Diameter in weighted random graphs

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WEIGHTED DIAMETER

Graph $G = (V, E)$:

- Distance $\text{dist}(a, b) = \min_{\pi \in \Pi(a,b)} |\pi|$, the number of edges in $E$ in the shortest path connecting $a$ and $b$.

- Diameter of $G$ defined by:

$$\text{diam}(G) = \max\{\text{dist}(a, b), \ a, b \in V, \ \text{dist}(a, b) < \infty\}.$$ 

- Weight associated to each edge $e \in E$: $w_e$.

- Weighted distance $\text{dist}_w(a, b) = \min_{\pi \in \Pi(a,b)} \sum_{e \in \pi} w_e$.

- Weighted diameter of $G$ defined by:

$$\text{diam}_w(G) = \max\{\text{dist}_w(a, b), \ a, b \in V, \ \text{dist}_w(a, b) < \infty\}.$$
CONFIGURATION MODEL

For $n \in \mathbb{N}$, let $(d_i)_1^n$ be a sequence of non-negative integers such that $\sum_{i=1}^{n} d_i$ is even.

We define $G^*(n, (d_i)_1^n)$ a random multigraph with given degree sequence $(d_i)_1^n$: to each node $i$ we associate $d_i$ labeled half-edges. All half-edges need to be paired to construct the graph, this is done by randomly matching them. When a half-edge of $i$ is paired with a half-edge of $j$, we interpret this as an edge between $i$ and $j$.

The graph $G^*(n, (d_i)_1^n)$ obtained following this procedure may not be simple. Conditional on the multigraph $G^*(n, (d_i)_1^n)$ being a simple graph, we obtain a uniformly distributed random graph with the given degree sequence, which we denote by $G(n, (d_i)_1^n)$,
ASSUMPTIONS ON THE DEGREE SEQUENCE

(i) There exists a distribution $p = \{p_k\}_{k=0}^{\infty}$ such that $|\{i, d_i = k\}|/n \to p_k$ for every $k \geq 0$ as $n \to \infty$;

(ii) $\lambda := \sum_{k \geq 0} kp_k \in (0, \infty)$;

(iii) $\sum_{i=1}^{n} d_i^2 = O(n)$,

(iv) for some $\tau > 0$, $\Delta_n := \max_{i \in V} d_i = O(n^{1/2-\tau})$. 
ASSUMPTIONS ON THE DEGREE SEQUENCE

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Let \( \hat{D}_n \) be the degree of a random vertex of \( G^*(n, (d_i)_1^n) \) and \( D \) a random variable with distribution \( p_k \), then (i) is equivalent to \( \hat{D}_n \overset{d}{\to} D \).
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(iii) ensures that \( \lim \inf \mathbb{P}(G^*(n, (d_i)_n) \text{ is simple}) > 0 \).
EXPLORATION PROCESS
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BRANCHING PROCESS APPROXIMATION

The first individual has offspring distribution $\{p_k\}$.

The other individuals have offspring distribution $\{q_k\}$.

Let $\{q_k\}_{k=0}^{\infty}$ the size-biased probability mass function corresponding to $\{p_k\}$, by

$$q_k = \frac{(k+1)p_{k+1}}{\lambda},$$

and,

$$\nu = \sum_{k=0}^{\infty} k q_k \in (0, \infty).$$

The mean of the size of generation $k$ is $\lambda \nu^{k-1}$.

The condition $\nu > 1$ is equivalent to the existence of a giant component.
TYPICAL GRAPH DISTANCE

**Theorem 1.** For \(a\) and \(b\) chosen uniformly at random in the giant component of \(G(n, (d_i)_1^n)\), we have

\[
\frac{\text{dist}(a, b)}{\log n} \xrightarrow{p} \frac{1}{\log \nu}.
\]

Van der Hofstad, Hooghiemstra, Van Mieghem 2005
A SIMPLE HEURISTIC

Let $Z_k^{(1)}$ be the number of free half-edges in the ball $B(a, k) = \{i, \text{dist}(1, i) \leq k\}$. $Z_0^{(1)}$ is the degree of node 1.

By the branching process approximation, $Z_k^{(1)}$ is close to $\lambda \nu^{k-1}$.
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A free half-edge of $Z_k^{(1)}$ is attached to a free half-edge of $Z_k^{(2)}$ with positive probability if $Z_k^{(1)} Z_k^{(2)}$ is of order the total number of free half-edges left after $k$ exploration steps.
A SIMPLE HEURISTIC

Let $Z^{(1)}_k$ be the number of free half-edges in the ball $B(a, k) = \{i, \text{dist}(1, i) \leq k\}$. $Z^{(1)}_0$ is the degree of node 1.

