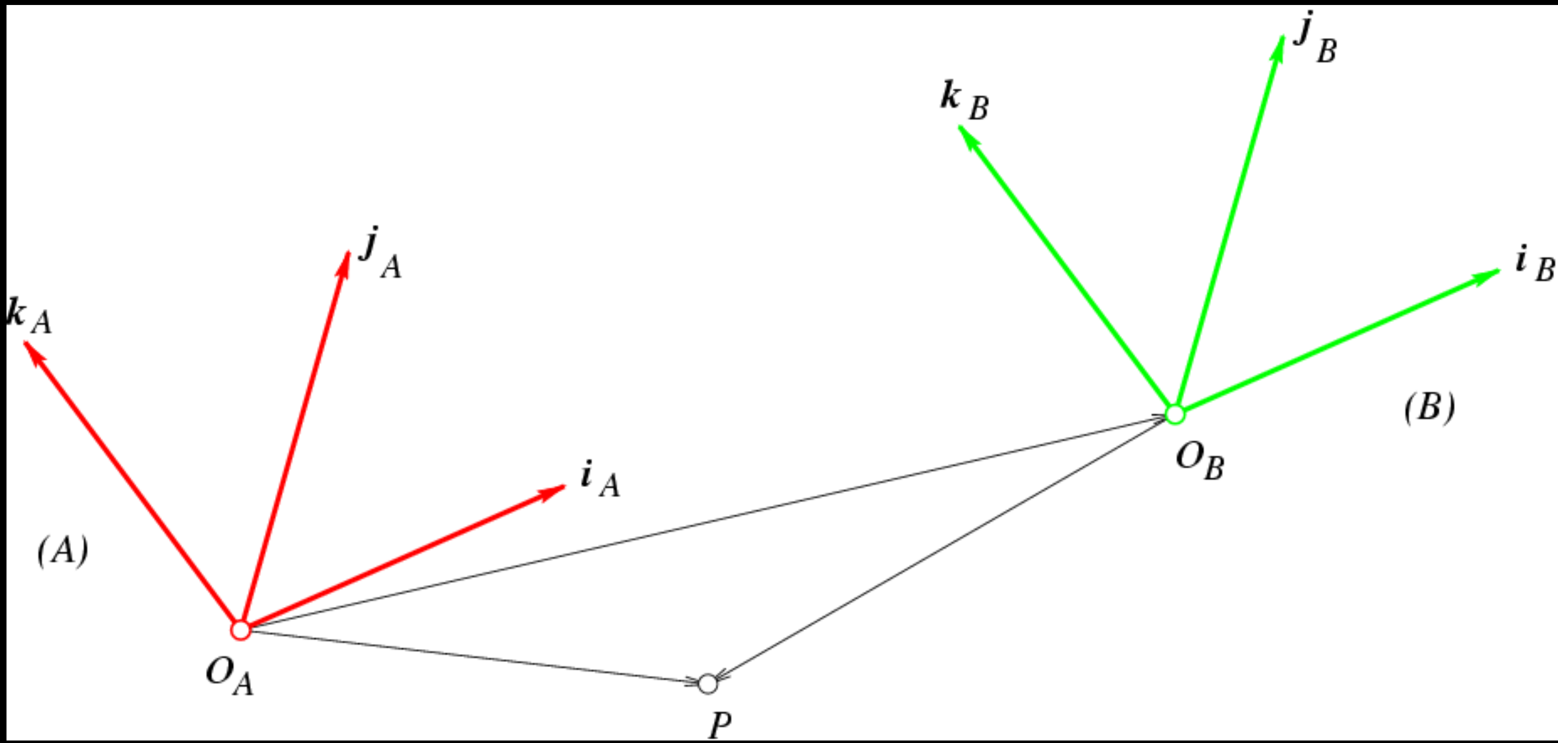
Planes



$$\overrightarrow{AP}.\mathbf{n} = 0 \Leftrightarrow ax + by + cz - d = 0 \Leftrightarrow \mathbf{\Pi}.\mathbf{P} = 0$$

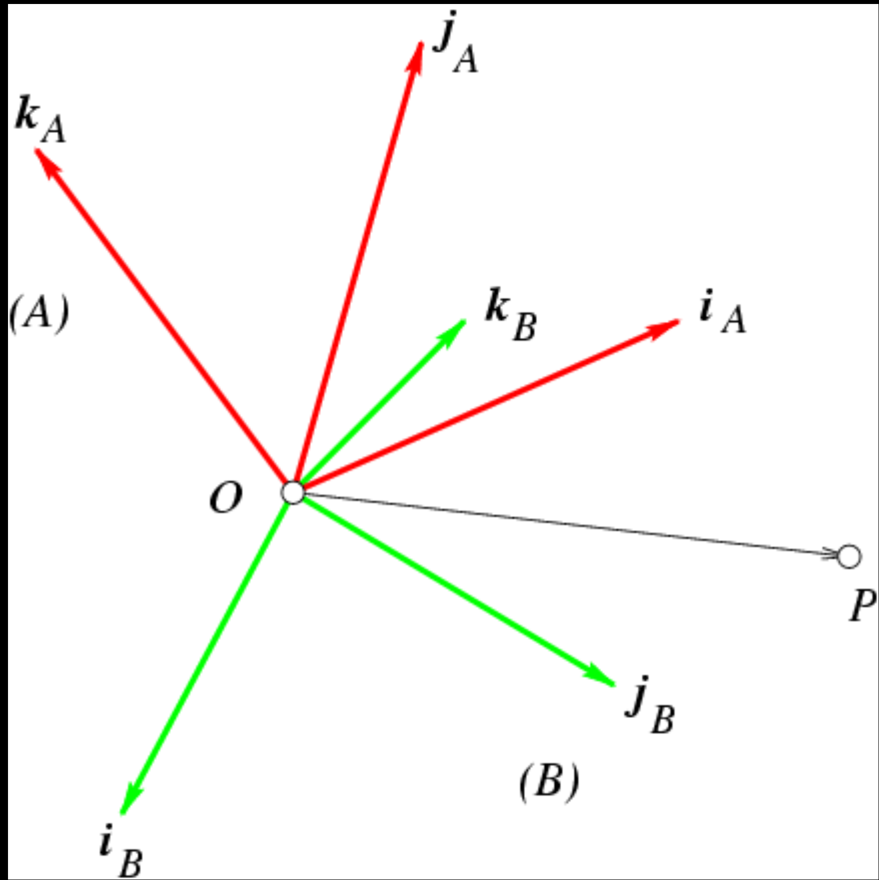
where $\mathbf{\Pi} = \begin{bmatrix} a \\ b \\ c \\ -d \end{bmatrix}$ and $\mathbf{P} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

