Matchings and rank for random diluted graphs

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INRIA & ENS

Joint work with Charles Bordenave (CNRS & Uni. Toulouse)
and Justin Salez (INRIA & ENS).

(i) GEOGRAPHY GAME

(ii) RANK

(iii) MATCHINGS

(i) GEOGRAPHY GAME

inspired from Johan Wästlund

(ii) RANK

(iii) MATCHINGS

GEOGRAPHY GAME

- 2 players game: Alice plays first and Bob follows.
- Rule: name geographic locations, each starting with the same letter with which the previous name ended.
- Winner: the last player who is able to give a name.
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UNDIRECTED VERTEX GEOGRAPHY

- General graph

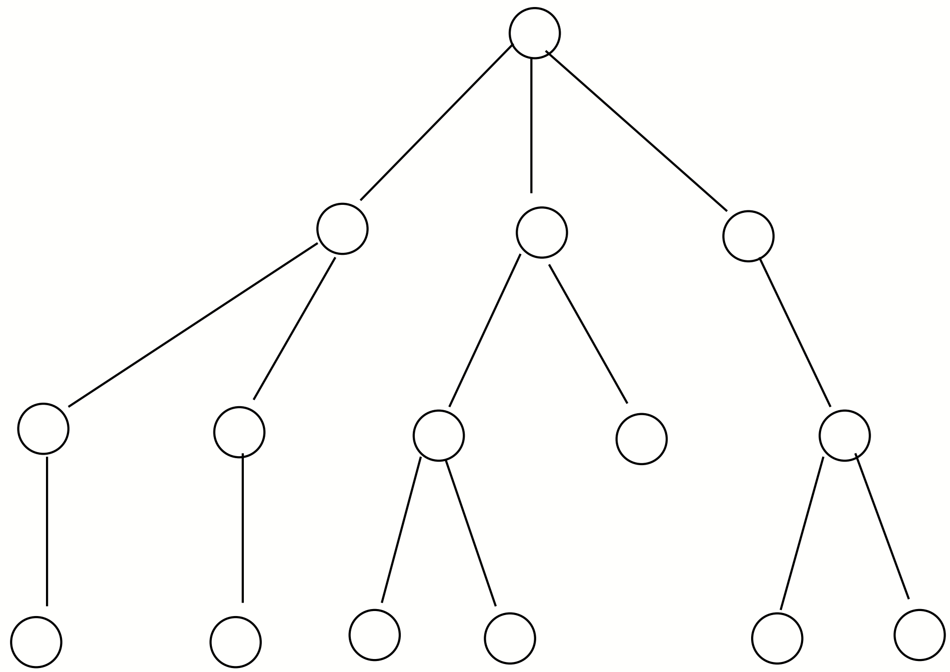
- Alice and Bob take turns choosing the edges of a self-avoiding walk

- Whoever gets stuck loses

- All players know all information about gameplay.

Random model: Geography on the Poisson Galton-Watson tree

Random rooted tree: Each node has $Po(\lambda)$-distributed number of children.



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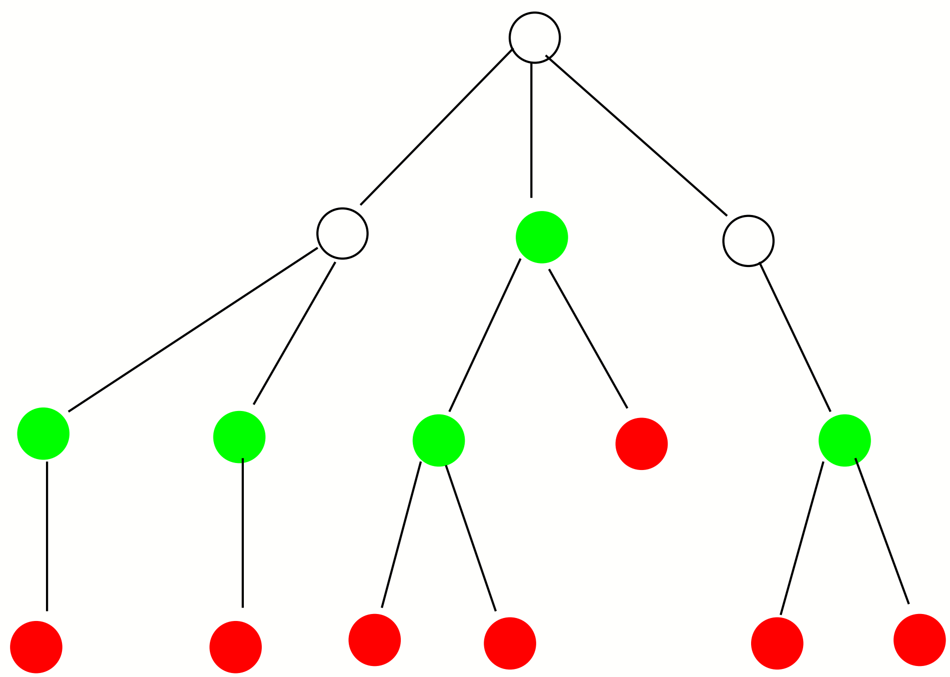

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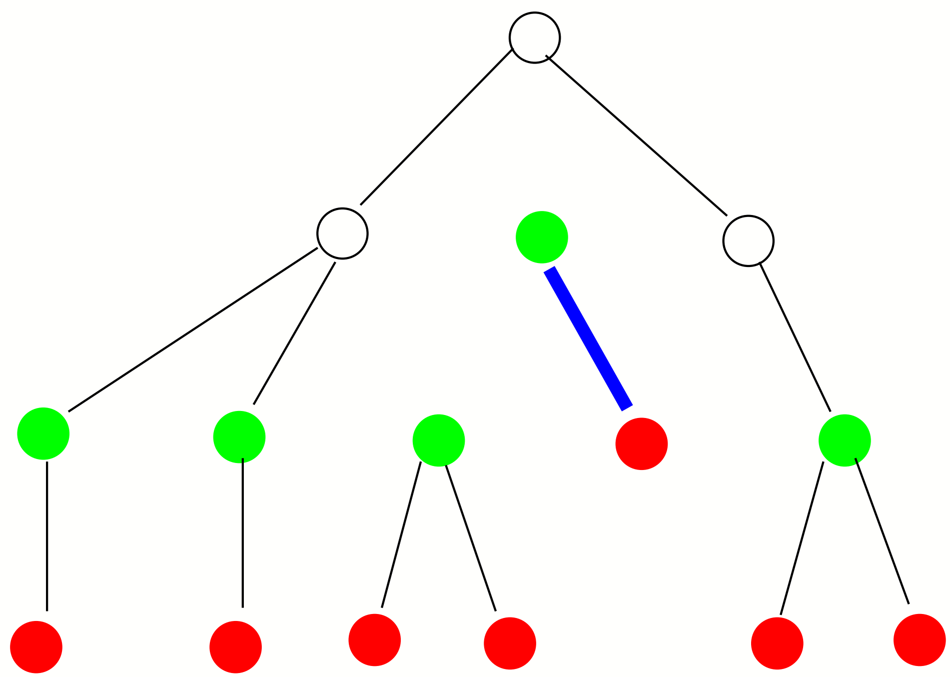

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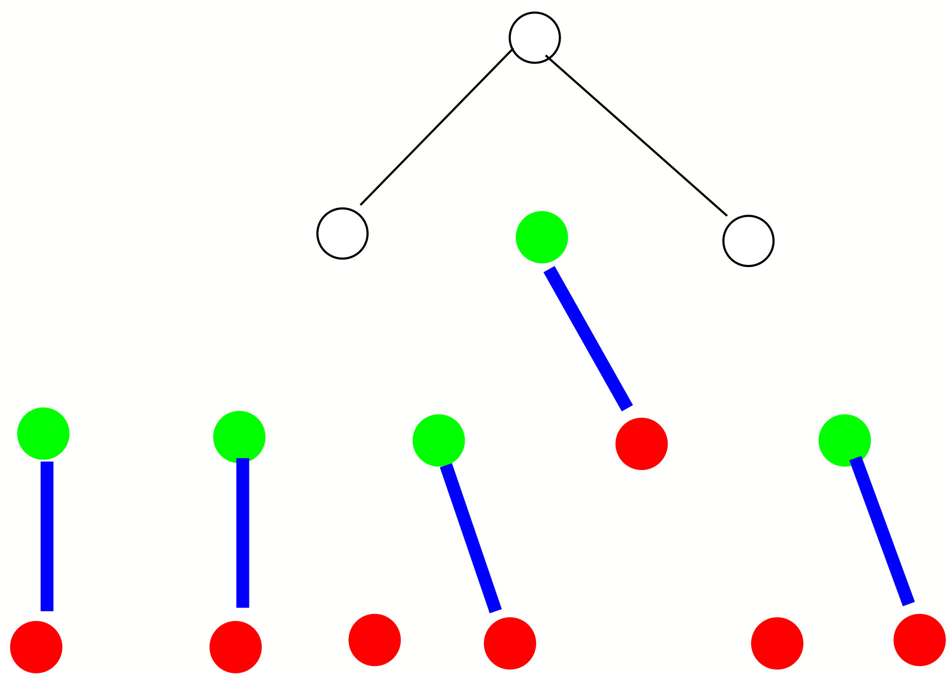