By the branching process approximation, $Z^{(1)}_k$ is close to $\lambda \nu^{k-1}$.

A free half-edge of $Z^{(1)}_k$ is attached to a free half-edge of $Z^{(2)}_k$ with positive probability if $Z^{(1)}_k Z^{(2)}_k$ is of order the total number of free half-edges left after $k$ exploration steps.

Take $k \approx \frac{1}{2} \frac{\log n}{\log \nu}$, then $Z^{(1)}_k \approx Z^{(2)}_k \approx \sqrt{n}$ and the number of free half-edges is $\approx n - 2\sqrt{n} \approx n$.

The typical distance between 1 and 2 is $\approx 2k = \frac{\log n}{\log \nu}$. 
Generating function of $\{q_k\}_{k=0}^{\infty}$: $G_q(z) = \sum_{k=0}^{\infty} q_k z^k$.

The extinction probability of the branching process with offspring distribution $\{q_k\}$ is the smallest solution in $[0, 1]$ of the fixed point equation $\beta = G_q(\beta)$. We define

$$\beta_* = G'_q(\beta) = \sum_{k=1}^{\infty} k q_k \beta^{k-1}.$$ 

$d_{\text{min}}$ is the minimum degree of the graph.

**Theorem 2.** We have

$$\frac{\text{diam}(G(n, (d_i)_1^n))}{\log n} \xrightarrow{p} \frac{1}{\log \nu} - \frac{1}{\log q_1} \mathbf{1}(d_{\text{min}} = 2) - 2 \frac{1}{\log \beta_*} \mathbf{1}(d_{\text{min}} = 1)$$

$$= \frac{1}{\log \nu} + \frac{2 \mathbf{1}(d_{\text{min}} \geq 2) - \mathbf{1}(d_{\text{min}} \geq 3)}{|\log \beta_*|}$$

Bollobás, de la Vega 1982 (random regular graphs)

Fernholz, Ramachandran 2007
A dichotomy:

Schröder case $\Leftrightarrow q_0 + q_1 > 0 \Rightarrow \lim_{n \to \infty} \beta_n^* P(Z_n = k) = \nu_k, \quad k \geq 1$

Böttcher case $\Leftrightarrow q_0 + q_1 = 0 \Rightarrow \mu = \min\{j, q_j > 0\} \geq 2$
LOWER DEVIATIONS FOR SUPERCRITICAL GWP

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Böttcher case $\iff q_0 + q_1 = 0 \implies \mu = \min\{j, q_j > 0\} \geq 2$

$\implies \mathbb{P}(Z_n < \mu^n) = 0$
LOWER DEVIATIONS FOR SUPERCritical GWp

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Schröder case ⇔ \( q_0 + q_1 > 0 \) \( \Rightarrow \) \( \lim_{n \to \infty} \beta_n^{-n} \mathbb{P}(Z_n = k) = \nu_k, \quad k \geq 1 \)

Böttcher case ⇔ \( q_0 + q_1 = 0 \) \( \Rightarrow \) \( \mu = \min\{j, q_j > 0\} \geq 2 \)
\( \Rightarrow \mathbb{P}(Z_n < \mu^n) = 0 \)

\[ \mathbb{P}(Z_n = \mu^n) = \prod_{j=0}^{n-1} q_{\mu^j} = \exp \left[ \frac{\mu^n - 1}{\mu - 1} \log q_\mu \right]. \]

More in: Fleischmann Wachtel 2005
Theorem 3. For \( a \) and \( b \) chosen uniformly at random in \( G(n, (d_i)_1^n) \) with \( d_{\min} \geq 2 \) and with i.i.d. exponential 1 weights on its edges, we have

\[
\frac{\text{dist}_w(a, b)}{\log n} \xrightarrow{p} \frac{1}{\nu - 1}.
\]

Bhamidi, Van der Hofstad, Hooghiemstra 2009

Bhamidi, Van der Hofstad, Hooghiemstra 2010 for Erdős-Rényi random graphs.
Theorem 4. For $a$ and $b$ chosen uniformly at random in $G(n, (d_i)_1^n)$ with $d_{\text{min}} \geq 2$ and with i.i.d. exponential 1 weights on its edges, we have

$$\frac{\text{dist}_w(a, b)}{\log n} \xrightarrow{p} \frac{1}{\nu - 1}.$$ 

Bhamidi, Van der Hofstad, Hooghiemstra 2009

Bhamidi, Van der Hofstad, Hooghiemstra 2010 for Erdős-Rényi random graphs.