Coordinate Changes: Pure Translations



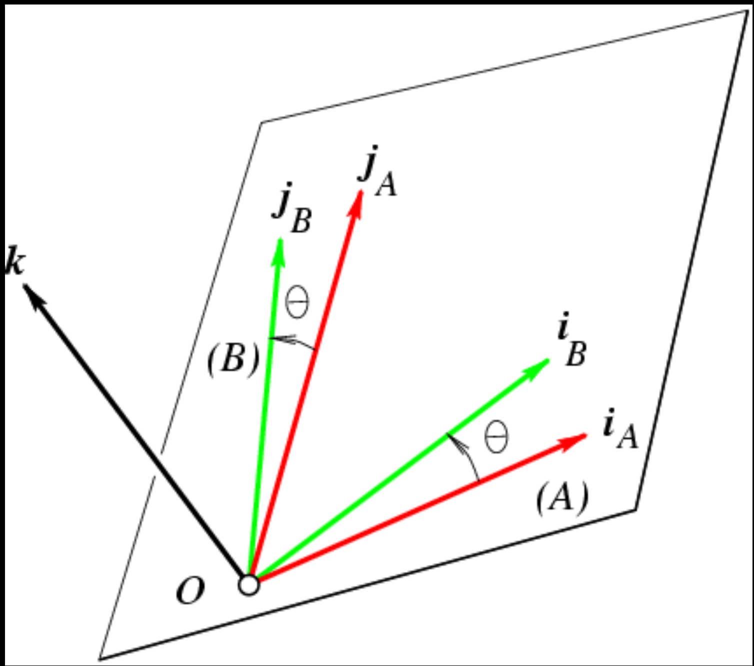
$$\overrightarrow{O_B P} = \overrightarrow{O_B O_A} + \overrightarrow{O_A P} \Leftrightarrow {}^B P = {}^A P + {}^B O_A$$

Coordinate Changes: Pure Rotations

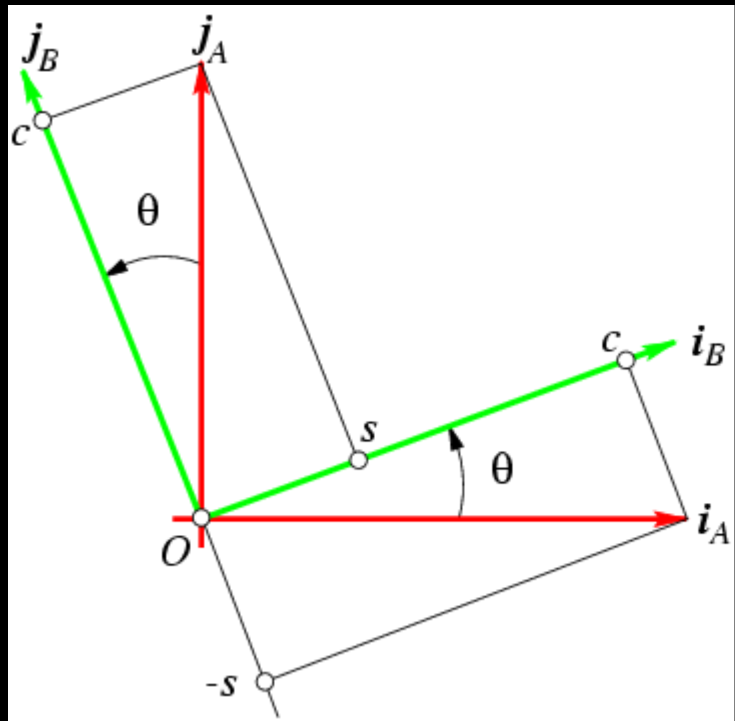


$$
{}^B_A R = \begin{bmatrix} \mathbf{i}_A.\mathbf{i}_B & \mathbf{j}_A.\mathbf{i}_B & \mathbf{k}_A.\mathbf{i}_B \\ \mathbf{i}_A.\mathbf{j}_B & \mathbf{j}_A.\mathbf{j}_B & \mathbf{k}_A.\mathbf{j}_B \\ \mathbf{i}_A.\mathbf{k}_B & \mathbf{j}_A.\mathbf{k}_B & \mathbf{k}_A.\mathbf{k}_B \end{bmatrix}
$$

Coordinate Changes: Rotations about the z Axis



$$
{}^{B}_{A}R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$



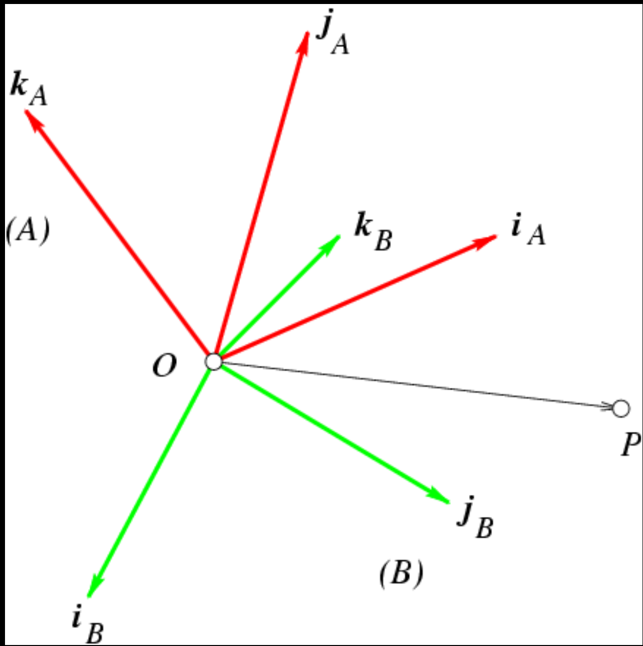
A rotation matrix is characterized by the following properties:

- Its inverse is equal to its transpose, and
- its determinant is equal to 1.

Or equivalently:

- Its rows (or columns) form a right-handed orthonormal coordinate system.

