# Diameter in weighted random graphs

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# WEIGHTED DIAMETER

Graph $G = (V, E)$:

- Distance dist$(a, b) = \min_{\pi \in \Pi(a,b)} |\pi|$, the number of edges in $E$ in the shortest path connecting $a$ and $b$.

- Diameter of $G$ defined by:

$$\text{diam}(G) = \max\{\text{dist}(a, b),\ a, b \in V,\ \text{dist}(a, b) < \infty\}.$$

- Weight associated to each edge $e \in E$: $w_e$.

- Weighted distance dist$_w(a, b) = \min_{\pi \in \Pi(a,b)} \sum_{e \in \pi} w_e$.

- Weighted diameter of $G$ defined by:

$$\text{diam}_w(G) = \max\{\text{dist}_w(a, b),\ a, b \in V,\ \text{dist}_w(a, b) < \infty\}\ .$$

# CONFIGURATION MODEL

For $n \in \mathbb{N}$, let $(d_i)_1^n$ be a sequence of non-negative integers such that $\sum_{i=1}^n d_i$ is even.

We define $G^*(n, (d_i)_1^n)$ a random multigraph with given degree sequence $(d_i)_1^n$: to each node $i$ we associate $d_i$ labeled half-edges. All half-edges need to be paired to construct the graph, this is done by randomly matching them. When a half-edge of $i$ is paired with a half-edge of $j$, we interpret this as an edge between $i$ and $j$.

The graph $G^*(n, (d_i)_1^n)$ obtained following this procedure may not be simple. Conditional on the multigraph $G^*(n, (d_i)_1^n)$ being a simple graph, we obtain a uniformly distributed random graph with the given degree sequence, which we denote by $G(n, (d_i)_1^n)$,

## ASSUMPTIONS ON THE DEGREE SEQUENCE

(i) There exists a distribution $p = \{p_k\}_{k=0}^{\infty}$ such that $|\{i,\ d_i = k\}|/n \to p_k$ for every $k \geq 0$ as $n \to \infty$;

(ii) $\lambda := \sum_{k \geq 0} k p_k \in (0, \infty)$;

(iii) $\sum_{i=1}^{n} d_i^2 = O(n)$,

(iv) for some $\tau > 0$, $\Delta_n := \max_{i \in V} d_i = O(n^{1/2-\tau})$.

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Let $\hat{D}_n$ be the degree of a random vertex of $G^*(n, (d_i)_1^n)$ and $D$ a random variable with distribution $p_k$, then (i) is equivalent to $\hat{D}_n \xrightarrow{d} D$.

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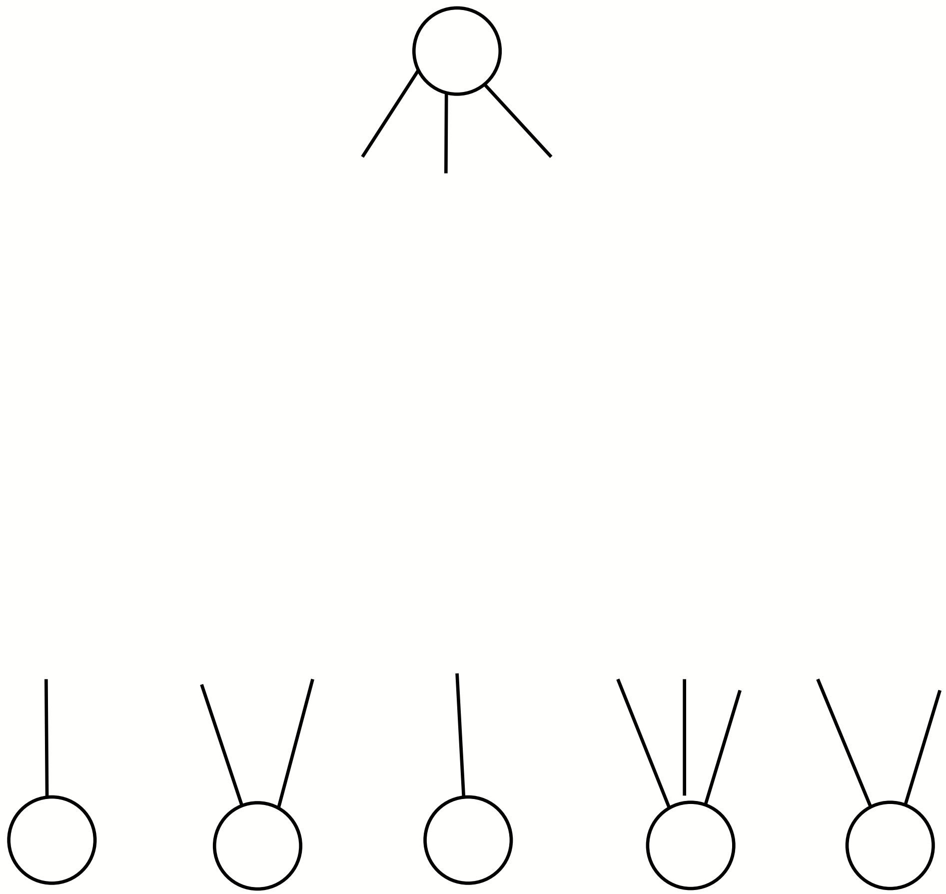
(iii) ensures that $\liminf \mathbb{P}(G^*(n, (d_i)_1^n) \text{ is simple}) > 0$.

