Matchings and rank for random diluted graphs

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(i) GEOGRAPHY GAME

(ii) RANK

(iii) MATCHINGS
(i) GEOGRAPHY GAME

inspired from Johan Wästlund

(ii) RANK

(iii) MATCHINGS
GEOGRAPHY GAME

- 2 players game: Alice plays first and Bob follows.
- Rule: name geographic locations, each starting with the same letter with which the previous name ended.
- Winner: the last player who is able to give a name.
- London
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- London Nicosia Ankara Amsterdam Madrid Dublin New Delhi...
UNDIRECTED VERTEX GEOGRAPHY

- General graph
- Alice and Bob take turns choosing the edges of a self-avoiding walk
- Whoever gets stuck loses
- All players know all information about gameplay.
Random model: Geography on the Poisson Galton-Watson tree

Random rooted tree: Each node has $Po(\lambda)$-distributed number of children.
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Random rooted tree: Each node has \( Po(\lambda) \)-distributed number of children.

- Let \( p = \mathbb{P}(\text{Bob wins under optimal play}) \)
Random model: Geography on the Poisson Galton-Watson tree

Random rooted tree: Each node has $Po(\lambda)$-distributed number of children.

- Let $p = \mathbb{P}(\text{Bob wins under optimal play}) = \mathbb{P}(\text{root is colored red})$. 
Random model: Geography on the Poisson Galton-Watson tree

Random rooted tree: Each node has $Po(\lambda)$-distributed number of children.

- Let $p = \mathbb{P}(\text{Bob wins under optimal play}) = \mathbb{P}(\text{root is colored red})$.

- Thanks to the branching property:

  \[ p = \mathbb{P}(\text{all children are green}) = \mathbb{P}(\text{no child is red}) = e^{-\lambda p}, \]

  and so

  \[ p = \frac{W(\lambda)}{\lambda}. \]
A NAIVE GUESS

The function $p = P(\text{Bob wins})$ as a function of the average offspring $\lambda$. 
The function \( p = \mathbb{P}(\text{Bob wins}) \) as a function of the average offspring \( \lambda \).
WHAT HAPPENED?

The solution $p = \frac{W(\lambda)}{\lambda}$ is the fixed-point of the map:

$$p \mapsto e^{-\lambda p}.$$

But the truth about the game comes from iterating that map.
Let $p_k = \mathbb{P}(\text{Bob wins if the tree is truncated after } k \text{ moves})$. 
TRUNCATED GAME

Let $p_k = \mathbb{P}(\text{Bob wins if the tree is truncated after } k \text{ moves})$.

- $p_0 = 1$
Let $p_k = \mathbb{P}(\text{Bob wins if the tree is truncated after } k \text{ moves}).$

- $p_0 = 1$
- $p_1 = \mathbb{P}(\text{Po}(\lambda) = 0) = e^{-\lambda}$
Let $p_k = \mathbb{P}(\text{Bob wins if the tree is truncated after } k \text{ moves})$.

- $p_0 = 1$
- $p_1 = \mathbb{P}(\text{Po}o(\lambda) = 0) = e^{-\lambda}$
- Then for $k \geq 0$, 

  \[ p_{k+1} = \mathbb{P}(\text{Bob looses from any child of the root in the } k\text{-truncated tree}) \]

  \[ = \mathbb{E}[(1 - p_k)^N] \text{ where, } N \sim \text{Po}(\lambda) \]

  \[ = e^{-\lambda p_k} \]
\( \lambda = 2.5 \)
\[ \lambda = 2.5 \]
$\lambda = 2.9$
TRUTH

\[ \lambda = 2.9 \]
APPARITION OF THE CORE

Do the leaf removal algorithm on the infinite tree.
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Do the leaf removal algorithm on the infinite tree.

For $\lambda > e$, some paths to infinity will never be 'touched' by the algorithm.

These paths constitute the core of the original tree.

Implications for the game:

- if the game starts outside of the core, there will be a winner.

- if the game starts in the core, there exists an infinite sequence of optimal moves staying in the core.
On the core, the parity of $k$ will determine the winner of the truncated game.