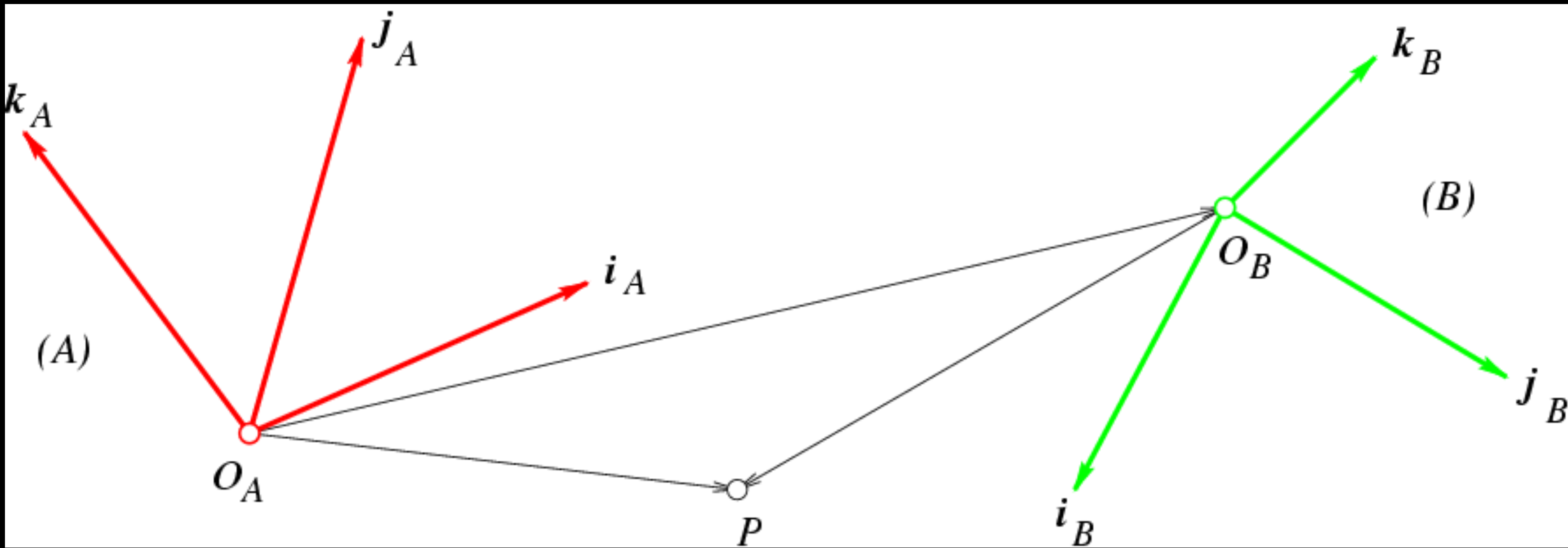
Coordinate Changes: Pure Rotations



$$\overrightarrow{OP} = \begin{bmatrix} \mathbf{i}_A & \mathbf{j}_A & \mathbf{k}_A \end{bmatrix} \begin{bmatrix} {}^{A}x \\ {}^{A}y \\ {}^{A}z \end{bmatrix} = \begin{bmatrix} \mathbf{i}_B & \mathbf{j}_B & \mathbf{k}_B \end{bmatrix} \begin{bmatrix} {}^{B}x \\ {}^{B}y \\ {}^{B}z \end{bmatrix}$$

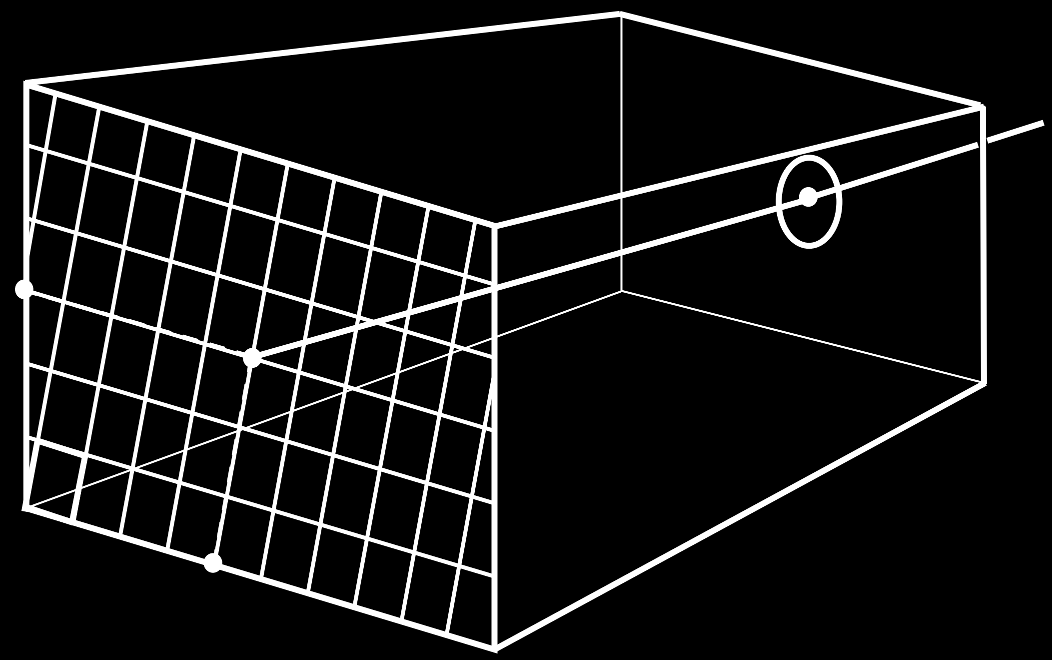
$$\Rightarrow \quad {}^{B}P = {}^{B}_{A}R\,{}^{A}P$$

Coordinate Changes: Rigid Transformations

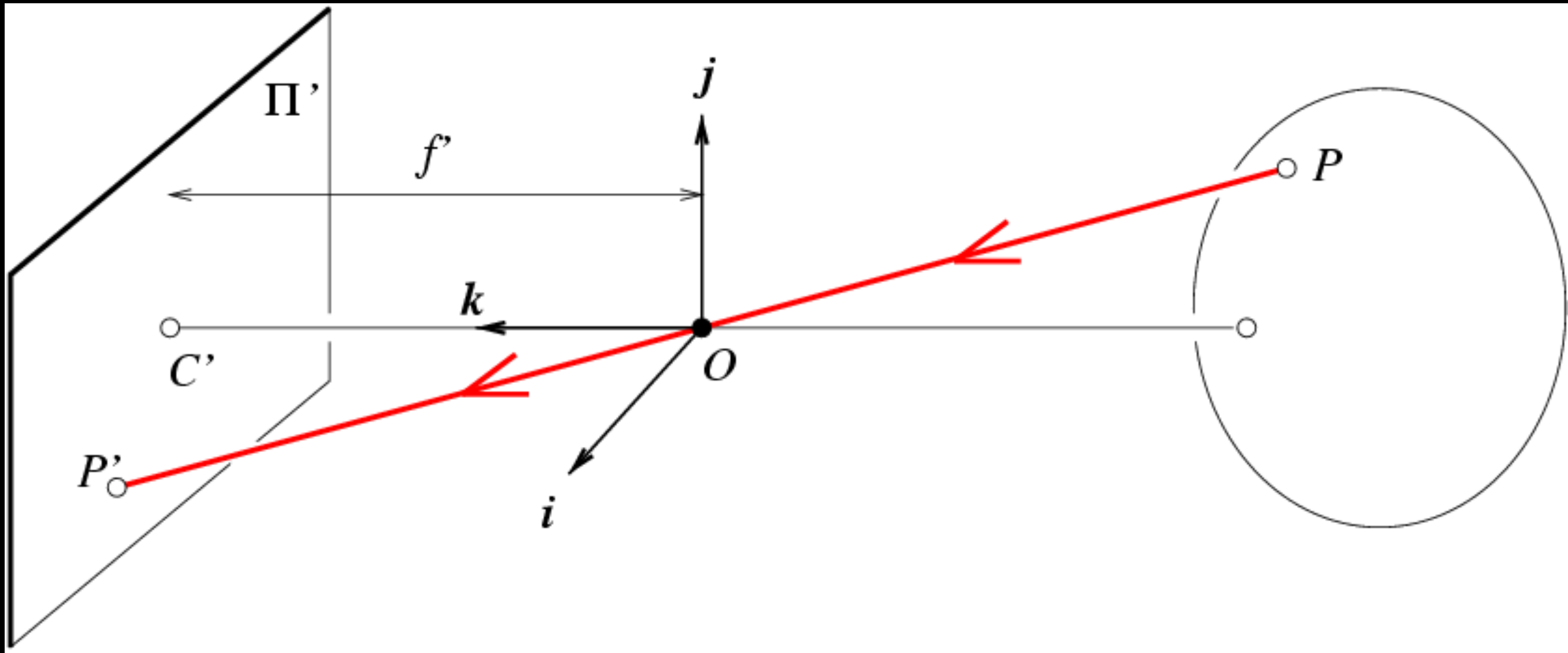


$$\begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R & {}^{B}O_{A} \\ \mathbf{0}^{T} & 1 \end{bmatrix} \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R\,{}^{A}P + {}^{B}O_{A} \\ 1 \end{bmatrix} = {}^{B}_{A}T \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix}$$