# EXPLORATION PROCESS



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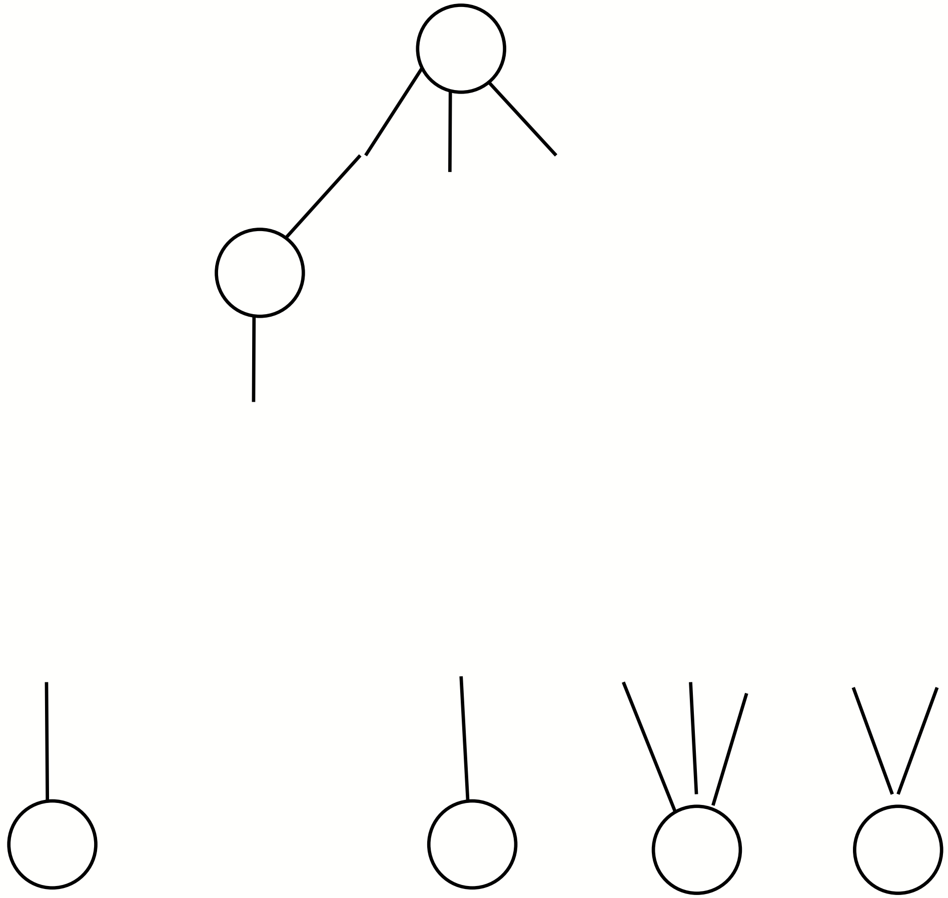