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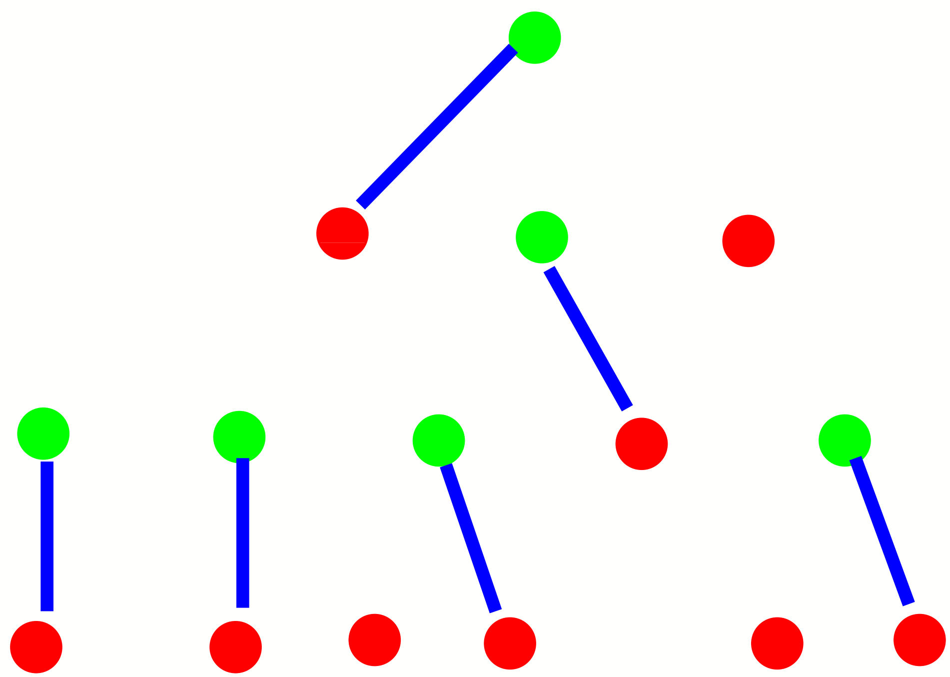

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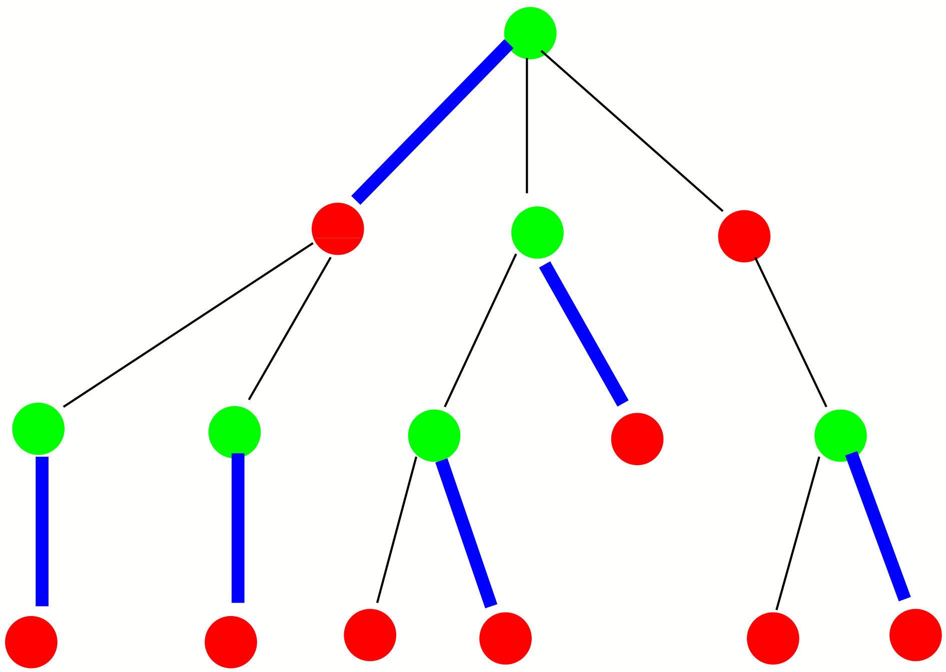

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- Let $p = \mathbb{P}(\text{Bob wins under optimal play}) = \mathbb{P}(\text{root is colored red})$.

Random model: Geography on the Poisson Galton-Watson tree

Random rooted tree: Each node has $Po(\lambda)$-distributed number of children.

- Let $p = \mathbb{P}($Bob wins under optimal play$) = \mathbb{P}($root is colored red$)$.

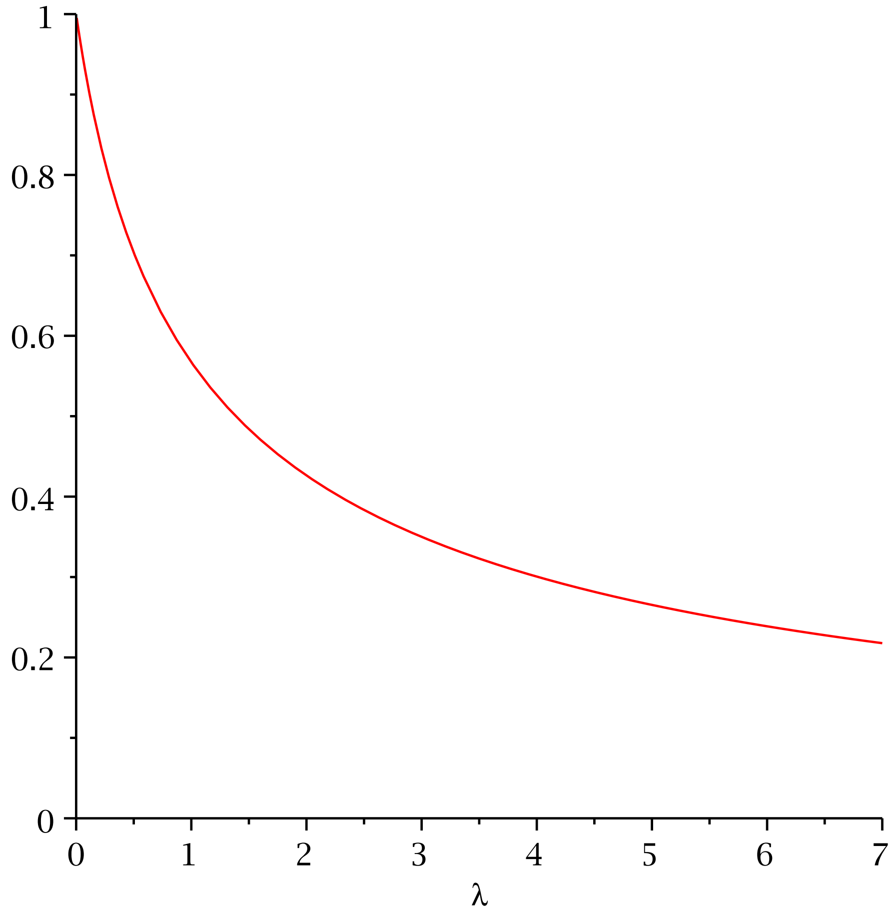
- Thanks to the branching property:

$$\begin{aligned} p &= \mathbb{P}(\text{ all children are green}) \\ &= \mathbb{P}(\text{ no child is red}) = e^{-\lambda p}, \end{aligned}$$

and so

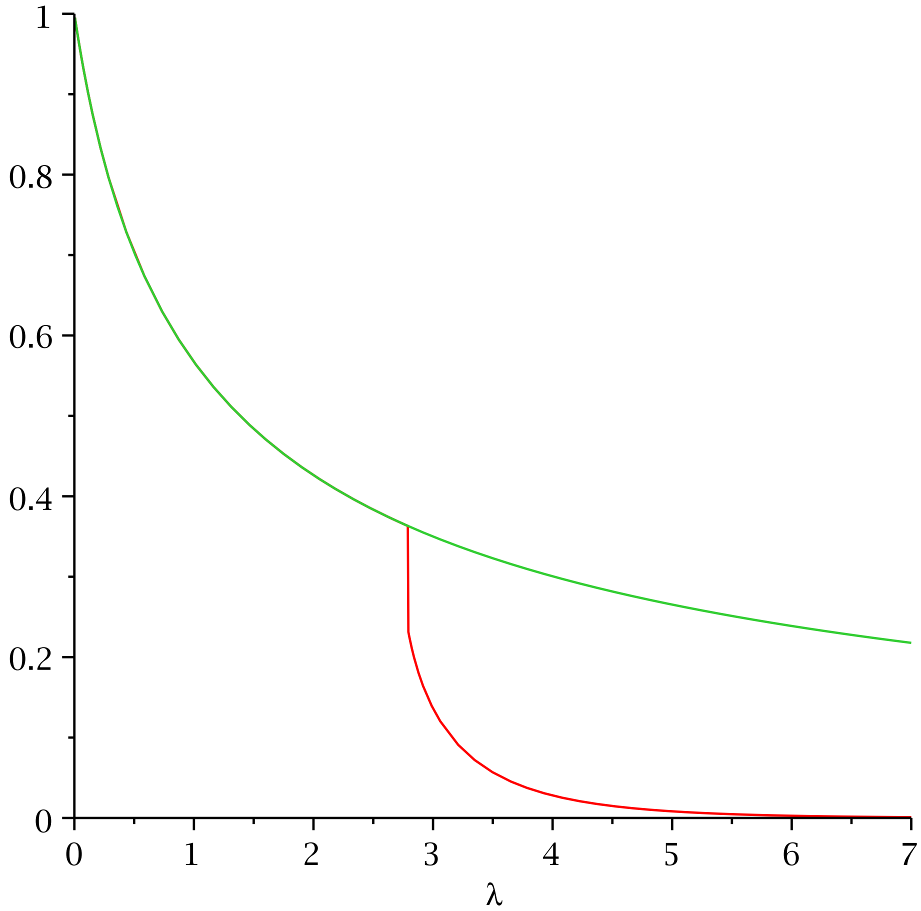
$$p = \frac{W(\lambda)}{\lambda}.$$

A NAIVE GUESS



The function $p = \mathbb{P}(\text{Bob wins})$ as a function of the average offspring λ.