Cameras and their parameters



Pinhole Perspective Equation



$$\begin{cases} x' = f'\dfrac{x}{z} \\ y' = f'\dfrac{y}{z} \end{cases}$$

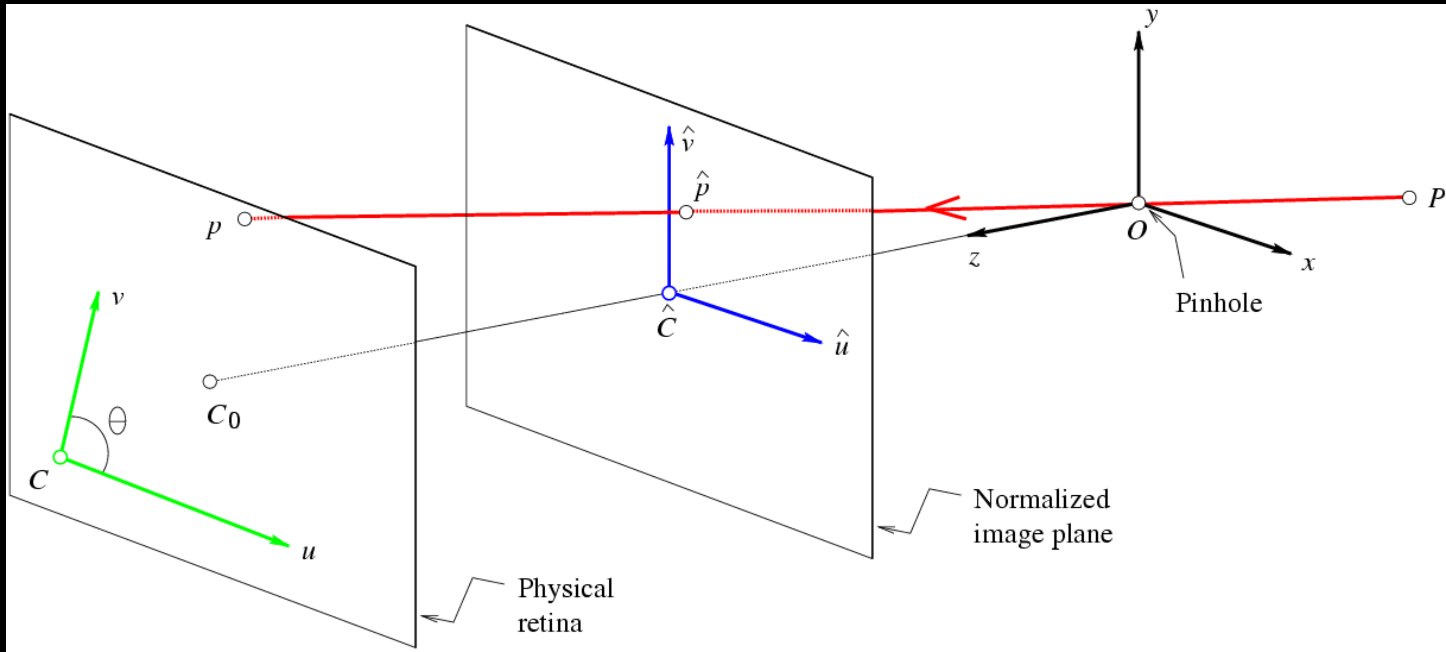
The Intrinsic Parameters of a Camera

Units:

k, l : pixel/m

f : m

α, β : pixel



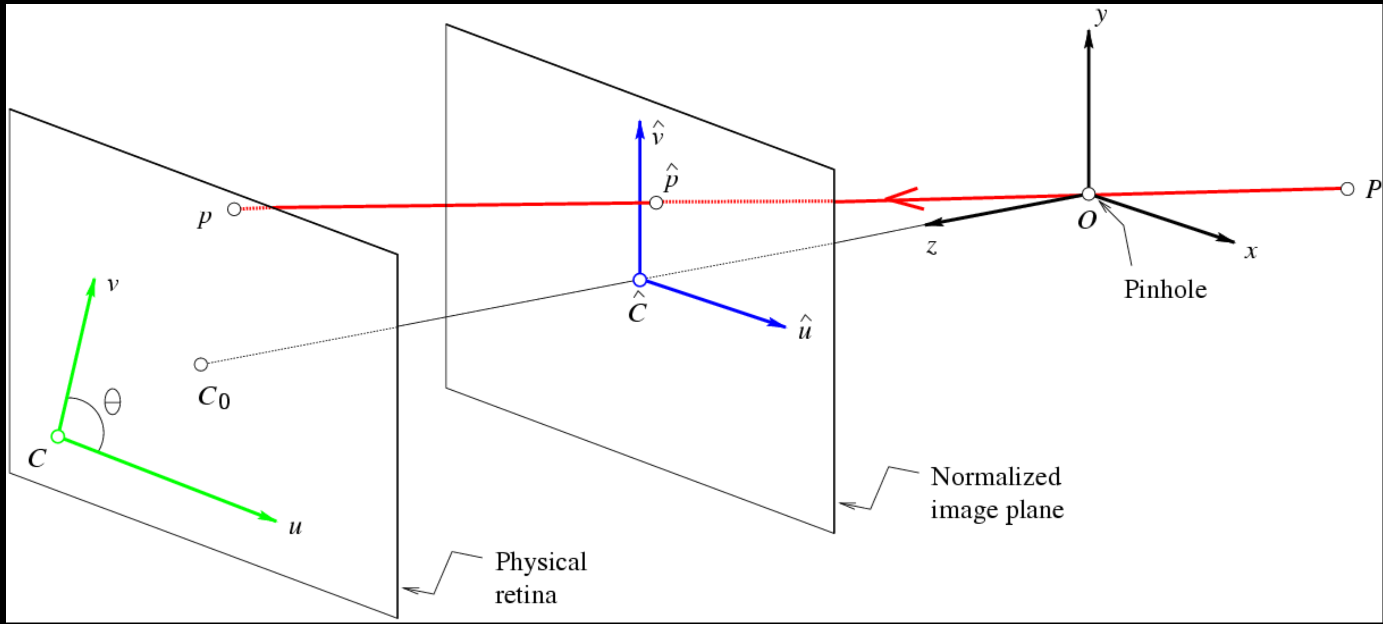
$$\begin{cases} \hat{u} = \dfrac{x}{z} \\ \hat{v} = \dfrac{y}{z} \end{cases} \iff \hat{\boldsymbol{p}} = \frac{1}{z} \left(\text{Id} \quad \mathbf{0} \right) \begin{pmatrix} \boldsymbol{P} \\ 1 \end{pmatrix}$$

Normalized Image Coordinates

Physical Image Coordinates

$$\begin{cases} u = kf\dfrac{x}{z} \\ v = lf\dfrac{y}{z} \end{cases} \rightarrow \begin{cases} u = \alpha\dfrac{x}{z} + u_0 \\ v = \beta\dfrac{y}{z} + v_0 \end{cases} \rightarrow \begin{cases} u = \alpha\dfrac{x}{z} - \alpha\cot\theta\dfrac{y}{z} + u_0 \\ v = \dfrac{\beta}{\sin\theta}\dfrac{y}{z} + v_0 \end{cases}$$

