RECURSIVE INTERFEROMETRIC REPRESENTATIONS

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ABSTRACT
Classification requires building invariant representations relatively to groups of deformations that preserve signal classes. Recursive interferometry computes invariants with a cascade of complex wavelet transforms and modulus operators. The resulting representation is stable relatively to elastic deformations and provides invariant representations of stationary processes. It maps signals to a manifold which preserves signal discriminability.

1. INTRODUCTION
Signal classes are usually invariant to certain types of deformations that may include translations, rotations, scalings or any other group of operators. Classification algorithms must then be invariant relatively to these deformations. The invariance often also applies to elastic deformations which define much larger Lie groups. However, building invariants reduces the representation dimension, which may affect its ability to discriminate different patterns. It is therefore necessary to construct representations that balance invariance, stability and discriminability requirements.

The Fourier transform modulus is translation invariant but the representation of high frequencies is highly not invariant to elastic deformations. Computer vision researchers have introduced histogram techniques to build local invariants by delocalizing high frequency information, which lead to efficient local descriptors for classification, when global invariants are not needed. Deep neural networks also provide efficient data dependent invariant representation [2, 3] in number of applications, but are not well understood.

This paper follows a harmonic analysis approach to invariant representations. Section 2 introduces a recursive interference representation, and analyzes the properties low frequency interferences computed with cascades of wavelet transform modulus. Section 3 studies interference invariance and discriminability and Section 4 provides a fast filter bank implementation. The paper concentrates on translation invariance but generalization to any other group can be found in [4].

2. RECURSIVE INTERFEROMETRY
Recursive interferometry maps signal high frequencies to lower frequencies, with a cascade of wavelet transforms and modulus operators, which yields a progressively more invariant representation.

2.1 Wavelet Transform Modulus
A modulus operator applied on a wavelet transform is shown to compute low frequency interferences. A wavelet transform filters a real multidimensional signal \( f \in L^2(\mathbb{R}^d) \) with a family of \( K-1 \) wavelets \( \{ \psi_d \}_{1 \leq d < K} \) which are scaled by \( 2^j \):

\[
\forall x \in \mathbb{R}^d, \quad W_{j,k}f(x) = f * \psi_{j,k}(x)
\]

with

\[
\psi_{j,k}(x) = 2^{-d/2} \psi_k(2^{-j}x).
\]

It is computed up to a coarse scale \( 2^j \) where the remaining low frequencies are carried by a low-pass filtering \( f * \psi_{0,0}(x) \), where \( \psi_{0,0}(x) \) is a real low frequency scaling function. Let \( \hat{f}(\omega) \) be the Fourier transform of \( f \) with \( \omega \in \mathbb{R}^d \). The modulus of \( \omega \in \mathbb{R}^d \) is written \( |\omega| \). Since

\[
\hat{W}_{j,k}f(\omega) = \hat{f}(\omega) \hat{\psi}_k(2^j \omega) \quad \text{and} \quad \hat{f}(-\omega) = \hat{f}^*(\omega), \text{if for all } \omega \in \mathbb{R}^d
\]

\[
(1-\delta) \leq |\hat{\psi}_0(2^j \omega)|^2 + \sum_{k=1}^{K-1} \sum_{j \geq 1} \left( |\hat{\psi}_k(2^j \omega)|^2 + |\hat{\psi}_k(-2^j \omega)|^2 \right) / 2 \leq 1
\]

then the wavelet transform is a complete contracting mapping

\[
||f||^2 (1-\delta) \leq ||f * \psi_{0,0}||^2 + \sum_{k=1}^{K-1} \sum_{j \geq 1} ||W_{j,k}f||^2 \leq ||f||^2,
\]
with \( \| f \|^2 = \int |f(x)|^2 \, dx \). We consider complex analytic wavelets such that \( \psi_0(\omega) = 0 \) if \( \psi_0(-\omega) \neq 0 \) for \( k \geq 1 \). At low frequencies, \( \psi_0(\omega) \) covers the domain \( |\omega| \leq \pi \), with \( \psi_0(2p\pi) = 0 \) for \( p \in \mathbb{Z}^d \), and \( \psi_0(\omega) \) for \( k > 1 \) is mostly non-negligible inside a 1 octave frequency annulus \( \pi \leq |\omega| \leq 2\pi \).