Draw $\leftrightarrow$ influence of boundary conditions remains positive.
(i) GEOGRAPHY GAME

(ii) RANK

(iii) MATCHINGS
SPECTRAL MEASURE OF FINITE GRAPHS

Let $G_n = (V_n, E_n)$ be a simple graph on $V_n = \{1, \cdots, n\}$. We define

$$A_n = \text{Adjacency matrix of } G_n = (\mathbb{1}(i,j \in E_n))_{1 \leq i,j \leq n}$$

Let

$$\lambda_n(A_n) \leq \cdots \leq \lambda_1(A_n)$$

denote the real eigenvalues of the symmetric matrix $A_n$.

The spectral measure of $G_n$ is

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(A_n)}.$$ 

The rank of $A_n$ is

$$\text{rank}(A_n) = 1 - \mu_n(\{0\}).$$
QUESTIONS

Let \((G_n), n \in \mathbb{N}\), be a sequence of graphs on \(V_n = \{1, \cdots, n\}\) such that \(G_n\) "converges" to a limit graph \(G\).

1. Does \(\mu_n\) converge to a measure \(\mu_G\) for the usual weak convergence topology?

2. Do we have a formula for \(\mu_G\) in some cases?

3. Does \(\text{rank} (A_n)\) converge? to \(\mu_G(\{0\})\)?
Let $G_n$ is a random graph drawn uniformly on the set of $d$-regular graphs on $n$ vertices.

**Theorem 1** (McKay 1981). For $d \geq 3$ the spectral measure $\mu_n$ converges weakly as $n$ goes to infinity to the deterministic measure $\mu_{KM}$ supported on $[-2\sqrt{d-1}, 2\sqrt{d-1}]$

$$\mu_{KM}(dx) = \frac{d}{2\pi} \frac{\sqrt{4(d-1) - x^2}}{d^2 - x^2} dx.$$

$\implies \mu_{KM}$ is the Kesten-McKay measure, it first appeared in (Kesten 1959) in the context of simple random walks on groups.

Khorunzhy, Shchrebina, Vengerovsky (04): convergence of the spectral measure for Erdős-Rényi graphs.

Bauer & Gollineli (00): asymptotic rank of the uniform spanning tree on the complete graph.

Bhamidi, Evans, Sen (09): convergence of the spectrum of the adjacency matrix of growing trees.
We define $\text{core}(G_n)$ as the remaining graph after iterative leaf removal and removal of all isolated vertices.

$$\dim \ker A_n = |\{\text{isolated vertices in leaf removal}\}| + \dim \ker A(\text{core}(G_n)).$$

For a finite tree, the core is empty: $\mu_n(\{0\}) = \frac{1}{n}|\{\text{isolated vertices in leaf removal}\}|$.

In general, $\mu_n(\{0\}) \geq \frac{1}{n}|\{\text{isolated vertices in leaf removal of } G_n\}|$. 

We define $\text{core}(G_n)$ as the remaining graph after iterative leaf removal and removal of all isolated vertices.
EXERCISE!

What is the size of the kernel of this tree?
The size of its kernel is 3.
We are interested in a sequence $G_n$ of random diluted graphs: $\deg(v; G_n) = O(1)$ as the number of vertices $n$ tends to infinity.

Important examples of random graphs on $\{1, \cdots, n\}$,

- Erdős-Rényi graphs with parameter $p = \lambda/n$.
- Uniform measure on $k$-regular graphs.
- Graphs with prescribed degree distribution $(\pi_k)$: independently for each vertex, we draw a random number of half-edges with distribution $(\pi_k)$. If the total number of half-edge is even, we match them uniformly.
SPECTRAL MEASURE ROOTED AT A VERTEX

$\ell^2(V_n)$ admits an orthonormal basis of eigenvectors $(b_1, \ldots, b_n)$, a priori different from the canonical orthonormal basis $(e_v)_{v \in V_n}$, such that:

$$\forall x \in \mathbb{C}^n, \quad A_n x = \sum_{i=1}^{n} \lambda_i(A_n) \langle x, b_i \rangle b_i.$$ 