The Intrinsic Parameters of a Camera



Calibration Matrix

$$\boldsymbol{p} = \mathcal{K}\hat{\boldsymbol{p}}, \quad \text{where} \quad \boldsymbol{p} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{K} \stackrel{\text{def}}{=} \begin{pmatrix} \alpha & -\alpha\cot\theta & u_0 \\ 0 & \dfrac{\beta}{\sin\theta} & v_0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Perspective Projection Equation

$$\boldsymbol{p} = \frac{1}{z}\mathcal{M}\boldsymbol{P}, \quad \text{where} \quad \mathcal{M} \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{K} & \boldsymbol{0} \end{pmatrix}$$

The Extrinsic Parameters of a Camera

- When the camera frame (C) is different from the world frame (W),

$$\begin{pmatrix} {}^C P \\ 1 \end{pmatrix} = \begin{pmatrix} {}^C_W \mathcal{R} & {}^C O_W \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} {}^W P \\ 1 \end{pmatrix}.$$

- Thus,

$$\boldsymbol{p} = \frac{1}{z} \mathcal{M} \boldsymbol{P}, \quad \text{where} \quad \begin{cases} \mathcal{M} = \mathcal{K} \begin{pmatrix} \mathcal{R} & \boldsymbol{t} \end{pmatrix}, \\ \mathcal{R} = {}^C_W \mathcal{R}, \\ \boldsymbol{t} = {}^C O_W, \\ \boldsymbol{P} = \begin{pmatrix} {}^W P \\ 1 \end{pmatrix}. \end{cases}$$

$$p \approx M\, P$$

- Note: z is *not* independent of $\mathcal{M}$ and $\boldsymbol{P}$:

$$\mathcal{M} = \begin{pmatrix} \boldsymbol{m}_1^T \\ \boldsymbol{m}_2^T \\ \boldsymbol{m}_3^T \end{pmatrix} \Longrightarrow z = \boldsymbol{m}_3 \cdot \boldsymbol{P}, \quad \text{or} \quad \begin{cases} u = \dfrac{\boldsymbol{m}_1 \cdot \boldsymbol{P}}{\boldsymbol{m}_3 \cdot \boldsymbol{P}}, \\ v = \dfrac{\boldsymbol{m}_2 \cdot \boldsymbol{P}}{\boldsymbol{m}_3 \cdot \boldsymbol{P}}. \end{cases}$$

Explicit Form of the Projection Matrix

$$\mathcal{M} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T & \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta} \boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T & \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ \boldsymbol{r}_3^T & t_z \end{pmatrix}$$

Note: If $\mathcal{M} = (\mathcal{A} \quad \boldsymbol{b})$ then $|\boldsymbol{a}_3| = 1$.

Replacing $\mathcal{M}$ by $\lambda\mathcal{M}$ in

$$\begin{cases} u = \dfrac{\boldsymbol{m}_1 \cdot \boldsymbol{P}}{\boldsymbol{m}_3 \cdot \boldsymbol{P}} \\ v = \dfrac{\boldsymbol{m}_2 \cdot \boldsymbol{P}}{\boldsymbol{m}_3 \cdot \boldsymbol{P}} \end{cases}$$

does not change u and v.

M is only defined up to scale in this setting!!

Theorem (Faugeras, 1993)

Let $\mathcal{M} = (\mathcal{A} \quad \boldsymbol{b})$ be a 3×4 matrix and let $\boldsymbol{a}_i^T$ $(i = 1, 2, 3)$ denote the rows of the matrix $\mathcal{A}$ formed by the three leftmost columns of $\mathcal{M}$.

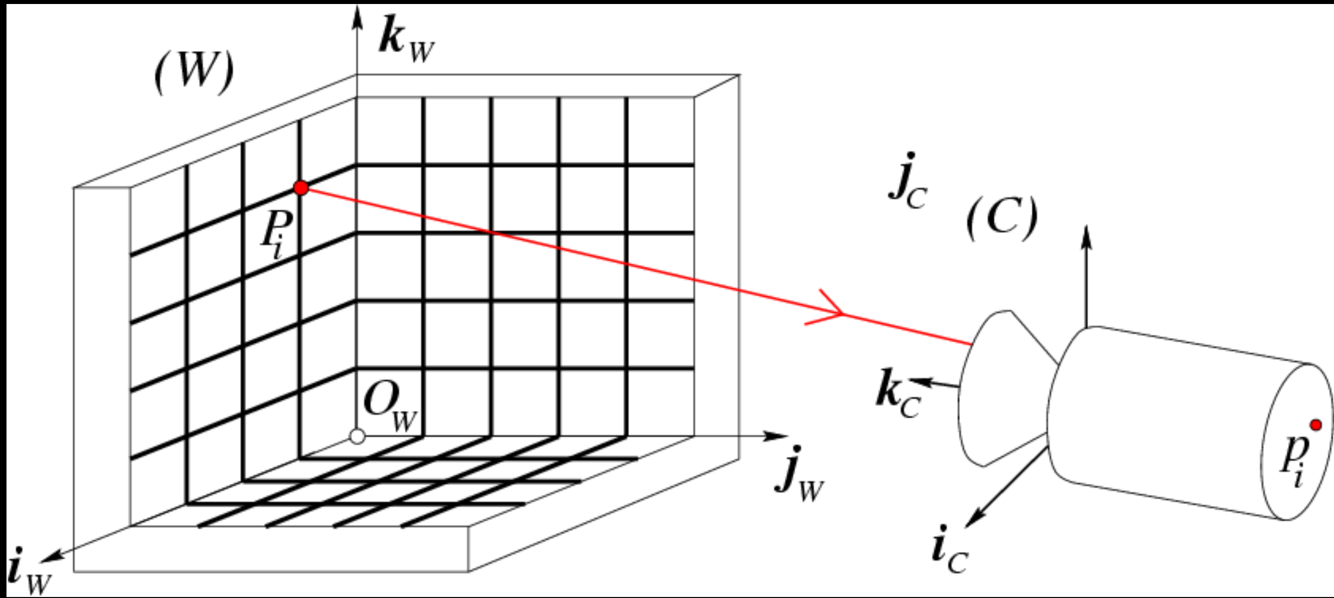
- A necessary and sufficient condition for $\mathcal{M}$ to be a perspective projection matrix is that $\text{Det}(\mathcal{A}) \neq 0$.
- A necessary and sufficient condition for $\mathcal{M}$ to be a zero-skew perspective projection matrix is that $\text{Det}(\mathcal{A}) \neq 0$ and

$$(\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = 0.$$

- A necessary and sufficient condition for $\mathcal{M}$ to be a perspective projection matrix with zero skew and unit aspect-ratio is that $\text{Det}(\mathcal{A}) \neq 0$ and

$$\begin{cases} (\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = 0, \\ (\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_3) = (\boldsymbol{a}_2 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3). \end{cases}$$

Calibration Problem



Given n points $P_1, \ldots, P_n$ with *known* positions and their images $p_1, \ldots, p_n$