TRUTH



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WHAT HAPPENED?

The solution $p = \frac{W(\lambda)}{\lambda}$ is the fixed-point of the map:

$$p \mapsto e^{-\lambda p}.$$

But the truth about the game comes from iterating that map.

TRUNCATED GAME

Let $p_k = \mathbb{P}(\text{Bob wins if the tree is truncated after } k \text{ moves})$.

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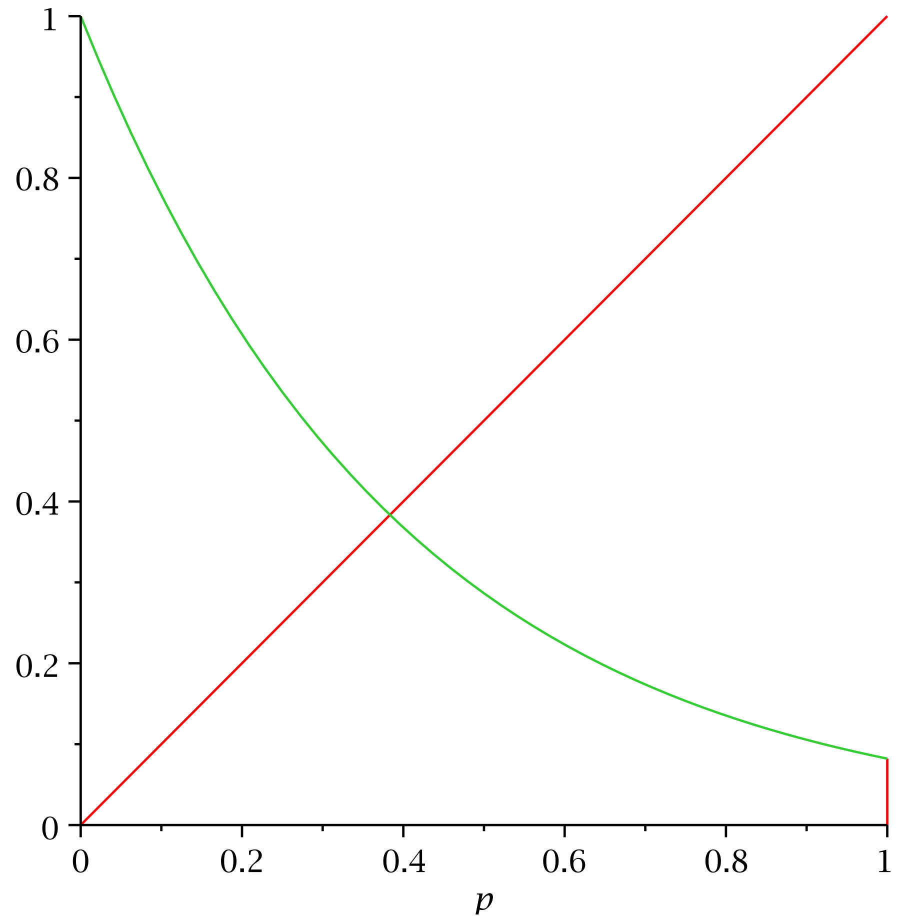
Let $p_k = \mathbb{P}(\text{Bob wins if the tree is truncated after } k \text{ moves})$.

- $p_0 = 1$
- $p_1 = \mathbb{P}(Po(\lambda) = 0) = e^{-\lambda}$
- Then for $k \geq 0$,

$$\begin{aligned} p_{k+1} &= \mathbb{P}(\text{Bob looses from any child of the root in the } k\text{-truncated tree}) \\ &= \mathbb{E}[(1 - p_k)^N] \text{ where, } N \sim Po(\lambda) \\ &= e^{-\lambda p_k} \end{aligned}$$

TRUTH



$$\lambda = 2.5$$

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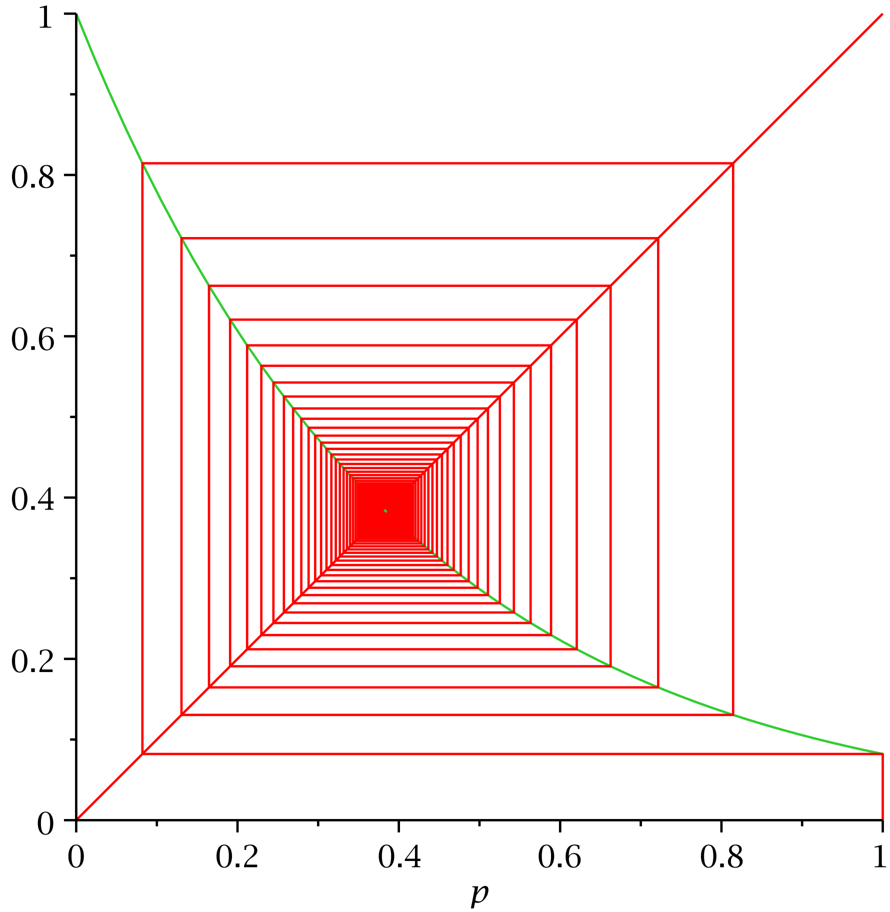

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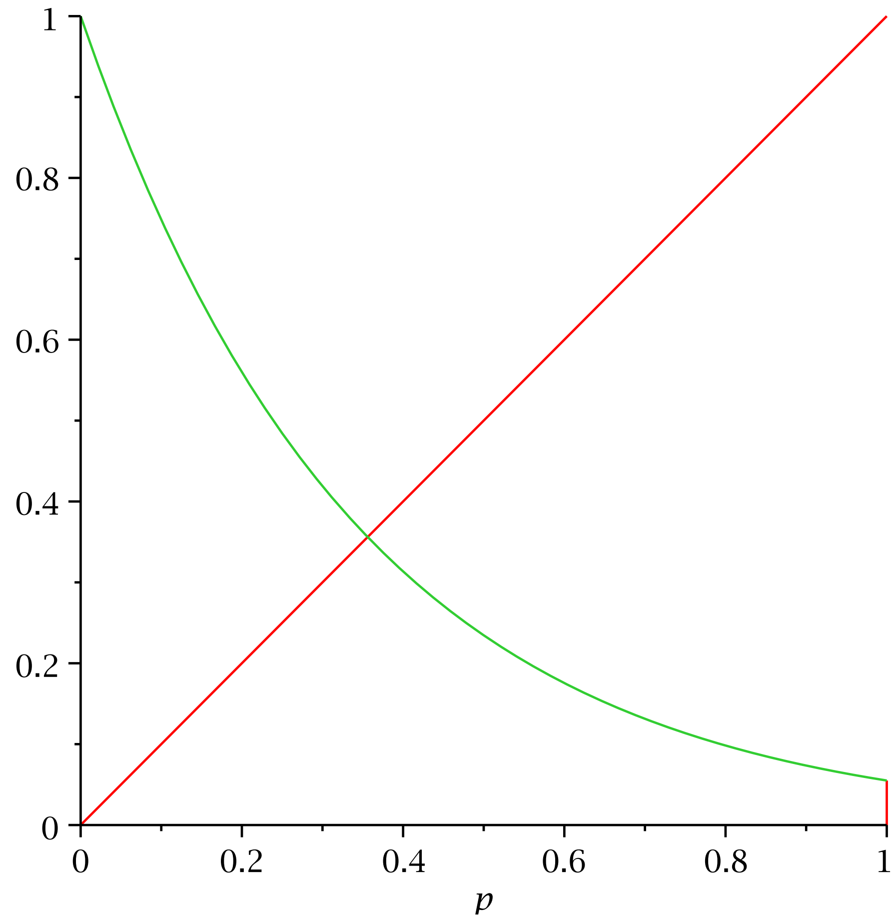