The spectral measure of $G_n$ when rooted at $v$ is simply

$$\mu(G_n, v) = \sum_{i=1}^{n} |\langle b_i, e_v \rangle|^2 \delta_{\lambda_i(A_n)},$$

so that for any measurable function $f$,

$$\langle f(A_n)e_v, e_v \rangle = \int_{\mathbb{R}} f(x) d\mu(G_n, v)(x).$$

$\mu(G_n, v)$ is the contribution of vertex $v$ to the empirical spectral measure:

$$\mu_n = \frac{1}{n} \sum_{v \in V_n} \mu(G_n, v).$$
Theorem 2. If $G_n$ converges weakly under uniform rooting to a rooted Galton-Watson tree (GWT) $T$ (with a finite first moment for the offspring distribution), then in probability,

$$
\mu_n = \frac{1}{n} \sum_{v \in V_n} \mu(G_n,v) \to E \mu_T
$$

Hence we have an upper bound on the size of the kernel:

$$
\limsup_{n \to \infty} \mu_n(\{0\}) \leq E \mu_T(\{0\}),
$$

so that the convergence of the rank is proved when

$$
\lim \frac{1}{n} \left| \{ \text{isolated vertices in leaf removal of } G_n \} \right| = E \mu_T(\{0\}).
$$
If $T$ is a GWT with finite branching, then its adjacency operator $A_T$ is self-adjoint a.s.

On $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im(z) > 0\}$, the Stieltjes transform of $\mu_T$ is defined by

$$s_T(z) = \int_{\mathbb{R}} \frac{\mu_T(dx)}{x - z} = \langle R_T(z)e_o, e_o \rangle,$$

where

$$R_T(z) = (A_T - zI)^{-1}$$

is the resolvent of the adjacency operator of $T$.

$\implies$ The Stieltjes transform characterizes the probability measure.

In particular, $\mu_T(\{0\}) = \lim_{t \downarrow 0} t\Im s_T(it)$. 
Schur’s identity for an invertible hermitian matrix $B = \begin{pmatrix} b_{11} & u^* \\ u & \tilde{B} \end{pmatrix}$, 

$$(B^{-1})_{11} = \left(b_{11} - u^* \tilde{B}^{-1} u\right)^{-1}.\] 

For a tree $T$, we get for $B = A_T - zI$, 

$$s_T(z) = \langle R_T(z) e_o, e_o \rangle = - \left(z + \sum_{v \sim o} \langle R_{T \setminus o}(z) e_v, e_v \rangle \right)^{-1}$$ 

$$= - \left(z + \sum_{v \sim o} s_{T_v}(z) \right)^{-1}.\]
Thanks to the branching property, we get a Recursive Distributional Equation (RDE) for the law of $s_T(.)$.

Let $\mathcal{H}$ be the set of holomorphic functions from $\mathbb{C}_+$ to $\mathbb{C}_+$ such that $|f(z)| \leq (\Im z)^{-1}$.

**Theorem 3.** (i) There exists an unique probability measure on $\mathcal{H}$ such that

$$s(.) \overset{d}{=} - \left( (. ) + \sum_{k=1}^{N} s_k(.) \right)^{-1},$$

where $s_k$, $s$ are i.i.d. copies independent of $N$ with law $(\pi^s_k)$.

(ii) The law of the Stieltjes transform of $\mu_T$ is given by,

$$s_T(.) \overset{d}{=} - \left( (. ) + \sum_{k=1}^{N_\ast} s_k(.) \right)^{-1},$$

where $N_\ast$ has distribution $(\pi^s_k)$ and $s_k$ are independents with law $s$. 
RDE FOR THE KERNEL

Recall that $\mu_T(\{0\}) = \lim_{t \downarrow 0} t \Im s_T(it)$.

Let $h(t) = \Im s(it)$, so that by definition,

$$h(t) \xrightarrow{d} \left( t + \sum_{i=1}^{N} h_i(t) \right)^{-1}$$

and then iterating once,

$$th(t) \xrightarrow{d} \left( 1 + \sum_{i=1}^{N} \left( t^2 + \sum_{j=1}^{N_i} th_{i,j}(t) \right)^{-1} \right)^{-1}.$$  

If $\xi = \lim_{t \downarrow 0} th(t) \in [0, 1]$, we get

$$\xi = \left( 1 + \sum_{i=1}^{N} \left( \sum_{j=1}^{N_i} \xi_{i,j} \right)^{-1} \right)^{-1}.$$
SOLVING THE RDE FOR THE KERNEL

If $\varphi$ is the generating function of the asymptotic degree distribution, let

$$F(x) = \varphi'(1)x\bar{x} + \varphi(1-x) + \varphi(1-\bar{x}) - 1,$$

where $\bar{x} = \varphi'(1-x)/\varphi'(1)$.