Find $\boldsymbol{i}$ and $\boldsymbol{e}$ such that

$$\sum_{i=1}^{n} \left[\left(u_i - \frac{\boldsymbol{m}_1(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i} \right)^2 + \left(v_i - \frac{\boldsymbol{m}_2(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i} \right)^2 \right] \quad \text{is minimized}$$

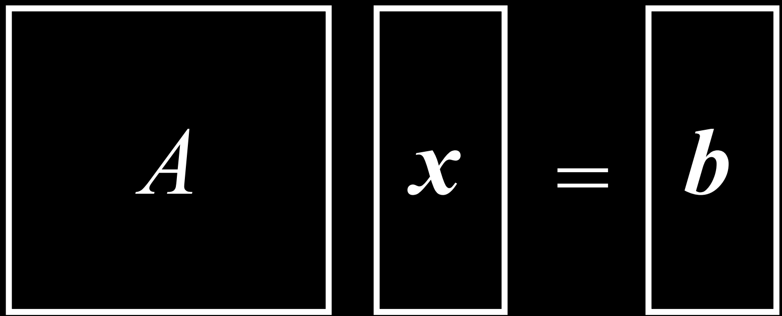
Linear Camera Calibration

Given n points $P_1, \ldots, P_n$ with *known* positions and their images $p_1, \ldots, p_n$

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{pmatrix} \dfrac{\boldsymbol{m}_1 \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3 \cdot \boldsymbol{P}_i} \\ \dfrac{\boldsymbol{m}_2 \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3 \cdot \boldsymbol{P}_i} \end{pmatrix} \iff \begin{pmatrix} \boldsymbol{m}_1 - u_i \boldsymbol{m}_3 \\ \boldsymbol{m}_2 - v_i \boldsymbol{m}_3 \end{pmatrix} \boldsymbol{P}_i = 0$$

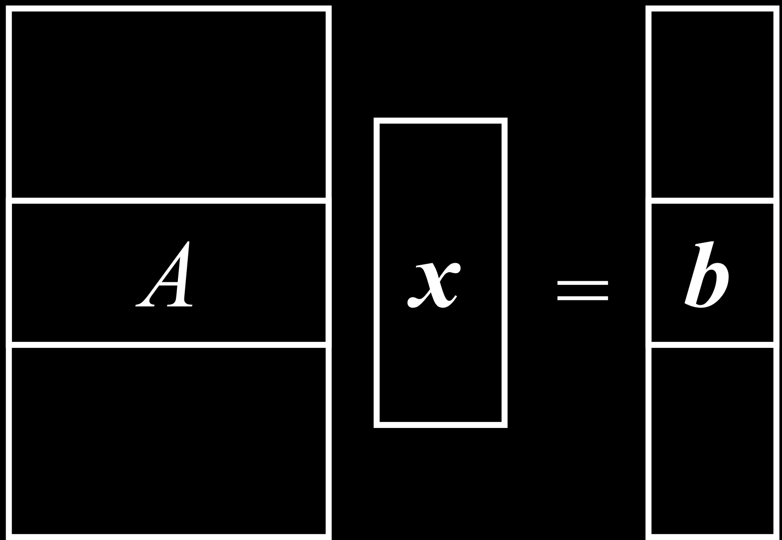
$$\mathcal{P}\boldsymbol{m} = 0 \text{ with } \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{P}_1^T & \boldsymbol{0}^T & -u_1 \boldsymbol{P}_1^T \\ \boldsymbol{0}^T & \boldsymbol{P}_1^T & -v_1 \boldsymbol{P}_1^T \\ \cdots & \cdots & \cdots \\ \boldsymbol{P}_n^T & \boldsymbol{0}^T & -u_n \boldsymbol{P}_n^T \\ \boldsymbol{0}^T & \boldsymbol{P}_n^T & -v_n \boldsymbol{P}_n^T \end{pmatrix} \text{ and } \boldsymbol{m} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{m}_1 \\ \boldsymbol{m}_2 \\ \boldsymbol{m}_3 \end{pmatrix} = 0$$

Linear Systems



Square system:

- unique solution
- Gaussian elimination



Rectangular system ??

- underconstrained: infinity of solutions
- overconstrained: no solution

Minimize $|Ax-b|^2$

How do you solve overconstrained linear equations ??

- Define $E = |\boldsymbol{e}|^2 = \boldsymbol{e} \cdot \boldsymbol{e}$ with

$$\begin{aligned} \boldsymbol{e} &= A\boldsymbol{x} - \boldsymbol{b} = \left[\boldsymbol{c}_1 \middle| \boldsymbol{c}_2 \middle| \ldots \middle| \boldsymbol{c}_n \right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \boldsymbol{b} \\ &= x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots x_n\boldsymbol{c}_n - \boldsymbol{b} \end{aligned}$$

- At a minimum,

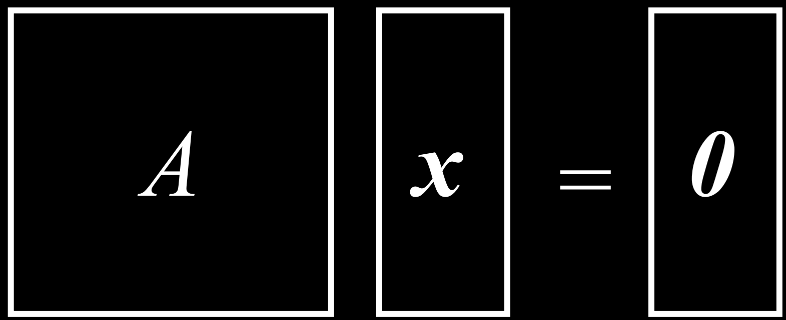
$$\begin{aligned} \frac{\partial E}{\partial x_i} &= \frac{\partial \boldsymbol{e}}{\partial x_i} \cdot \boldsymbol{e} + \boldsymbol{e} \cdot \frac{\partial \boldsymbol{e}}{\partial x_i} = 2\frac{\partial \boldsymbol{e}}{\partial x_i} \cdot \boldsymbol{e} \\ &= 2\frac{\partial}{\partial x_i}(x_1\boldsymbol{c}_1 + \cdots + x_n\boldsymbol{c}_n - \boldsymbol{b}) \cdot \boldsymbol{e} = 2\boldsymbol{c}_i \cdot \boldsymbol{e} \\ &= 2\boldsymbol{c}_i^T(A\boldsymbol{x} - \boldsymbol{b}) = 0 \end{aligned}$$

- or

$$0 = \begin{bmatrix} \boldsymbol{c}_i^T \\ \vdots \\ \boldsymbol{c}_n^T \end{bmatrix} (A\boldsymbol{x} - \boldsymbol{b}) = A^T(A\boldsymbol{x} - \boldsymbol{b}) \Rightarrow A^TA\boldsymbol{x} = A^T\boldsymbol{b},$$