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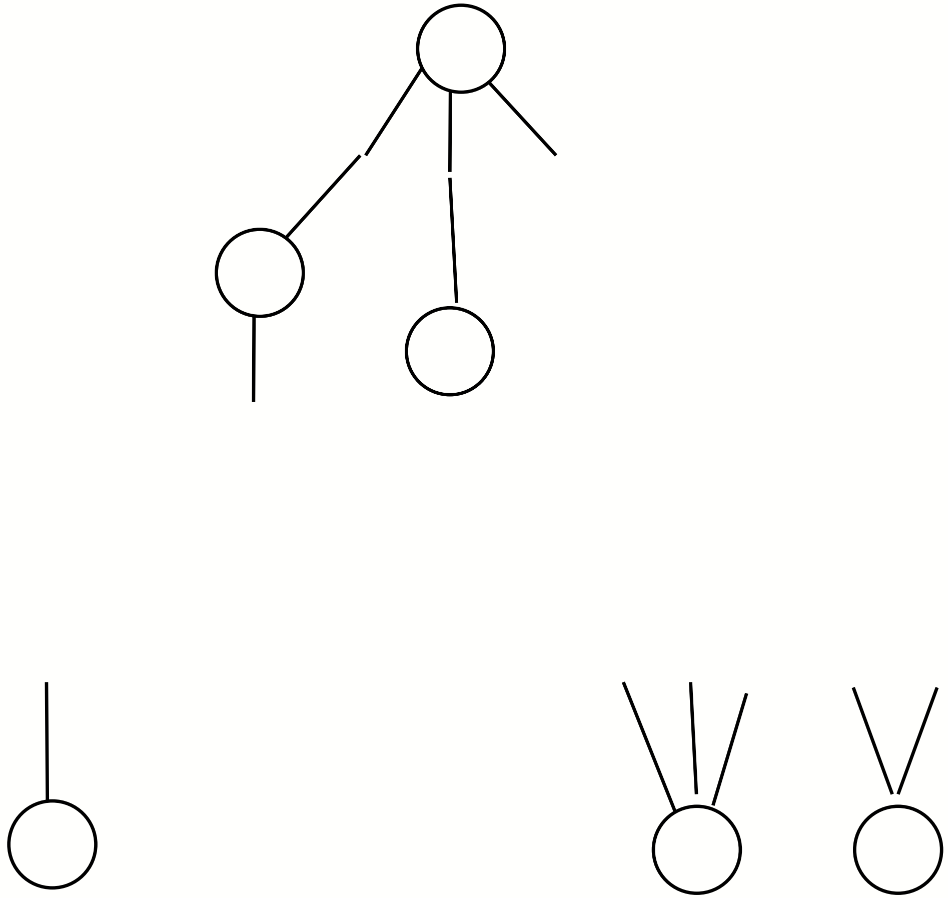

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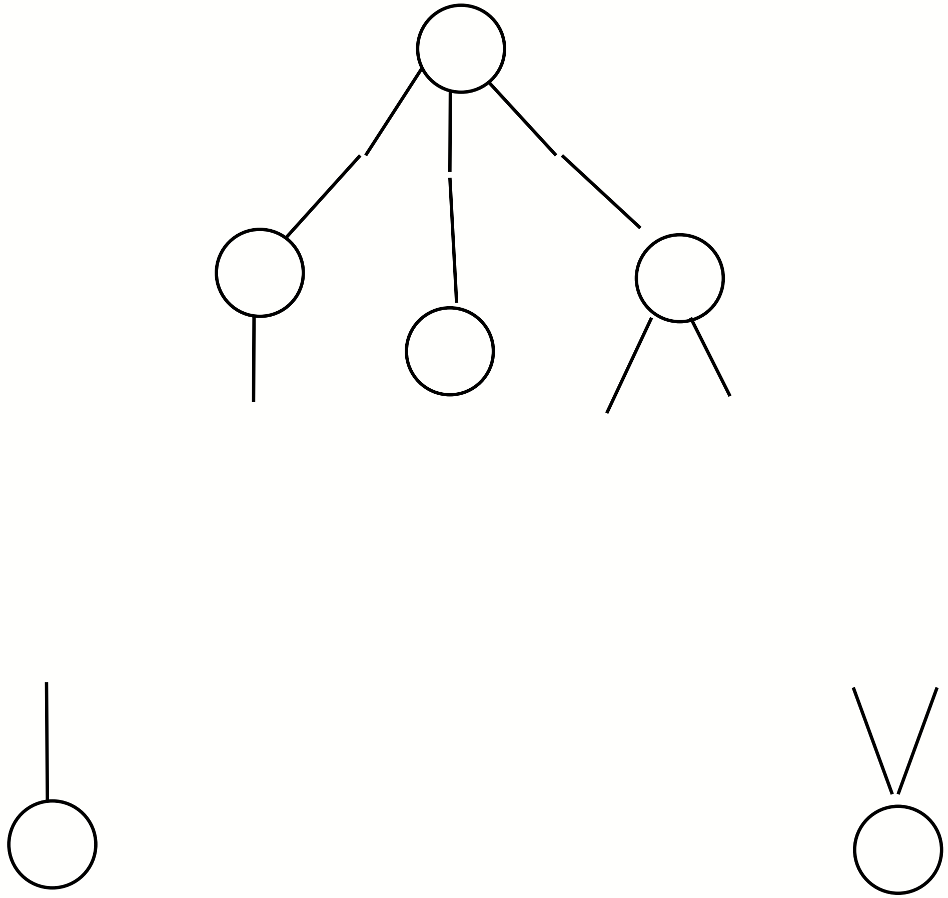


# BRANCHING PROCESS APPROXIMATION

The first individual has offspring distribution $\{p_k\}$.