$$\lambda = 2.9$$

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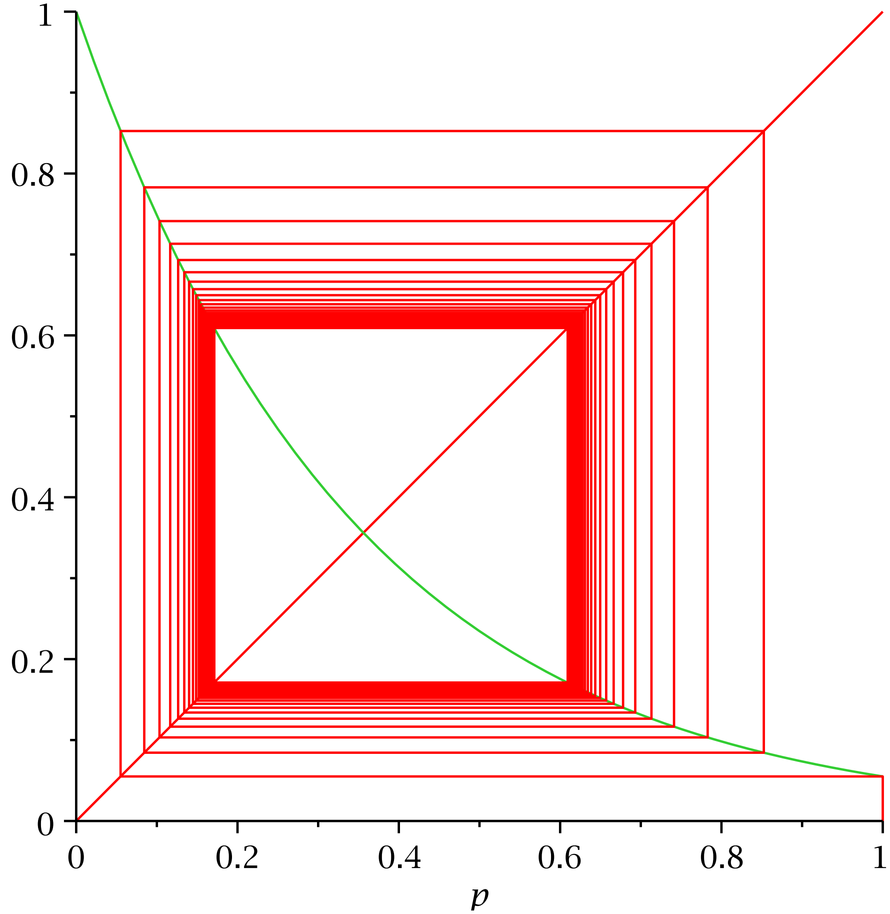
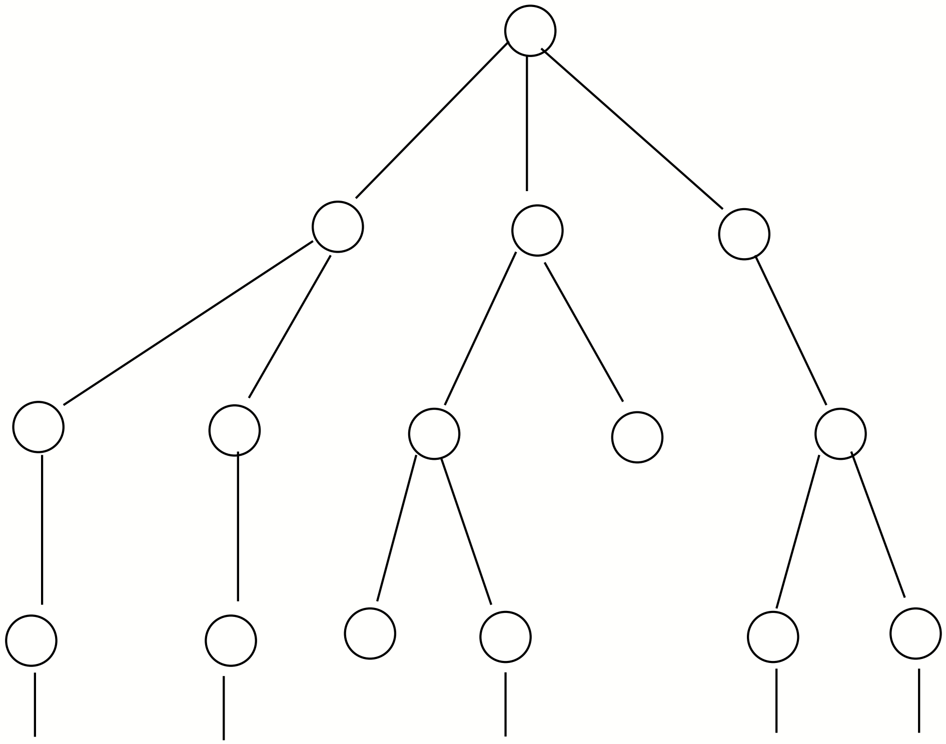


$$\lambda = 2.9$$

APPARITION OF THE CORE

Do the leaf removal algorithm on the infinite tree.



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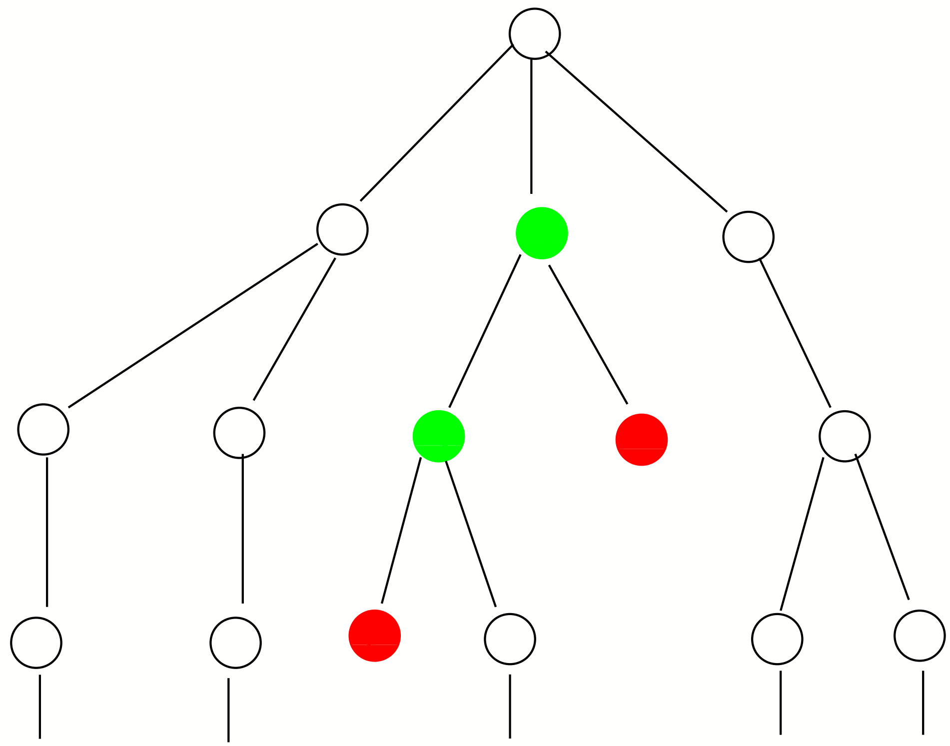

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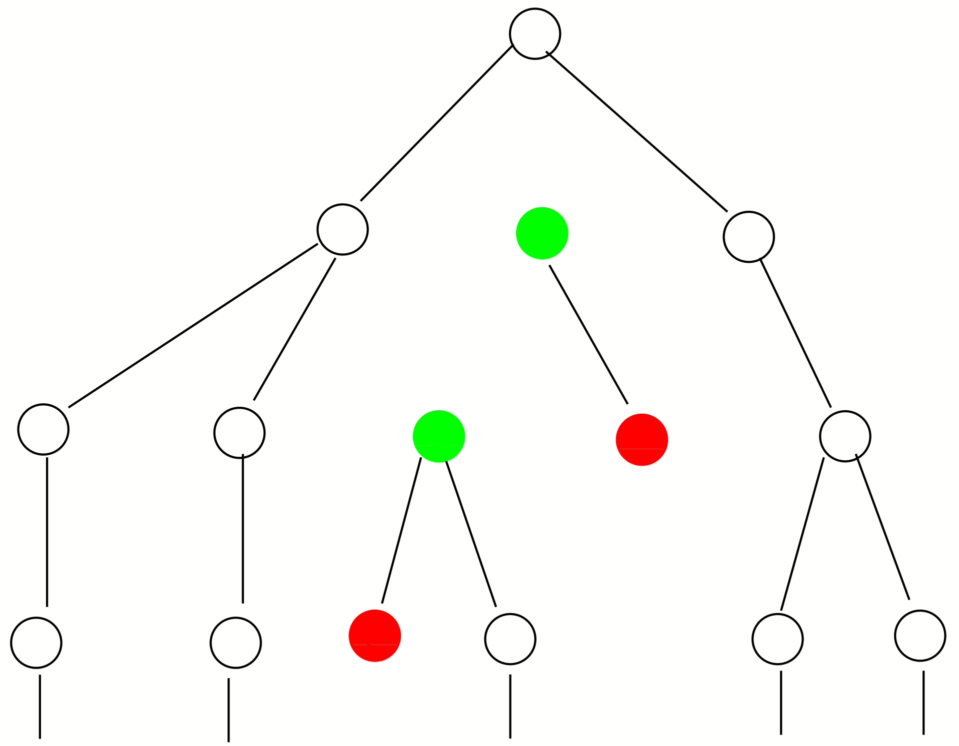


APPARITION OF THE CORE

Do the leaf removal algorithm on the infinite tree.