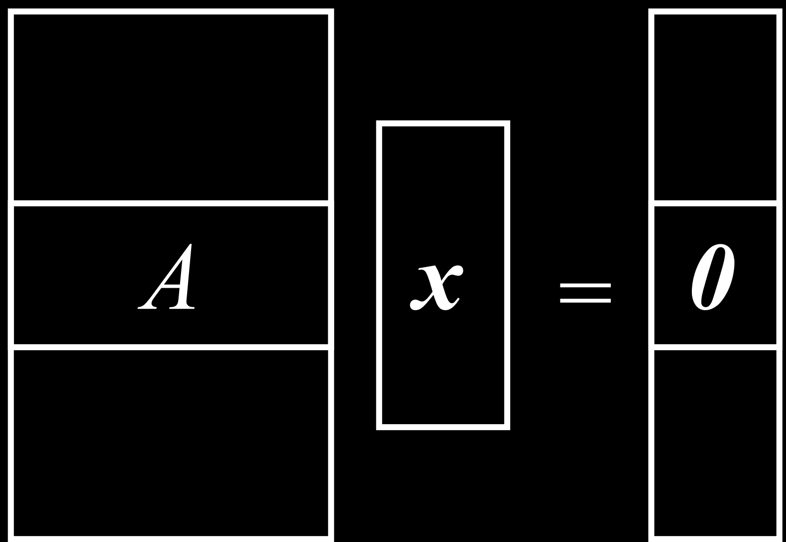
where $\boldsymbol{x} = A^{\dagger}\boldsymbol{b}$ and $A^{\dagger} = (A^TA)^{-1}A^T$ is the *pseudoinverse* of A !

Homogeneous Linear Systems



Square system:

- unique solution: 0
- unless Det(A)=0



Rectangular system ??

- 0 is always a solution

Minimize $|Ax|^2$ under the constraint $|x|^2 = 1$

How do you solve overconstrained homogeneous linear equations ??

$$E = |\mathcal{U}\boldsymbol{x}|^2 = \boldsymbol{x}^T(\mathcal{U}^T\mathcal{U})\boldsymbol{x}$$

- Orthonormal basis of eigenvectors: $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_q$.
- Associated eigenvalues: $0 \le \lambda_1 \le \ldots \le \lambda_q$.
- Any vector can be written as

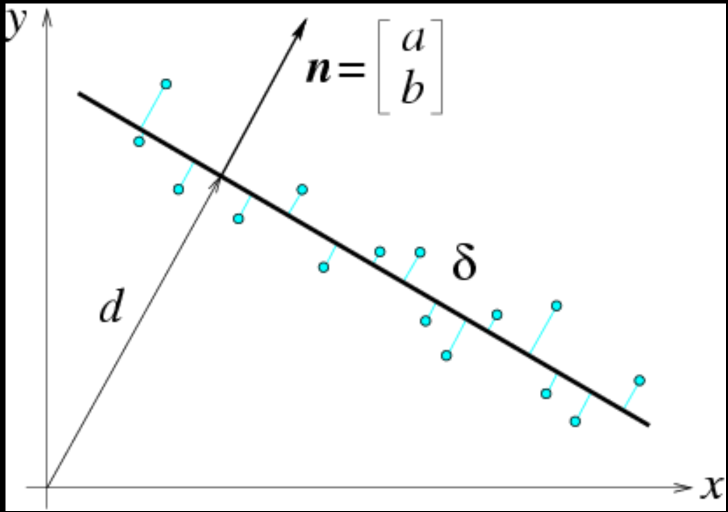
$$\boldsymbol{x} = \mu_1\boldsymbol{e}_1 + \ldots + \mu_q\boldsymbol{e}_q$$

for some μ_i $(i = 1, \ldots, q)$ such that $\mu_1^2 + \ldots + \mu_q^2 = 1$.

$$\begin{aligned} E(\boldsymbol{x})-E(\boldsymbol{e}_1) &= \boldsymbol{x}^T(U^TU)\boldsymbol{x}-\boldsymbol{e}_1^T(U^TU)\boldsymbol{e}_1 \\ &= \lambda_1\mu_1^2+ \ldots +\lambda_q\mu_q^2-\lambda_1 \\ &\geq \lambda_1(\mu_1^2+ \ldots +\mu_q^2-1)=0 \end{aligned}$$

The solution is $\boldsymbol{e}_1$.

Example: Line Fitting



Problem: minimize

$$E(a,b,d) = \sum_{i=1}^{n} (ax_i + by_i - d)^2$$

with respect to (*a*,*b*,*d*).

- Minimize *E* with respect to *d*:

$$\frac{\partial E}{\partial d} = 0 \Longrightarrow d = \sum_{i=1}^{n} \frac{ax_i + by_i}{n} = a\bar{x} + b\bar{y}$$

- Minimize *E* with respect to *a*,*b*:

$$E = \sum_{i=1}^{n} [a(x_i - \bar{x}) + b(y_i - \bar{y})]^2 = |\mathcal{U}\boldsymbol{n}|^2$$ where $$\mathcal{U} = \begin{pmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ \dots & \dots \\ x_n - \bar{x} & y_n - \bar{y} \end{pmatrix}$$

- Done !!

Note:

$$\mathcal{U}^T\mathcal{U} = \begin{pmatrix} \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 & \sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y} \\ \sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y} & \sum_{i=1}^{n} y_i^2 - n\bar{y}^2 \end{pmatrix}$$

- Matrix of second moments of inertia
- Axis of least inertia

Linear Camera Calibration

Given n points $P_1, \ldots, P_n$ with *known* positions and their images $p_1, \ldots, p_n$

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{pmatrix} \dfrac{\boldsymbol{m}_1 \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3 \cdot \boldsymbol{P}_i} \\ \dfrac{\boldsymbol{m}_2 \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3 \cdot \boldsymbol{P}_i} \end{pmatrix} \iff \begin{pmatrix} \boldsymbol{m}_1 - u_i \boldsymbol{m}_3 \\ \boldsymbol{m}_2 - v_i \boldsymbol{m}_3 \end{pmatrix} \boldsymbol{P}_i = 0$$

Linear least squares for $n > 5$!

$$\mathcal{P}\boldsymbol{m} = 0 \text{ with } \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{P}_1^T & \boldsymbol{0}^T & -u_1 \boldsymbol{P}_1^T \\ \boldsymbol{0}^T & \boldsymbol{P}_1^T & -v_1 \boldsymbol{P}_1^T \\ \ldots & \ldots & \ldots \\ \boldsymbol{P}_n^T & \boldsymbol{0}^T & -u_n \boldsymbol{P}_n^T \\ \boldsymbol{0}^T & \boldsymbol{P}_n^T & -v_n \boldsymbol{P}_n^T \end{pmatrix} \text{ and } \boldsymbol{m} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{m}_1 \\ \boldsymbol{m}_2 \\ \boldsymbol{m}_3 \end{pmatrix} = 0$$

Once M is known, you still got to recover the intrinsic and extrinsic parameters !!!