Recall:

$$\frac{\text{dist}(a, b)}{\log n} \xrightarrow{p} \frac{1}{\log \nu} \geq \frac{1}{\nu - 1}.$$
HEURISTIC: SPLIT TIMES

\[ \hat{S}_k(a) = d_a + \hat{d}_1 + \ldots + \hat{d}_k - k \]

\[ T_k(a) = \sum_{i=0}^{k-1} \frac{E_i}{\hat{S}_i(a)} \]

We have \( \mathbb{E}[\hat{S}_i(a)] \approx (\nu - 1)i \) and

\[
\lim_{k \to \infty} \frac{T_k}{\log k} = \frac{1}{\nu - 1}.
\]
Theorem 5. Consider a random graph $G(n, (d_i)^n_1)$ with i.i.d. exponential 1 weights on its edges, then

$$\frac{\text{diam}_w(G(n, (d_i)^n_1))}{\log n} \xrightarrow{p} \frac{1}{\nu - 1} + \frac{2}{d_{\min}} 1(d_{\min} \geq 3) + \frac{1(d_{\min}=2)}{1 - q_1} + \frac{2}{1 - \beta_*} 1(d_{\min}=1).$$

Ding, Han Kim, Lubetzky, Peres 2010 (random regular graphs)
Amini, Draief, L. 2010
WEIGHTED DIAMETER

Theorem 6. Consider a random graph $G(n, (d_i)_1^n)$ with i.i.d. exponential 1 weights on its edges, then

$$\frac{\text{diam}_w(G(n, (d_i)_1^n))}{\log n} \xrightarrow{p} \frac{1}{\nu - 1} + \frac{2}{d_{\min}} 1(d_{\min} \geq 3) + \frac{1}{1 - q_1} 1(d_{\min} = 2) + \frac{2}{1 - \beta_*} 1(d_{\min} = 1).$$

Ding, Han Kim, Lubetzky, Peres 2010 (random regular graphs)
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Recall:

$$\frac{\text{dist}_w(a, b)}{\log n} \xrightarrow{p} \frac{1}{\nu - 1}$$

$$\frac{\text{diam}(G(n, (d_i)_1^n))}{\log n} \xrightarrow{p} \frac{1}{\log \nu} - \frac{1}{\log q_1} 1(d_{\min} = 2) - 2 \frac{1}{\log \beta_*} 1(d_{\min} = 1).$$
LARGE DEVIATIONS FOR SPLIT TIMES

\[ \hat{S}_k(a) = d_a + \hat{d}_1 + \ldots + \hat{d}_k - k \quad T_k(a) = \sum_{i=0}^{k-1} \frac{E_i}{\hat{S}_i(a)} \]

Laplace transform:

\[ \mathbb{E} \left[ e^{\theta T_k(a)} \right] = \prod_{i=0}^{k-1} \left( 1 + \frac{\theta}{\hat{S}_i(a) - \theta} \right) \]

Case \( d_{\text{min}} \geq 3 \),

\[ \mathbb{P}(T_{\alpha_n}(a) \geq x \log n) \leq \mathbb{E} \left[ e^{d_{\text{min}} T_{\alpha_n}(a)} \right] \exp(-x d_{\text{min}} \log n) \approx \alpha_n^{d_{\text{min}} - 2} n^{-x d_{\text{min}}} \]

\[ \mathbb{P}(T_{\beta_n}(a) - T_{\alpha_n}(a) \geq y \log n) \leq \mathbb{E} \left[ e^{\theta T_{\beta_n}(a) - T_{\alpha_n}(a)} \right] \exp(-\theta y \log n) \approx n^{\theta \left( \frac{1}{2(\nu - 1 - \epsilon)} - y \right)} \]
Each node has a Poisson clock.

One chunk - PUSH

**Corollary 1.** Let $G \sim \mathcal{G}(n, r)$ be a random $r$-regular graph with $n$ vertices. Then w.h.p.

$$ABT(G) = 2\left(\frac{r - 1}{r - 2}\right) \log n + o(\log n).$$

Fountoulakis Panagtotou 2010
THANK YOU!