High frequency wavelet coefficients are mapped to low frequencies with a complex modulus which computes frequency interferences. The Fourier transform of \( M_{j,k}(x) = |W_{j,k}(x)|^2 \) is the convolution of \( W_{j,k}(\omega) \) with itself:

\[
\tilde{M}_{j,k}(\omega) = (2\pi)^{-d} \int W_{j,k}(\xi) \tilde{W}_{j,k}^*(\xi - \omega) \, d\xi.
\] (2)

This convolution measures the correlation between frequencies that are \( \omega \) apart. In quantum physics, where probabilities are calculated as the squared modulus of complex wave functions, it is interpreted as interferences. Although \( \tilde{W}_{j,k}(\omega) \) is non-negligible inside a frequency annulus \( 2^{-j}\pi \leq |\omega| \leq 2^{-j+1}\pi \), (2) shows that \( \tilde{M}_{j,k}(\omega) \) is a correlation measure which is mostly non-zero at lower-frequencies \( |\omega| \leq 2^{-j}\pi \).

To iterate this mapping and guarantee stability, the squared complex modulus is replaced by a modulus, which is contracting. It involves a square root operator \( |W_{j,k}(x)| = \sqrt{M_{j,k}(x)} \), which is singular when \( W_{j,k}(x) \) vanishes. Let us write

\[
|W_{j,k}(x)|^2 = \|W_{j,k}\|^2 \omega^2(x) \left(1 + \varepsilon(x)\right),
\]

where \( \omega(x) \) which is constant over the support of \( f \) with \( \|w\| = 1 \). A series expansion of \( \sqrt{1 + \varepsilon} \) gives

\[
|W_{j,k}(x)| = \|W_{j,k}\| \omega(x) \left(1 + \frac{1}{2} \varepsilon(x) + O(\varepsilon^2(x)) \right).
\]

The lower frequencies of \( |W_{j,k}(x)| \) are dominated by the squared modulus interferences term \( \varepsilon(x) \) and the \( O(\varepsilon^2(x)) \) higher order terms produce higher frequency harmonics of low amplitude. As a result, \( |W_{j,k}(x)| \) has a Fourier transform which is also mostly located at the lower frequencies \( |\omega| \leq 2^{-j}\pi \).

### 2.2 Recursive Interference Tree

Recursive interferometry computes a progressively lower frequency representation by iteratively calculating complex wavelet transforms and modulus operators, which produce “interferences of interferences”.

An interference tree up to a scale \( 2^J \) is a set of signals \( \hat{I}_J f(x, \alpha) \) located at the nodes of a tree, where \( j \leq J \) gives the depth of a node and \( \alpha \) its horizontal position in a left to right order. The wavelet transform modulus of \( f \) builds a first tree branch with \( K - 1 \) leaves per level, which carry 1st order interferences at each scale

\[
\hat{I}_j f(x, k) = |f * \psi_{j,k}(x)| \quad \text{for} \quad j \leq J \quad \text{and} \quad 1 \leq k < K
\]

plus the low signal frequencies at the last level

\[
\hat{I}_J f(x, 0) = f * \psi_{J,0}(x) .
\]

Each of the \( K - 1 \) leaves of depths \( m < J \) are sub-decomposed with a second wavelet transform and modulus operator, which computes second order interferences located at the leaves of a new tree of depth \( J \).

The interference tree is progressively constructed by decomposing the signals \( \hat{I}_m f(x, \alpha) \) at the leaves of a previously calculated tree, with a wavelet transform modulus up to a scale \( 2^J \), until all the tree leaves are at the depth \( J \), as illustrated in Figure 1. The wavelet transform modulus of \( \hat{I}_m f(x, \alpha) \) up to the level \( J \) defines a new tree whose leaves are

\[
\hat{I}_j f(x, \alpha K^{J-m} + k) = |\hat{I}_m f(., \alpha) * \psi_{j,k}(x)| \quad \text{for} \quad l < j \leq J ,
\]

and

\[
\hat{I}_J f(x, \alpha K^{J-m}) = \hat{I}_m f(., \alpha) * \psi_{0,J}(x) .
\]

The signals \( \hat{I}_J f(x, \alpha) \) are recursive interferences, computed with \( p(\alpha) \) wavelet transforms and modulus operators. The interference order \( p(\alpha) \) at a node \( \alpha \) is the number of non-zero digit of \( \alpha \) written in base \( K \).

All tree signals \( \hat{I}_J f(x, \alpha) \) are further filtered with the low-pass filter \( \psi_{0,j}(x) \) to eliminate high frequency harmonics resulting from the last modulus computation:

\[
I_j f(x, \alpha) = \hat{I}_j f(., \alpha) * \psi_{0,j}(x) .
\]

If \( f(x) \in L^2[0,1]^d \) has a period 1 along the \( d \) directions, then interference signals \( \hat{I}_J f(x, \alpha) \) have also a period 1. Since \( \psi_0(2p\pi) = 0 \) for \( p \in \mathbb{Z}^d \), at the maximum scale \( 2^J = 1 \), all \( I_0 f(x, \alpha) \) are constant in \( x \). The tree leaves stores a single value \( I_0 f(\alpha) \) providing a delocatedized information on the whole support of \( f \).
Figure 1: A recursive interference tree computes a first wavelet transform modulus (in black), and iteratively computes wavelet transform modulus of its leaves (2nd order in green and 3rd order in red), until all leaves are at the maximum depth.