$F$ admits an historical record at $x$ if $x = \bar{x}$ and $F(x) > F(y)$ for any $0 \leq y < x$.

**Theorem 4.** If $p_1 < \ldots < p_r$ are the locations of the historical records of $F$, then the RDE admits exactly $r$ solutions, say $0 \leq X_1 <_{st} \ldots <_{st} X_r \leq 1$, and for any $i \in \{1, \ldots, r\}$, $\mathbb{E}[X_i] = F(p_i)$. 
ERDŐS RÉNYI GRAPHS

The core does not contribute to the size of the kernel as $n$ tends to infinity:

$$
\mu_n(\{0\}) = n^{-1}|\{\text{isolated vertices in LR}\}| + n^{-1} \dim \ker \text{core}(G_n)
$$

$$
= \mathbb{E}\mu_T(\{0\}) + o(n)
$$

$$
= p + e^{-\lambda p} + \lambda pe^{-\lambda p} - 1,
$$

where $p$ is the smallest root of $x = e^{-\lambda e^{-\lambda x}}$.

Function $F$ for $\lambda = 2$, $\lambda = e$ and $\lambda = 3$. 
When the RDE has more than one solution, then \( n^{-1} |\{\text{isolated vertices in LR}\}| \) converges to the smallest one and \( \mathbb{E} \mu_T(\{0\}) \) to the largest one!

Function \( F \) for \( \pi_d = \frac{d}{1+d} \) and \( \pi_d^3 = \frac{1}{1+d} \).
(i) GEOGRAPHY GAME

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MAXIMUM MATCHINGS

Let $G = (V, E)$ be a simple finite graph. A matching of $G$ is a subset of edges with no common end-vertex.

- The matching is **perfect** if all vertices are covered.

- The matching is **maximum** if the number of covered vertices is maximum.
MAXIMUM MATCHING IN ERDŐS-RÉNYI GRAPHS

Let $G_n$ be an Erdős-Rényi graph on $\{1, \cdots, n\}$ with parameter $\lambda/n$.

**Theorem 5** (Karp & Sipser (82)). *In probability,*

$$\lim_{n \to \infty} \frac{|\{\text{uncovered vertices in a maximal matching}\}|}{n} = p + e^{-\lambda p} + \lambda pe^{-\lambda p} - 1,$$

where $p$ is the smallest root of $x = e^{-\lambda e^{-\lambda x}}$.

If $0 \leq \lambda \leq e$ then $p = e^{-\lambda p}$ is solution to the Lambert equation. If $\lambda > e$, $x = e^{-\lambda e^{-\lambda x}}$ has two solutions in $[0, 1]$. 
Left: $\lim_{n \to \infty} \frac{|\{\text{uncovered vertices in a maximal matching}\}|}{n}$ as function of $\lambda$.

Right: $q$, smallest root of $x = e^{-\lambda} e^{-\lambda x}$, as function of $\lambda$. 
EXERCISE!

What is the number of uncovered vertices in a maximum matching of this tree?
The minimal number of uncovered vertices is 3.
BOLTZMANN MEASURE

(Zdeborová & Mézard 06)

Let $G = (V, E)$ be a simple finite graph with $M$ a matching of $G$.

The defect (energy) of $M$, denoted $H(M)$, is the number of uncovered vertices in $M$.

The associated Boltzmann measure with temperature $z \geq 0$ is

$$
\mu_G^z(M) = \frac{z^{H(M)}}{P_G(z)}
$$

where $P_G(z)$ is the matching defect polynomial:

$$
P_G(z) = \sum_{M : \text{matchings}} z^{H(M)}.
$$
Do we have convergence of the energy density?

\[
\frac{1}{n} \langle H(M) \rangle_{z,G} = \frac{1}{n} \sum_{M : \text{matchings}} H(M) \mu_G(M)
\]

YES in the following cases:

- zero temperature \( z = 0 \):
  - Karp & Sipser (82) for Erdős-Rényi graph.
  - Bohman & Frieze (09) with a ’log-concave’ condition on the degree distribution.