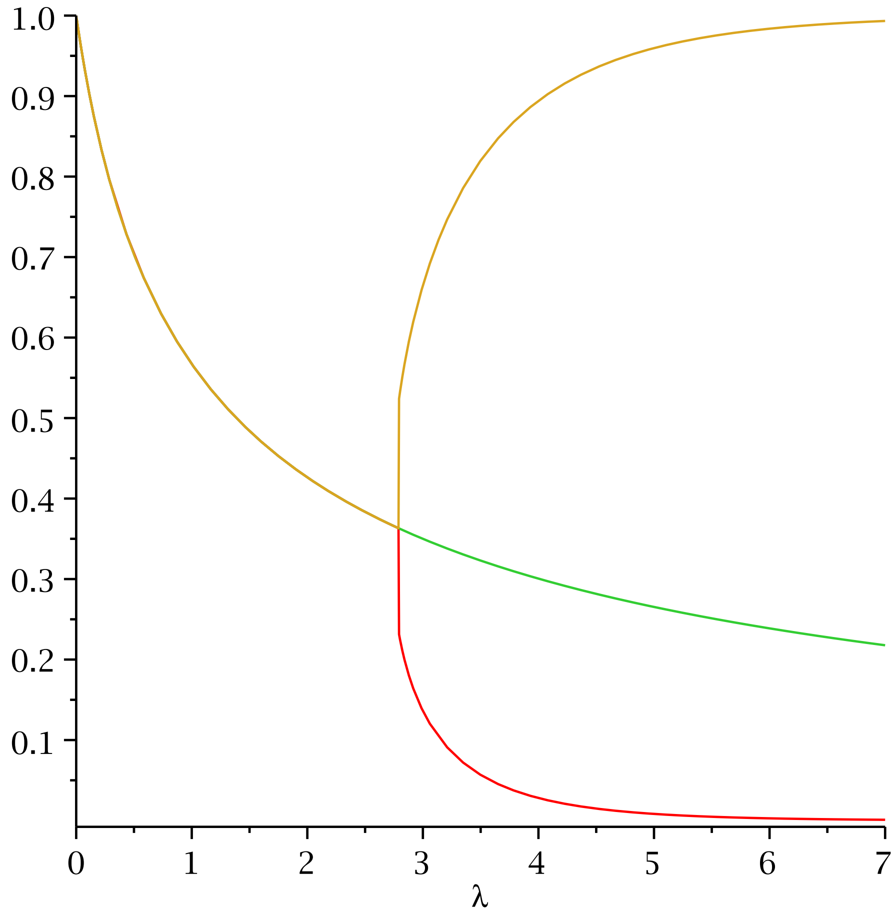
For $\lambda > e$, some paths to infinity will never be 'touched' by the algorithm.

These paths constitute the core of the original tree.

Implications for the game:

- if the game starts outside of the core, there will be a winner.
- if the game starts in the core, there exists an infinite sequence of optimal moves staying in the core.

TRUTH: DRAW HAPPENS



On the core, the parity of k will determine the winner of the truncated game.

Draw $\leftrightarrow$ influence of boundary conditions remains positive.

(i) GEOGRAPHY GAME

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SPECTRAL MEASURE OF FINITE GRAPHS

Let $G_n = (V_n, E_n)$ be a simple graph on $V_n = \{1, \cdots, n\}$. We define

$$A_n = \text{ Adjacency matrix of } G_n = (\mathbb{1}((i,j) \in E_n))_{1 \le i,j \le n}$$

Let

$$\lambda_n(A_n) \le \cdots \le \lambda_1(A_n)$$

denote the real eigenvalues of the symmetric matrix A_n.

The spectral measure of G_n is

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(A_n)}.$$

The rank of A_n is

$$\mathrm{rank}\,(A_n) = 1 - \mu_n(\{0\}).$$

QUESTIONS

Let $(G_n), n \in \mathbb{N},$ *be a sequence of graphs on* $V_n = \{1, \cdots, n\}$ *such that* G_n *"converges" to a limit graph* G.

1. *Does* μ_n *converge to a measure* μ_G *for the usual weak convergence topology ?*
2. *Do we have a formula for* μ_G *in some cases ?*
3. *Does* $\mathrm{rank}\,(A_n)$ *converge? to* $\mu_G(\{0\})$ *?*

KNOWN RESULTS

Let G_n is a random graph drawn uniformly on the set of d-regular graphs on n vertices.

Theorem 1 (McKay 1981). *For $d \geq 3$ the spectral measure μ_n converges weakly as n goes to infinity to the deterministic measure μ_{KM} supported on $[-2\sqrt{d-1}, 2\sqrt{d-1}]$*

$$\mu_{KM}(dx) = \frac{d}{2\pi} \frac{\sqrt{4(d-1) - x^2}}{d^2 - x^2} dx.$$

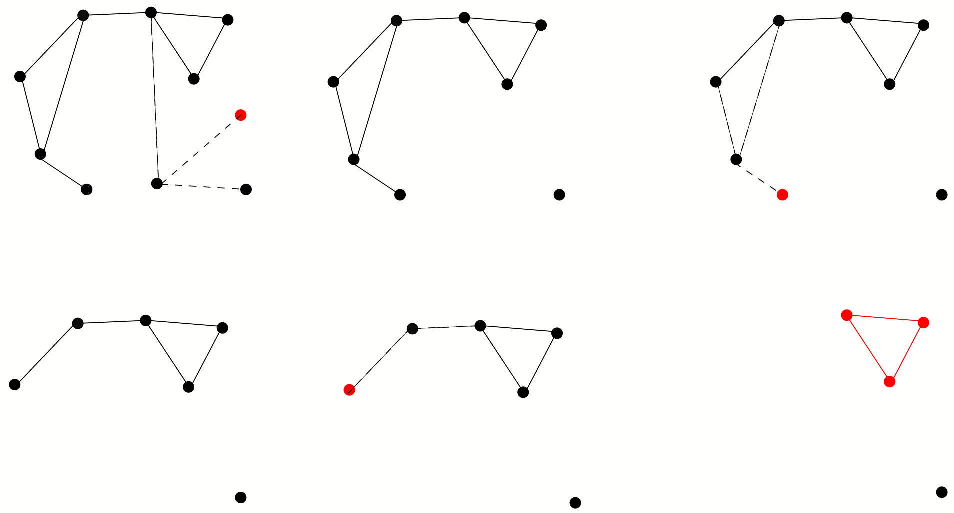
$\Longrightarrow$ μ_{KM} is the Kesten-McKay measure, it first appeared in (Kesten 1959) in the context of simple random walks on groups.

Khorunzhy, Shchrebina, Vengerovsky (04): convergence of the spectral measure for Erdős-Rényi graphs.

Bauer & Gollineli (00): asymptotic rank of the uniform spanning tree on the complete graph.

Bhamidi, Evans, Sen (09): convergence of the spectrum of the adjacency matrix of growing trees.

LEAF REMOVAL AND RANK



We define $\mathrm{core}(G_n)$ as the remaining graph after iterative leaf removal and removal of all isolated vertices.

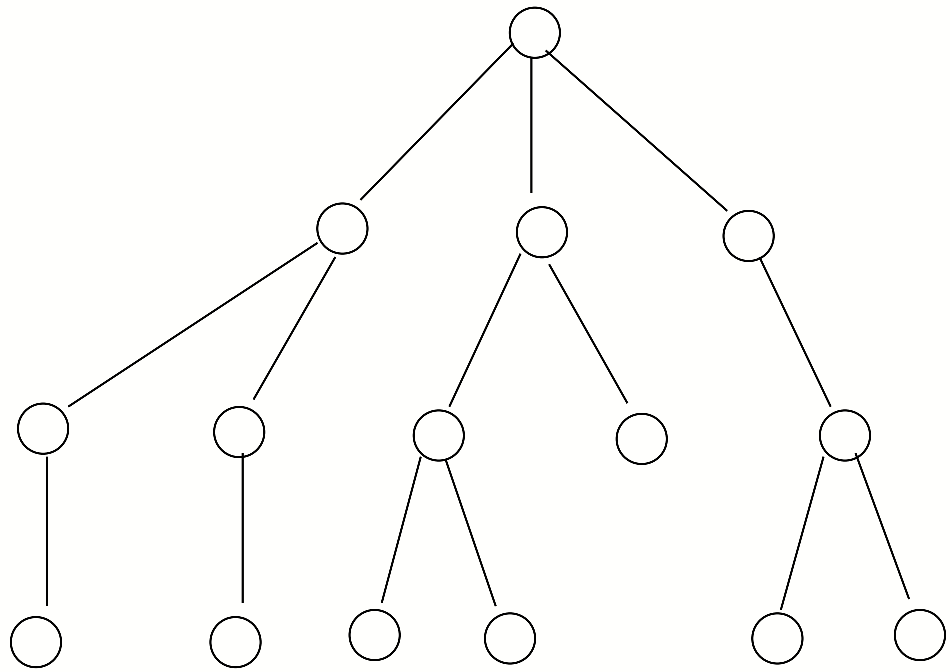
$$\dim\ker A_n \quad = \quad |\{\text{isolated vertices in leaf removal}\}| + \dim\ker A(\mathrm{core}(G_n)).$$

For a finite tree, the core is empty: $\mu_n(\{0\}) = \frac{1}{n}|\{\text{isolated vertices in leaf removal}\}|$.