This is a decomposition problem, not an estimation problem.

$$\rho\,\mathcal{M} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T & \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta} \boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T & \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ \boldsymbol{r}_3^T & t_z \end{pmatrix}$$

- Intrinsic parameters
- Extrinsic parameters

Degenerate Point Configurations

Are there other solutions besides *M* ??

$$\mathbf{0} = \mathcal{P}\boldsymbol{l} = \begin{pmatrix} \boldsymbol{P}_1^T & \mathbf{0}^T & -u_1\boldsymbol{P}_1^T \\ \mathbf{0}^T & \boldsymbol{P}_1^T & -v_1\boldsymbol{P}_1^T \\ \dots & \dots & \dots \\ \boldsymbol{P}_n^T & \mathbf{0}^T & -u_n\boldsymbol{P}_n^T \\ \mathbf{0}^T & \boldsymbol{P}_n^T & -v_n\boldsymbol{P}_n^T \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \\ \boldsymbol{\nu} \end{pmatrix} = \begin{pmatrix} \boldsymbol{P}_1^T\boldsymbol{\lambda} - u_1\boldsymbol{P}_1^T\boldsymbol{\nu} \\ \boldsymbol{P}_1^T\boldsymbol{\mu} - v_1\boldsymbol{P}_1^T\boldsymbol{\nu} \\ \dots \\ \boldsymbol{P}_n^T\boldsymbol{\lambda} - u_n\boldsymbol{P}_n^T\boldsymbol{\nu} \\ \boldsymbol{P}_n^T\boldsymbol{\mu} - v_n\boldsymbol{P}_n^T\boldsymbol{\nu} \end{pmatrix}$$

$$\begin{cases} \boldsymbol{P}_i^T\boldsymbol{\lambda} - \dfrac{\boldsymbol{m}_1^T\boldsymbol{P}_i}{\boldsymbol{m}_3^T\boldsymbol{P}_i}\boldsymbol{P}_i^T\boldsymbol{\nu} = 0 \\ \boldsymbol{P}_i^T\boldsymbol{\mu} - \dfrac{\boldsymbol{m}_2^T\boldsymbol{P}_i}{\boldsymbol{m}_3^T\boldsymbol{P}_i}\boldsymbol{P}_i^T\boldsymbol{\nu} = 0 \end{cases} \Rightarrow \begin{cases} \boldsymbol{P}_i^T(\boldsymbol{\lambda}\boldsymbol{m}_3^T - \boldsymbol{m}_1\boldsymbol{\nu}^T)\boldsymbol{P}_i = 0 \\ \boldsymbol{P}_i^T(\boldsymbol{\mu}\boldsymbol{m}_3^T - \boldsymbol{m}_2\boldsymbol{\nu}^T)\boldsymbol{P}_i = 0 \end{cases}$$

- Coplanar points: $(\lambda,\mu,\nu)=(\Pi,0,0)$ or $(0,\Pi,0)$ or $(0,0,\Pi)$
- Points lying on the intersection curve of two quadric surfaces = straight line + twisted cubic

Does not happen for 6 or more random points!

Analytical Photogrammetry

Given n points $P_1, \ldots, P_n$ with *known* positions and their images $p_1, \ldots, p_n$

Find $\boldsymbol{i}$ and $\boldsymbol{e}$ such that

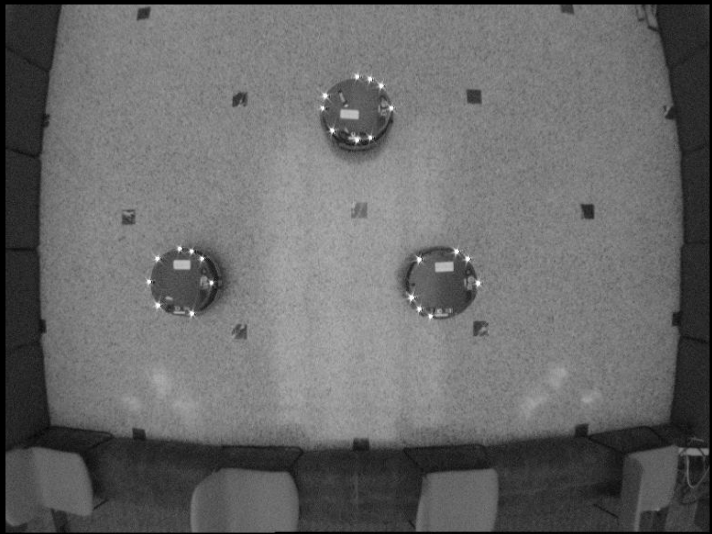
$$\sum_{i=1}^{n} \left[\left(u_i - \frac{\boldsymbol{m}_1(\boldsymbol{i}, \boldsymbol{e}) \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3(\boldsymbol{i}, \boldsymbol{e}) \cdot \boldsymbol{P}_i} \right)^2 + \left(v_i - \frac{\boldsymbol{m}_2(\boldsymbol{i}, \boldsymbol{e}) \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3(\boldsymbol{i}, \boldsymbol{e}) \cdot \boldsymbol{P}_i} \right)^2 \right] \quad \text{is minimized}$$

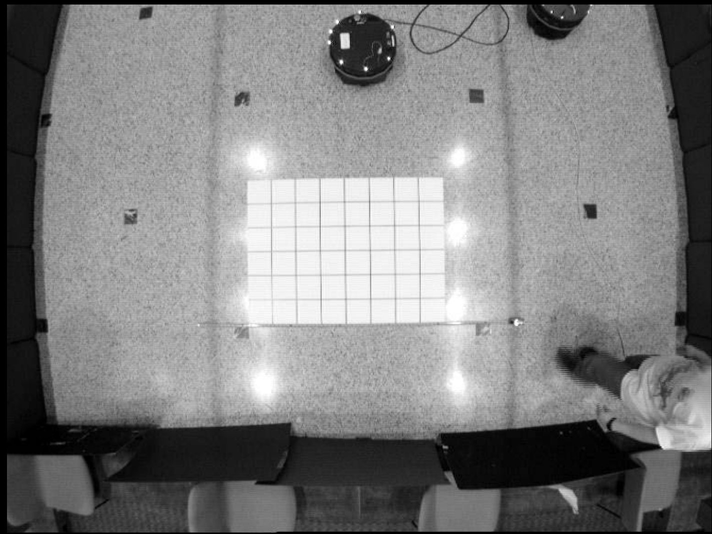
Non-Linear Least-Squares Methods

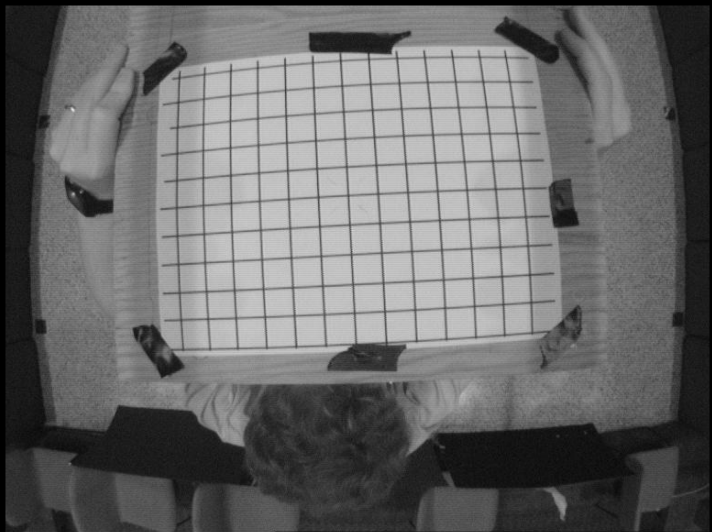
- Newton
- Gauss-Newton
- Levenberg-Marquardt

Iterative, quadratically convergent in favorable situations

Mobile Robot Localization (Devy *et al.*, 1997)







(Rothganger, Sudsang, & Ponce, 2002)