- positive temperature \( z > 0 \):
  - Bayati & Nair (06) under a restrictive large girth condition.
  - Bayati, Gamarnik, Katz, Nair, Tetali (07) for very high temperature.

For the random assignment problem: it converges at zero temperature to \( \zeta(2) \), Aldous (01) and at very high temperature, Talagrand (03).
THE CAVITY METHOD

Rewrite the energy density as follows:

\[
\frac{1}{n} \langle H(M) \rangle_{z,G} = \frac{1}{n} \sum_{o \in V} \mu_G^z(o \text{ is uncovered}).
\] (1)

Use the fundamental recursion relation:

\[
P_G(z) = \sum_{M: \text{matchings}} z^{H(M)} = zP_{G-o}(z) + \sum_{v \sim o} P_{G-\{o,v\}}(z).
\]

Multiply by \( zP_{G-o}(z)^{-1} \), to derive a recursive identity for the marginal probability of exposure:

\[
\mu_G^z(o \text{ is uncovered}) = \frac{zP_{G-o}(z)}{P_G(z)} = f_{[G,o]}(z) = \frac{z^2}{z^2 + \sum_{v \sim o} f_{[G-o,v]}(z)}.
\]

On a Galton-Watson tree (GWT), we obtain a Recursive Distributional Equation (RDE) and 'the mean of its solution is the limit of (1)', Aldous & Steele (03).
Fixed point equation for the distribution of $f(z) \in [0, 1]$:

$$f(z) \overset{d}{=} d \frac{z^2}{z^2 + \sum_{i=1}^{N} f_i(z)},$$

where $N \sim$ the standard size biased degree distribution of the random graph.

By iterating once

$$f(z) \overset{d}{=} d \frac{1}{1 + \sum_{i=1}^{N} \frac{1}{z^2 + \sum_{j=1}^{N_i} f_{ij}(z)}}$$

and letting $z \to 0$:

$$f(0) \overset{d}{=} d \frac{1}{1 + \sum_{i=1}^{N} \left(\sum_{j=1}^{N_i} f_{ij}(0)\right)^{-1}}$$
If $\phi$ is the generating function of the asymptotic degree distribution, let

$$F(x) = \phi'(1)x\overline{x} + \phi(1 - x) + \phi(1 - \overline{x}) - 1,$$

where $\overline{x} = \phi'(1 - x)/\phi'(1)$.

$F$ admits an historical record at $x$ if $x = \overline{x}$ and $F(x) > F(y)$ for any $0 \leq y < x$.

**Theorem 6.** If $p_1 < \ldots < p_r$ are the locations of the historical records of $F$, then the RDE admits exactly $r$ solutions, say $0 \leq X_1 <_s \ldots <_s X_r \leq 1$, and for any $i \in \{1, \ldots, r\}$, $E[X_i] = F(p_i)$.

For Erdős-Rényi random graphs, $\phi(x) = e^{\lambda(1-x)}$, $F$ has a unique record and we recover the result of Karp and Sipser.

The log-concave condition of Bohman and Frieze ensures that $F$ has a unique record.
What happens when the RDE has more than one solution?

No correlation decay: influence of boundary conditions remains positive.

\[ F \quad \text{for} \quad \pi_d = \frac{d}{1+d} \quad \text{and} \quad \pi_{d^3} = \frac{1}{1+d}. \]
What happens when the RDE has more than one solution?

Look at positive temperature:

:-)
THE RDE AT POSITIVE TEMPERATURE

A contracting property:

\[
|f(z) - g(z)| = \left| \frac{z^2}{z^2 + \sum_{i=1}^{N} f_i(z)} - \frac{z^2}{z^2 + \sum_{i=1}^{N} g_i(z)} \right| \\
\leq \sum_{i=1}^{N} \left| \frac{z^2}{z^2} f_i(z) - g_i(z) \right|.
\]

Hence with the Wasserstein distance:

\[
W(f(z), g(z)) \leq \frac{\mathbb{E}[N]}{z^2} W(f(z), g(z)),
\]

so that for \( z > \sqrt{\mathbb{E}[N]} \), the RDE has a unique solution by the Banach fixed point Theorem.
HEILMANN-LIEB THEOREM

We denote by $\mathcal{H}$ the space of holomorphic functions on the half-plane $\mathbb{H}_+ = \{ z \in \mathbb{C}, \Re(z) > 0 \}$ taking real non-negative values on $(0, \infty)$.