3. INVARiance AND DISCRIMINABILITY

The classification ability of recursive interferometry relies on its invariance and discriminability properties that are reviewed. The norm of interference signals at a depth $j$ is

$$||I_j f||$$
realizations of two random processes \( F_1 = f_1(x - \tau_1(x)) \) and \( F_2(x) = f_2(x - \tau_2(x)) \). The template signals \( f_1 \) and \( f_2 \) are deformed with two elastic random deformations \( \tau_1(x) \) and \( \tau_2(x) \) satisfying \( \| \tau_1(x) \| \leq a < 1 \) and \( \| \tau_2(x) \| \leq a < 1 \). Let \( \tilde{F}_i = F_i + W \) be a noisy realization of \( F_i \) with an additive Gaussian white noise \( W \). Figures 2(a,b) show two realizations of \( \tilde{F}_1 \) and \( \tilde{F}_2 \).

The probability distributions of \( \| \Phi(\tilde{F}_1) - \Phi(\tilde{F}_2) \|^2 \) is shown in Figure 2(c) for \( \Phi(f) = f \) and in Figure 2(d) for a Fourier modulus \( \Phi(f) = |\hat{f}| \). In these two cases, the intra class distance for \( i = i' \) is of the same order as the distances across classes when \( i \neq i' \). Indeed, if \( \Phi(f) = f \) then the signal representation is not invariant to translation and \( \Phi(f) = |\hat{f}| \) is not stable relatively to elastic deformations. Both classes can therefore not be discriminated with these distances.

Figure 2(e) gives the distribution of \( \| \Phi(\tilde{F}_i) - \Phi(\tilde{F}_j) \|^2 \) for \( \Phi(f) = I_0 f \). Recursive interferences are computed with a one-dimensional Gabor wavelet \( \psi(x) = \theta(x) e^{i2\pi x} \), where \( \theta \) is a Gaussian. The distance is larger across classes (\( i \neq i' \)) then within classes (\( i = i' \)), so both classes can be discriminated by thresholding the distance on recursive interferences.

### 3.1 Stationary Processes Interferences

Not all signal classes may be obtained as deformations of a deterministic template signal. In particular, realizations of a stationary texture are not elastic deformations of a single signal. Recursive interferences map the realizations of a stationary process to a small ball in the transformed space. Discriminating the realizations of two stationary processes is thus possible through the Euclidean distance of their interference representation.

If \( F \) is a zero-mean stationary process then \( I_j \tilde{F}(x, \alpha) \) remains stationary in \( x \). Indeed, it is computed with a cascade of wavelet transforms which are convolutions and modulus operators, which both preserve stationarity. Let \( \sigma^2 = E \{ |F(x) - E \{ F(x) \}|^2 \} \). Wavelet signals \( F * \psi_{j,k}(x) \) are stationary processes, and (1) implies that their variance \( \sigma^2_{j,k} \) satisfy

\[
(1 - \delta) \sigma^2 \leq \sum_{j,k} \sigma^2_{j,k} \leq \sigma^2.
\]

However, the modulus operator reduce these variances because of the complex phase suppression.

![Figure 2](image-url)
where as \( \| E \{ I_0F \} \|^2 \sim E \{ \| F \|^2 \} \). It shows that \( I_0F \)
remains in a ball whose spread is much smaller
then its distance to 0. Realizations of two station-
ary processes are discriminated by measuring the
distance of their interference transform.

Figures 2(f,g) show the realizations of two dif-
ferent white noise processes \( F_1 \) and \( F_2 \). Their
supports are defined by two Bernoulli distributions
\( \text{Prob}\{\{F_1(n) = 0\} = p_1 \) and \( \text{Prob}\{\{F_1(n) \neq 0\} = 1 - p_1 \), with \( p_1 = 2p_2 \). Over its support, each
\( F_i(n) \) is a Gaussian white noise. For \( \Phi(f) = f \)
and \( \Phi(f) = |f| \). Figures 2(h,i) show that the dis-
tribution of \( \| \Phi(F_i) - \Phi(F_j) \| \) are similar within the
same class \( i = i' \) and and across classes \( i \neq i' \), when
\( \Phi(f) = f \) and \( \Phi(f) = |f| \). On the opposit, intra
class and across class distances are well separated by
an interference representation \( \Phi(f) = I_0f \).