The other individuals have offspring distribution $\{q_k\}$.

Let $\{q_k\}_{k=0}^{\infty}$ the size-biased probability mass function corresponding to $\{p_k\}$, by

$$q_k = \frac{(k+1)p_{k+1}}{\lambda}\,, \text{ and, } \nu = \sum_{k=0}^{\infty} k q_k \in (0, \infty).$$

The mean of the size of generation $k$ is $\lambda\nu^{k-1}$.

The condition $\nu > 1$ is equivalent to the existence of a giant component.

# TYPICAL GRAPH DISTANCE

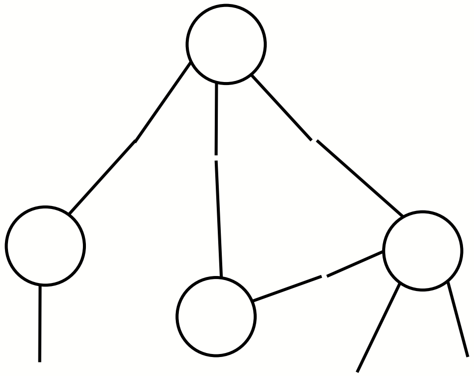
**Theorem 1.** *For $a$ and $b$ chosen uniformly at random in the giant component of $G(n,(d_i)_1^n)$, we have*

$$\frac{\textit{dist}(a,b)}{\log n} \xrightarrow{p} \frac{1}{\log \nu}.$$

Van der Hofstad, Hooghiemstra, Van Mieghem 2005

## A SIMPLE HEURISTIC

Let $Z_k^{(1)}$ be the number of free half-edges in the ball $B(a,k) = \{i, \text{dist}(1,i) \leq k\}$.
$Z_0^{(1)}$ is the degree of node $1$.



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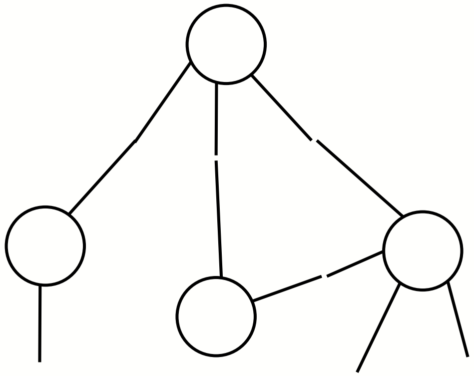

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A free half-edge of $Z_k^{(1)}$ is attached to a free half-edge of $Z_k^{(2)}$ with positive probability if $Z_k^{(1)} Z_k^{(2)}$ is of order the total number of free half-edges left after $k$ exploration steps.

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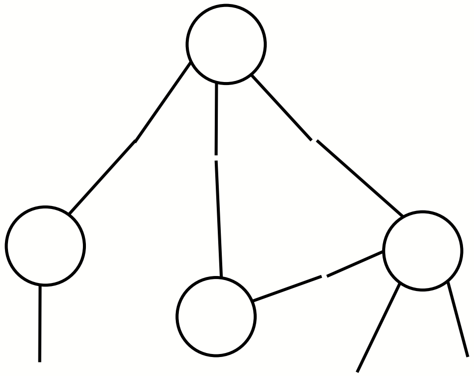

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Take $k \approx \frac{1}{2}\frac{\log n}{\log \nu}$, then $Z_k^{(1)} \approx Z_k^{(2)} \approx \sqrt{n}$ and the number of free half-edges is $\approx n - 2\sqrt{n} \approx n$.

The typical distance between $1$ and $2$ is $\approx 2k = \frac{\log n}{\log \nu}$.

# DIAMETER

Generating function of $\{q_k\}_{k=0}^{\infty}$: $G_q(z) = \sum_{k=0}^{\infty} q_k z^k$.