**Theorem 7** (Heilmann-Lieb (72)). $z \in \mathbb{H}_+ \implies P_G(z) \in \mathbb{H}_+$.

Hence, the marginal probability of exposure

$$f_{[G,o]}(z) = \mu_G^z (o \text{ is uncovered}) = \frac{zP_{G-o}(z)}{P_G(z)}$$

belongs to $\mathcal{H}$.

**Corollary 1.** The RDE on $\mathcal{H}$:

$$f(.) = \frac{d}{(.)^2 + \sum_{i=1}^{N} f_i(.)},$$

has a unique solution and the energy density converges to:

$$\frac{1}{n} \langle H(M) \rangle_{z,G_n} = \frac{1}{n} \sum_{o \in V} f_{[G_n,o]}(z) \rightarrow \mathbb{E} \left[ \frac{z^2}{z^2 + \sum_{i=1}^{N'} f_i(z)} \right],$$

where $f_i$ are iid solutions of the RDE.
EXCHANGE OF LIMITS

Let $U_G^z$ be the (random) set of vertices uncovered in a Boltzmann matching of $G$ at temperature $z$.

\[
\begin{align*}
U_{G_n}^z \xrightarrow{n\to\infty} U_T^z \\
\downarrow z\to0 & \hspace{1cm} \downarrow z\to0 \\
U_{G_n}^0 \xrightarrow{n\to\infty} U_T^0
\end{align*}
\]

A priori,

\[
\mathbb{E}[f_T(0)] = \max_{x\in[0,1]} F(x) \geq \limsup \frac{1}{n} \langle H(M) \rangle_{0,G_n} = \limsup \frac{1}{n} \sum_{o\in V} f_{[G_n,o]}(0)
\]

This bound is tight!
The entropy of the law $\mu_G^z$ is defined by:

$$S_G(z) = -\sum_M \mu_G^z(M) \ln \mu_G^z(M).$$

A simple rewriting of the matching defect polynomial gives:

$$\ln P_G(z) \ln z = \langle H(M) \rangle_{z,G} + \frac{S_G(z)}{\ln z} \leq \langle H(M) \rangle_{0,G},$$

and

$$S_G(z) \leq \ln(\text{number of matchings}).$$

Hence dividing by $n$, we get

$$\lim_{z \to 0} \lim_{n \to \infty} \frac{1}{n} \langle H(M) \rangle_{z,G_n} + \frac{1}{n} \frac{S_{G_n}(z)}{\ln z} = \mathbb{E}[f_T(0)] \leq \lim \inf \frac{1}{n} \langle H(M) \rangle_{0,G_n}.$$

A SIMPLE TRICK
SUMMARY OF STORY (iii)

- If $G_n$ converges weakly under uniform rooting to a rooted Galton-Watson tree (GWT) $T$ (with a finite first moment for the offspring distribution), then its matching number converges to the largest solution of the zero temperature RDE associated with $T$.

- If there is only one solution to the zero temperature RDE, either there is no core or there exists a (almost) perfect matching on the core.

- If there are more than one solution to the zero temperature RDE, a positive fraction of the vertices of the core are uncovered in any maximum matching.
  → disprove a conjecture of Wormald on Hamiltonicity of random graphs.
Our proof for the convergence of the rank works only when the ‘zero temperature’ RDE has a unique solution, i.e. when there is a (almost) perfect matching on the core.
Theorem 8 (Wästlund). *On a finite graph, Alice wins if and only if every maximum matching covers the starting point.*
Theorem 9 (Wästlund). On a finite graph, Alice wins if and only if every maximum matching covers the starting point.
SUMMARY OF STORY (i)

Undirected Vertex Geography on an Erdős-Rényi graph:

- if $\lambda \leq e$, the core is asymptotically negligible so the average probability of loss is given by the Lambert function.

- if $\lambda > e$, ??
END OF THE STORIES

THANK YOU!