4. FAST ALGORITHM WITH MODULUS
FILTER BANK

This section describes a fast filter bank algorithm
which computes the recursive interference transform
of a multidimensional discrete signal \( f[n] \) of
size \( N \), with \( n = (n_1, \ldots, n_d) \). The computational
structure involves a cascade of convolutions and
modulus operators as in deep neural architectures
[2, 3] but involves no learning.

We consider \( f[n] \) as a signal obtained by sam-
pling a 1 period function \( f(x) \) at intervals \( 2^L = N^{-1} \). Each \( I_f, f(x, \alpha) \) has a frequency support
mostly concentrated at frequencies \( |\omega| \leq 2^{-j}\pi \) but
may go beyond, and it is thus uniformly sampled
at intervals \( 2^L \). We write \( I_f[n, \alpha] = \hat{I}_j f(2^n, \alpha) \).

The discrete wavelet transform of \( f \) is com-
puted at scales \( 2^j < 2^L = N^{-1} \). The root of the
tree is at the level \( L \) and \( \hat{I}_j f[n, 0] = f[n] \). The
finest scale wavelet transform of \( f[n] \) is computed
without subsampling, using discrete wavelet filters
\( \psi_{1,k}[n] = 2^{-1} \psi(2^{-1}n) \):

\[
\hat{I}_{L+1} f[n, 0] = f \star \psi_{1,0}[n]
\]

and

\[
\hat{I}_{L+1} f[n, k] = |f \star \psi_{1,k}[n]| \text{ for } 0 < k < K.
\]

These signals are nearly oversampled by a factor 2
relatively to their frequency spread.

The filtering algorithm continues recursively
by computing the \( K \) children of each \( \hat{I}_j f[n, \alpha] \), with
one low-pass filter and \( K - 1 \) complex band-pass
filters, which are subsampled by a factor 2. If
\( \alpha \neq 0 \mod K \) and is thus a band-pass filter output
then the wavelet transform is calculated with over-
sampled wavelets filters \( \psi_{2,k}[n] = 2^{-2} \psi_k(2^{-2}n) \):

\[
\hat{I}_{L+1} f[n, \alpha K] = I_f[n, \alpha] \star \psi_{2,0}[2n]
\]

and

\[
\hat{I}_{L+1} f[n, \alpha K + k] = |I_f[n, \alpha] \star \psi_{2,k}[2n]| \text{ for } 0 < k < K.
\]

If \( \alpha = 0 \mod K \) and is thus a low-pass filter output
then since \( \hat{I}_{L+1} f[n, \alpha] \) was already obtained through a
convolution with \( \psi_{1,0} \), the next wavelet scale is cal-
culated with the filters \( g_k[n] \) whose transfer func-
tion \( \hat{g}_k(\omega) \) satisfies:

\[
\hat{g}_k(2\omega) = \hat{g}_k(\omega/2) \hat{g}_0(\omega).
\]

Children are then computed with

\[
\hat{I}_{L+1} f[n, \alpha K] = I_f[n, \alpha] \star g_0[2n]
\]

and

\[
\hat{I}_{L+1} f[n, \alpha K + k] = |I_f[n, \alpha] \star g_k[2n]| \text{ for } 0 < k < K.
\]

The oversampling factor 2 is finally removed by
filtering \( I_f[n, \alpha] \) with the low-pass filter \( \psi_{1,0}[n] \)
and by subsampling the output

\[
I_f[n, \alpha] = \hat{I}_j f[\alpha] \star \psi_{1,0}[2n].
\]

At each level \( j \) of the tree, there are \( K^{j-L} \) in-
dices \( \alpha \) and each signal \( I_f[n, \alpha] \) has \( 2^{-d_j} \) sam-
ple, so there is a total of \( 2^{-d_j} K^{j-L} \) coefficients,
with \( 2^L = N^{-1} \). If \( d = 1 \) and \( K = 2 \) there are
\( N \) coefficients. The filter bank algorithm is im-
plemented with \( O(N \log_2 N) \) operations. If \( K > 2 \)
then the \( 2^{-d_j} K^{j-L} \) coefficients are computed with
\( O(2^{-d_j} K^{j-L}) \) operations, which makes \( O(\log_2(K)) \)
at the bottom of the tree.

REFERENCES

[1] Y. Amit., U. Grenander and M. Piccioni; “Struc-
tural image restoration through deformable tem-
plates”. Journal of the American Statistical Asso-

LeCun: What is the Best Multi-Stage Architecture

[3] Jake Bouvrie, Lorenzo Rosasco, Tomasso Pog-
gio: On Invariance in Hierarchical Models. NIPS
2009

[4] S. Mallat “Invariance by Recursive Interfero-