The extinction probability of the branching process with offspring distribution $\{q_k\}$ is the smallest solution in $[0,1]$ of the fixed point equation $\beta = G_q(\beta)$. We define

$$\beta_* = G_q'(\beta) = \sum_{k=1}^{\infty} k q_k \beta^{k-1}.$$

$d_{\min}$ is the minimum degree of the graph.

**Theorem 2.** *We have*

$$\frac{\textit{diam}(G(n,(d_i)_1^n))}{\log n} \xrightarrow{p} \frac{1}{\log \nu} - \frac{\mathbf{1}(d_{\min}=2)}{\log q_1} - 2\frac{\mathbf{1}(d_{\min}=1)}{\log \beta_*}$$
$$= \frac{1}{\log \nu} + \frac{2 - \mathbf{1}(d_{\min} \geq 2) - \mathbf{1}(d_{\min} \geq 3)}{|\log \beta_*|}$$

Bollobás, de la Vega 1982 (random regular graphs)

Fernholz, Ramachandran 2007

# LOWER DEVIATIONS FOR SUPERCRITICAL GWP

A dichotomy:

Schröder case $\Leftrightarrow \quad q_0 + q_1 > 0 \quad \Rightarrow \lim_{n\to\infty} \beta_*^{-n} \mathbb{P}(Z_n = k) = \nu_k, \quad k \geq 1$

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$$\mathbb{P}(Z_n = \mu^n) = \prod_{j=0}^{n-1} q_\mu^{(\mu^j)} = \exp\left[\frac{\mu^n - 1}{\mu - 1} \log q_\mu\right].$$

More in: Fleischmann Wachtel 2005

## TYPICAL WEIGHTED DISTANCE

**Theorem 3.** *For $a$ and $b$ chosen uniformly at random in $G(n, (d_i)_1^n)$ with $d_{\min} \geq 2$ and with i.i.d. exponential $1$ weights on its edges, we have*

$$\frac{\mathit{dist}_w(a,b)}{\log n} \xrightarrow{p} \frac{1}{\nu - 1}.$$

Bhamidi, Van der Hofstad, Hooghiemstra 2009

Bhamidi, Van der Hofstad, Hooghiemstra 2010 for Erdős-Rényi random graphs.

# TYPICAL WEIGHTED DISTANCE

**Theorem 4.** *For $a$ and $b$ chosen uniformly at random in $G(n,(d_i)_1^n)$ with $d_{\min} \geq 2$ and with i.i.d. exponential $1$ weights on its edges, we have*

$$\frac{\textit{dist}_w(a,b)}{\log n} \xrightarrow{p} \frac{1}{\nu - 1}.$$

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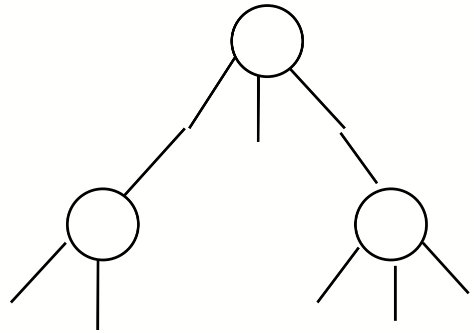
Recall:

$$\frac{\text{dist}(a,b)}{\log n} \xrightarrow{p} \frac{1}{\log \nu} \geq \frac{1}{\nu - 1}.$$

# HEURISTIC: SPLIT TIMES



$$\widehat{S}_k(a) = d_a + \widehat{d}_1 + ... + \widehat{d}_k - k$$

$$T_k(a) = \sum_{i=0}^{k-1} \frac{E_i}{\widehat{S}_i(a)}$$

We have $\mathbb{E}[\widehat{S}_i(a)] \approx (\nu - 1)i$ and

$$\lim_{k\to\infty} \frac{T_k}{\log k} = \frac{1}{\nu - 1}.$$

# WEIGHTED DIAMETER

**Theorem 5.** *Consider a random graph $G(n,(d_i)_1^n)$ with i.i.d. exponential $1$ weights on its edges, then*

$$\frac{\textit{diam}_w(G(n,(d_i)_1^n))}{\log n} \xrightarrow{p} \frac{1}{\nu-1} + \frac{2}{d_{\min}}\mathbf{1}_{(d_{\min}\geq 3)} + \frac{\mathbf{1}_{(d_{\min}=2)}}{1-q_1} + \frac{2}{1-\beta_*}\mathbf{1}_{(d_{\min}=1)}.$$

Ding, Han Kim, Lubetzky, Peres 2010 (random regular graphs)