In general, $\mu_n(\{0\}) \geq \frac{1}{n}|\{\text{isolated vertices in leaf removal of } G_n\}|$.

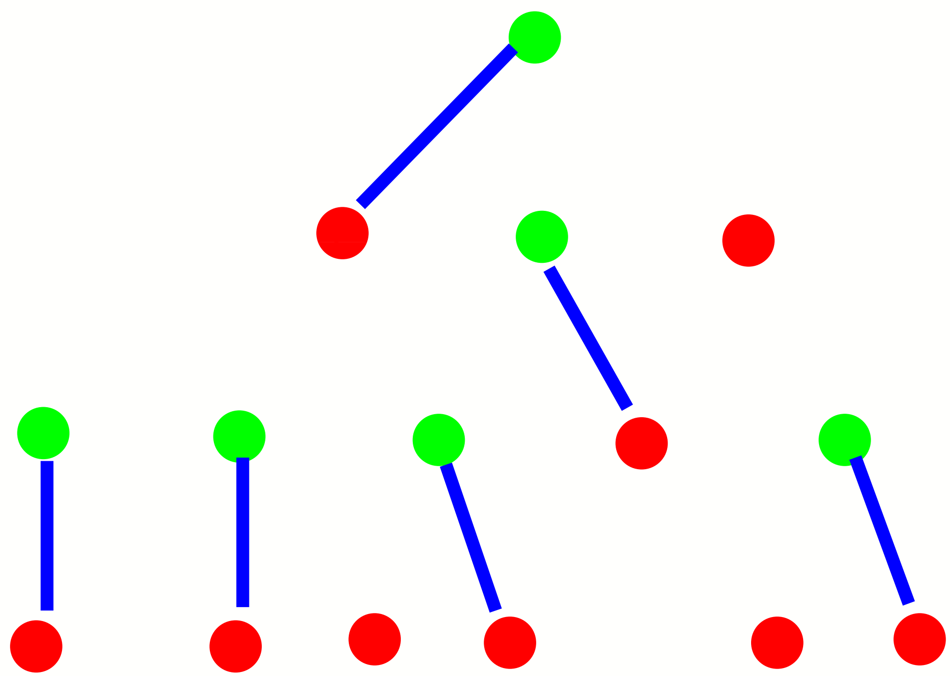
EXERCISE!

What is the size of the kernel of this tree?



SOLUTION

The size of its kernel is 3.



GRAPH ENSEMBLES

We are interested in a sequence G_n of random diluted graphs : $\deg(v; G_n) = O(1)$ as the number of vertices n tends to infinity.

Important examples of random graphs on $\{1, \cdots, n\}$,

- Erdős-Rényi graphs with parameter $p = \lambda/n$.
- Uniform measure on k-regular graphs.
- Graphs with prescribed degree distribution (π_k) : independently for each vertex, we draw a random number of half-edges with distribution (π_k). If the total number of half-edge is even, we match them uniformly.

SPECTRAL MEASURE ROOTED AT A VERTEX

$\ell^2(V_n)$ admits an orthonormal basis of eigenvectors $(b_1, \ldots, b_n)$, a priori different from the canonical orthonormal basis $(e_v)_{v \in V_n}$, such that:

$$\forall x \in \mathbb{C}^n, \quad A_n x = \sum_{i=1}^{n} \lambda_i(A_n) \langle x, b_i \rangle b_i.$$

The spectral measure of G_n when rooted at v is simply

$$\mu_{(G_n,v)} = \sum_{i=1}^{n} |\langle b_i, e_v \rangle|^2 \delta_{\lambda_i(A_n)},$$

so that for any measurable function f,

$$\langle f(A_n) e_v, e_v \rangle = \int_{\mathbb{R}} f(x) d\mu_{(G_n,v)}(x).$$

$\mu_{(G_n,v)}$ is the contribution of vertex v to the empirical spectral measure:

$$\mu_n = \frac{1}{n} \sum_{v \in V_n} \mu_{(G_n,v)}.$$

CONVERGENCE OF THE SPECTRAL MEASURE

Theorem 2. *If* G_n *converges weakly under uniform rooting to a rooted Galton-Watson tree (GWT)* $\mathcal{T}$ *(with a finite first moment for the offspring distribution), then in probability,*

$$\mu_n = \frac{1}{n} \sum_{v \in V_n} \mu_{(G_n,v)} \to \mathbb{E}\mu_{\mathcal{T}}$$

Hence we have an upper bound on the size of the kernel:

$$\limsup_{n\to\infty} \mu_n(\{0\}) \leq \mathbb{E}\mu_{\mathcal{T}}(\{0\}),$$

so that the convergence of the rank is proved when

$$\lim_{n\to\infty} \frac{1}{n} |\{\text{isolated vertices in leaf removal of } G_n\}| = \mathbb{E}\mu_{\mathcal{T}}(\{0\}).$$

RESOLVENT AND STIELJES TRANSFORM

If $\mathcal{T}$ is a GWT with finite branching, then its adjacency operator $A_{\mathcal{T}}$ is self-adjoint a.s.

On $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im(z) > 0\}$, the Stieljes transform of $\mu_{\mathcal{T}}$ is defined by

$$s_{\mathcal{T}}(z) = \int_{\mathbb{R}} \frac{\mu_{\mathcal{T}}(dx)}{x - z} = \langle R_{\mathcal{T}}(z) e_o, e_o \rangle,$$

where

$$R_{\mathcal{T}}(z) = (A_{\mathcal{T}} - zI)^{-1}$$

is the resolvent of the adjacency operator of $\mathcal{T}$.

$\Longrightarrow$ *The Stieltjes transform characterizes the probability measure.*

In particular, $\mu_{\mathcal{T}}(\{0\}) = \lim_{t \downarrow 0} t \Im s_{\mathcal{T}}(it)$.

STIELTJES TRANSFORM FOR INFINITE TREES

Schur's identity for an invertible hermitian matrix $B = \begin{pmatrix} b_{11} & u^* \\ u & \tilde{B} \end{pmatrix}$,

$$(B^{-1})_{11} = \left(b_{11} - u^* \tilde{B}^{-1} u\right)^{-1}.$$

For a tree $\mathcal{T}$, we get for $B = A_{\mathcal{T}} - zI$,

$$\begin{aligned} s_{\mathcal{T}}(z) &= \langle R_{\mathcal{T}}(z) e_o, e_o \rangle = -\left(z + \sum_{v \overset{\mathcal{T}}{\sim} o} \langle R_{\mathcal{T} \setminus o}(z) e_v, e_v \rangle \right)^{-1} \\ &= -\left(z + \sum_{v \overset{\mathcal{T}}{\sim} o} s_{\mathcal{T}_v}(z) \right)^{-1}. \end{aligned}$$

RDE FOR STIELTJES TRANSFORM

Thanks to the branching property, we get a Recursive Distributional Equation (RDE) for the law of $s_{\mathcal{T}}(.)$.

Let $\mathcal{H}$ be the set of holomorphic functions from $\mathbb{C}_+$ to $\mathbb{C}_+$ such that $|f(z)| \leq (\Im z)^{-1}$.

Theorem 3. *(i) There exists an unique probability measure on $\mathcal{H}$ such that*

$$s(.) \stackrel{d}{=} -\left((.) + \sum_{k=1}^{N} s_k(.)\right)^{-1},$$

where s_k, s are i.i.d. copies independent of N with law (π_k^s).

(ii) The law of the Stieltjes transform of $\mu_{\mathcal{T}}$ is given by,

$$s_{\mathcal{T}}(.) \stackrel{d}{=} -\left((.) + \sum_{k=1}^{N_*} s_k(.)\right)^{-1},$$

where N_* has distribution (π_k) and s_k are independents with law s.

RDE FOR THE KERNEL

Recall that $\mu_{\mathcal{T}}(\{0\}) = \lim_{t\downarrow 0} t\Im s_{\mathcal{T}}(it)$.

Let $h(t) = \Im s(it)$, so that by definition,

$$h(t) \stackrel{d}{=} \left(t + \sum_{i=1}^{N} h_i(t) \right)^{-1}$$

and then iterating once,

$$th(t) \stackrel{d}{=} \left(1 + \sum_{i=1}^{N} \left(t^2 + \sum_{j=1}^{N_i} th_{i,j}(t) \right)^{-1} \right)^{-1}.$$

If $\xi = \lim_{t\downarrow 0} th(t) \in [0,1]$, we get

$$\xi = \left(1 + \sum_{i=1}^{N} \left(\sum_{j=1}^{N_i} \xi_{i,j} \right)^{-1} \right)^{-1}.$$

SOLVING THE RDE FOR THE KERNEL

If φ is the generating function of the asymptotic degree distribution, let

$$F(x) = \varphi'(1)x\overline{x} + \varphi(1-x) + \varphi(1-\overline{x}) - 1,$$

where $\overline{x} = \varphi'(1-x)/\varphi'(1)$.

F admits an historical record at x if $x = \overline{\overline{x}}$ and $F(x) > F(y)$ for any $0 \leq y < x$.

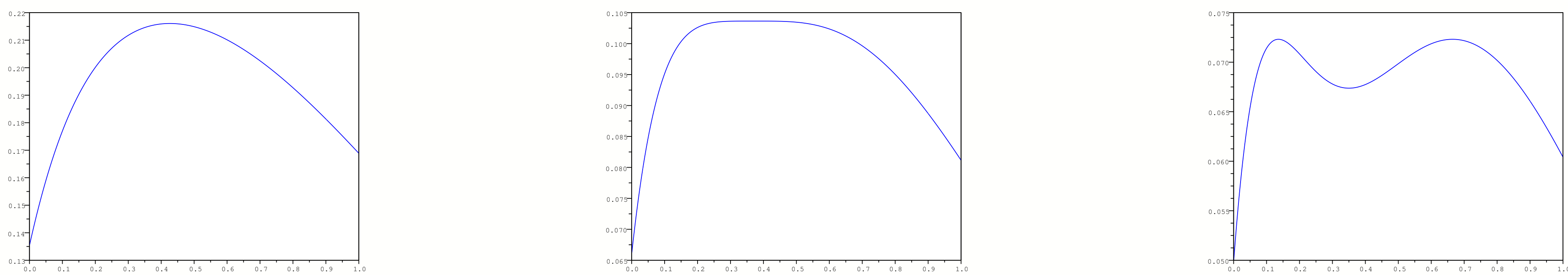
Theorem 4. *If $p_1 < \ldots < p_r$ are the locations of the historical records of F, then the RDE admits exactly r solutions, say $0 \leq X_1 <_{st} \ldots <_{st} X_r \leq 1$, and for any $i \in \{1, \ldots, r\}$, $\mathbb{E}[X_i] = F(p_i)$.*

ERDŐS RÉNYI GRAPHS

The core does not contribute to the size of the kernel as n tends to infinity:

$$\begin{aligned}\mu_n(\{0\}) &= n^{-1}|\{\text{isolated vertices in LR}\}| + n^{-1}\dim\ker\operatorname{core}(G_n) \\ &= \mathbb{E}\mu_{\mathcal{T}}(\{0\}) + o(n) \\ &= p + e^{-\lambda p} + \lambda p e^{-\lambda p} - 1,\end{aligned}$$

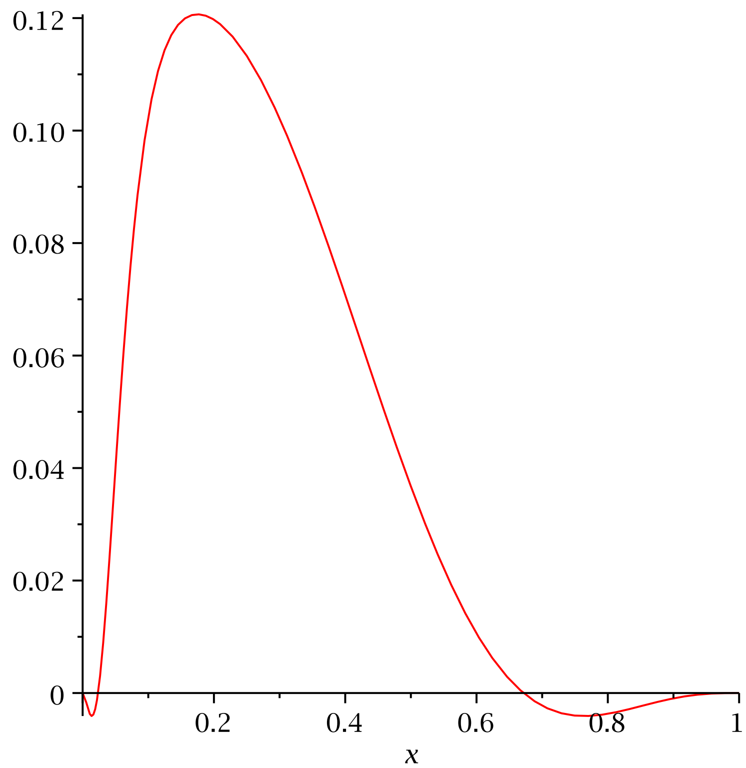
where p is the smallest root of $x = e^{-\lambda e^{-\lambda x}}$.



Function F for $\lambda = 2$, $\lambda = e$ and $\lambda = 3$.

OPEN PROBLEM

When the RDE has more than one solution, then $n^{-1}|\{\text{isolated vertices in LR}\}|$ converges to the smallest one and $\mathbb{E}\mu_{\mathcal{T}}(\{0\})$ to the largest one!



Function F for $\pi_d = \frac{d}{1+d}$ and $\pi_{d^3} = \frac{1}{1+d}$.

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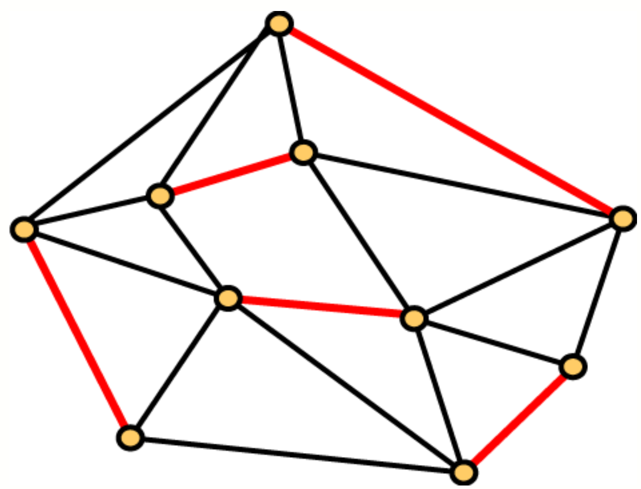
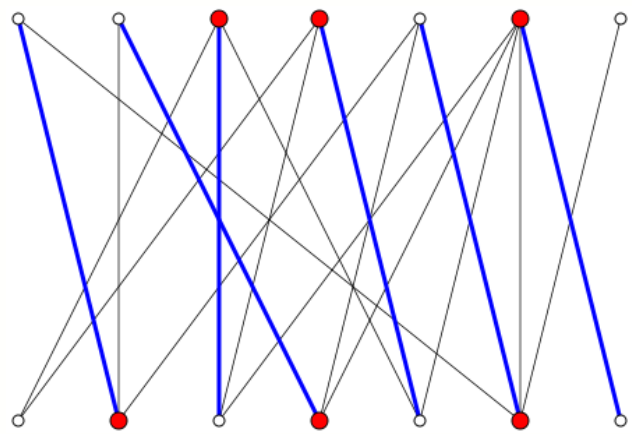
MAXIMUM MATCHINGS

Let $G = (V, E)$ be a simple finite graph. A matching of G is a subset of edges with no common end-vertex.

- The matching is perfect if all vertices are covered.

- The matching is maximum if the number of covered vertices is maximum.





MAXIMUM MATCHING IN ERDŐS-RÉNYI GRAPHS

Let G_n be an Erdős-Rényi graph on $\{1, \cdots, n\}$ with parameter λ/n.

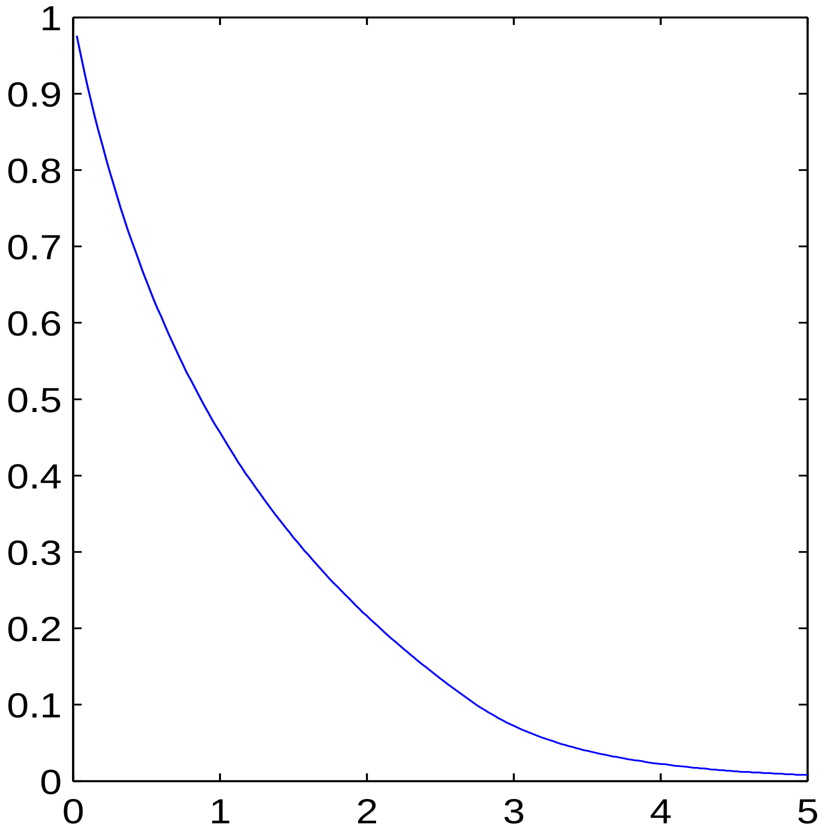
Theorem 5 (Karp & Sipser (82))**.** *In probability,*