Amini, Draief, L. 2010

# WEIGHTED DIAMETER

**Theorem 6.** *Consider a random graph $G(n,(d_i)_1^n)$ with i.i.d. exponential $1$ weights on its edges, then*

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Amini, Draief, L. 2010

Recall:

$$\frac{\text{dist}_w(a,b)}{\log n} \xrightarrow{p} \frac{1}{\nu-1}$$

$$\frac{\text{diam}(G(n,(d_i)_1^n))}{\log n} \xrightarrow{p} \frac{1}{\log\nu} - \frac{\mathbf{1}(d_{\min}=2)}{\log q_1} - 2\frac{\mathbf{1}(d_{\min}=1)}{\log \beta_*}.$$

# LARGE DEVIATIONS FOR SPLIT TIMES

$$\widehat{S}_k(a) = d_a + \widehat{d}_1 + \ldots + \widehat{d}_k - k \qquad T_k(a) = \sum_{i=0}^{k-1} \frac{E_i}{\widehat{S}_i(a)}$$

Laplace transform:

$$\mathbb{E}\left[e^{\theta T_k(a)}\right] = \prod_{i=0}^{k-1} \left(1 + \frac{\theta}{\widehat{S}_i(a) - \theta}\right)$$

Case $d_{\min} \geq 3$,

$$\begin{aligned}
\mathbb{P}(T_{\alpha_n}(a) \geq x \log n) &\leq \mathbb{E}\left[e^{d_{\min} T_{\alpha_n}(a)}\right] \exp(-x d_{\min} \log n) \\
&\approx \alpha_n^{\frac{d_{\min}}{d_{\min}-2}} n^{-x d_{\min}} \\
\mathbb{P}(T_{\beta_n}(a) - T_{\alpha_n}(a) \geq y \log n) &\leq \mathbb{E}\left[e^{\theta T_{\beta_n}(a) - T_{\alpha_n}(a)}\right] \exp(-\theta y \log n) \\
&\approx n^{\theta\left(\frac{1}{2(\nu-1-\epsilon)} - y\right)}
\end{aligned}$$

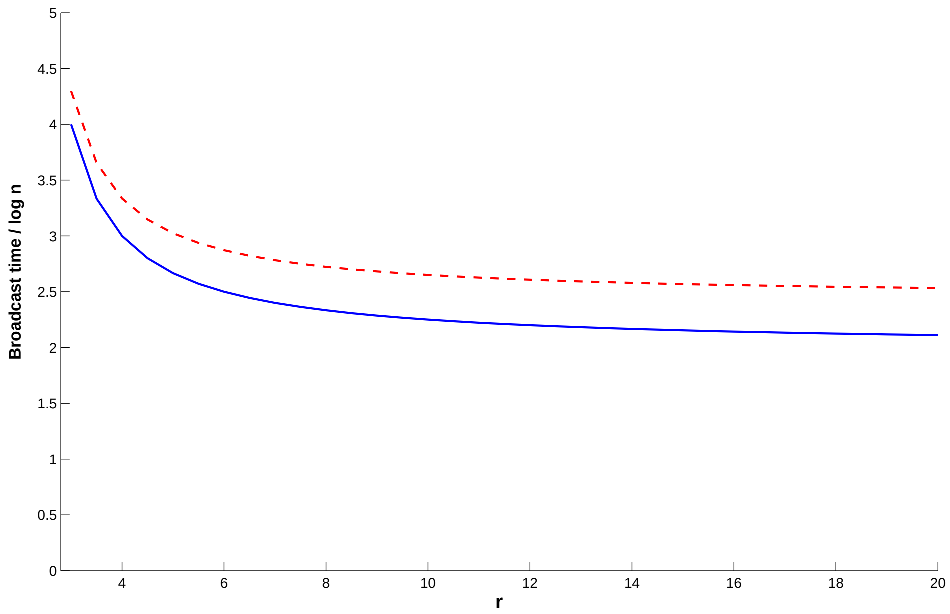
# A SIMPLE APPLICATION: ASYNC. BROADCAST IN RRG

Each node has a Poisson clock.

One chunk - PUSH

**Corollary 1.** *Let $G \sim \mathcal{G}(n, r)$ be a random $r$-regular graph with $n$ vertices. Then w.h.p.*

$$\mathit{ABT}(G) = 2\left(\frac{r-1}{r-2}\right)\log n + o(\log n).$$




Fountoulakis Panagtotou 2010

THANK YOU!