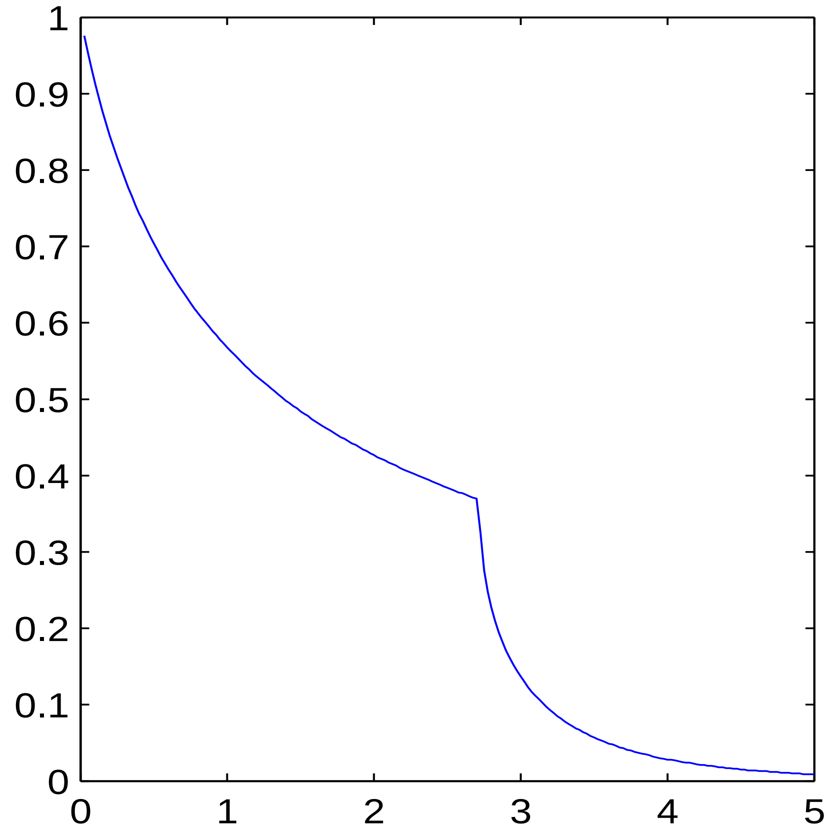
$$\lim_{n\to\infty} \frac{|\{\textit{uncovered vertices in a maximal matching}\}|}{n} = p + e^{-\lambda p} + \lambda p e^{-\lambda p} - 1,$$

where p is the smallest root of $x = e^{-\lambda e^{-\lambda x}}$.

If $0 \leq \lambda \leq e$ then $p = e^{-\lambda p}$ is solution to the Lambert equation. If $\lambda > e$, $x = e^{-\lambda e^{-\lambda x}}$ has two solutions in $[0, 1]$.

MAXIMUM MATCHING AS FUNCTION OF THE INTENSITY



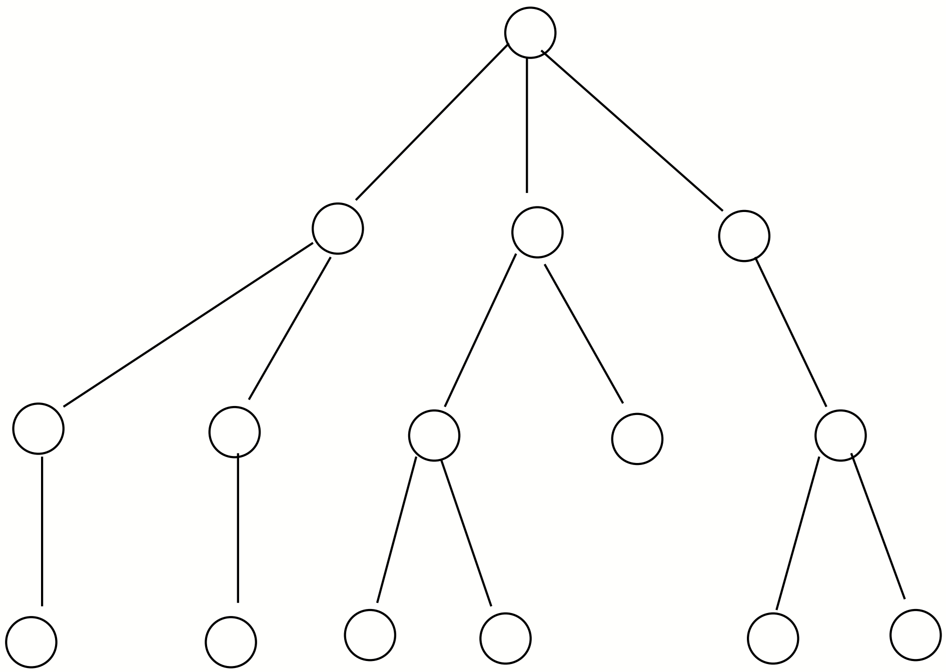


Left: $\lim_{n\to\infty} |\{\text{uncovered vertices in a maximal matching}\}|/n$ as function of λ.

Right: q, smallest root of $x = e^{-\lambda e^{-\lambda x}}$, as function of λ.

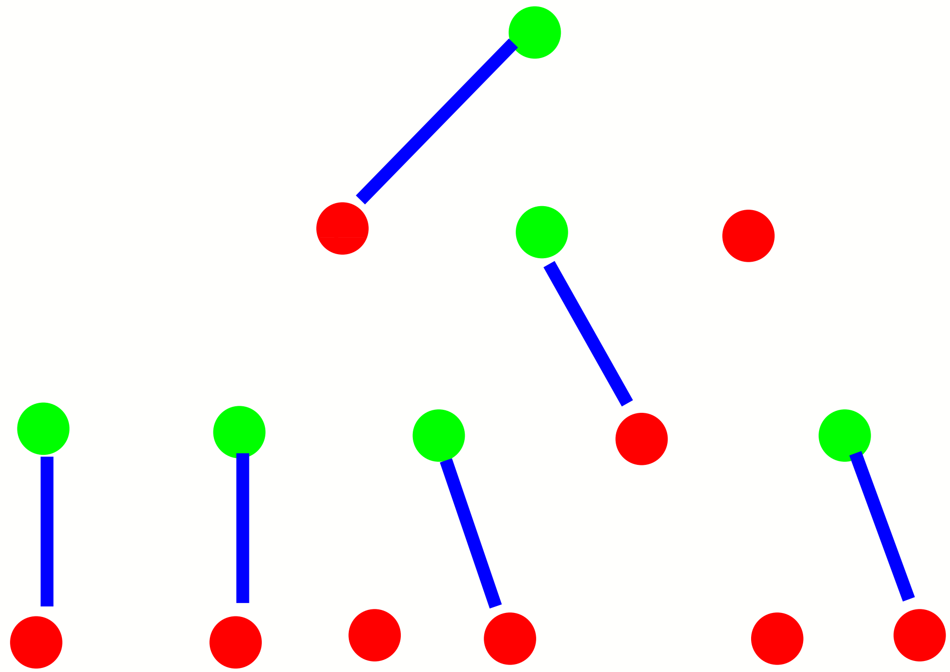
EXERCISE!

What is the number of uncovered vertices in a maximum matching of this tree?



SOLUTION

The minimal number of uncovered vertices is 3.



BOLTZMANN MEASURE

(Zdeborová & Mézard 06)

Let $G = (V, E)$ be a simple finite graph with M a matching of G.

The defect (energy) of M, denoted $H(M)$, is the number of uncovered vertices in M.

The associated Boltzmann measure with temperature $z \geq 0$ is

$$\mu_G^z(M) = \frac{z^{H(M)}}{P_G(z)}$$

where $P_G(z)$ is the matching defect polynomial:

$$P_G(z) \quad = \quad \sum_{M \text{ : matchings}} z^{H(M)}.$$

? CONVERGENCES ?

Do we have convergence of the energy density ?

$$\frac{1}{n}\langle H(M)\rangle_{z,G} = \frac{1}{n}\sum_{M\ :\ \text{matchings}} H(M)\mu_G^z(M)$$

YES in the following cases:

- zero temperature $z = 0$:
  - Karp & Sipser (82) for Erdős-Rényi graph.
  - Bohman & Frieze (09) with a 'log-concave' condition on the degree distribution.
- positive temperature $z > 0$:
  - Bayati & Nair (06) under a restrictive large girth condition.
  - Bayati, Gamarnik, Katz, Nair, Tetali (07) for very high temperature.

For the random assignment problem: it converges at zero temperature to $\zeta(2)$, Aldous (01) and at very high temperature, Talagrand (03).

THE CAVITY METHOD

Rewrite the energy density as follows:

$$\frac{1}{n}\langle H(M)\rangle_{z,G} = \frac{1}{n}\sum_{o\in V}\mu_G^z(o \text{ is uncovered}). \qquad (1)$$

Use the fundamental recursion relation:

$$P_G(z) = \sum_{M\text{ : matchings}} z^{H(M)} = zP_{G-o}(z) + \sum_{v\sim o} P_{G-\{o,v\}}(z).$$

Multiply by $zP_{G-o}(z)^{-1}$, to derive a recursive identity for the marginal probability of exposure:

$$\mu_G^z(o \text{ is uncovered}) = \frac{zP_{G-o}(z)}{P_G(z)} = f_{[G,o]}(z) = \frac{z^2}{z^2 + \sum_{v\sim o} f_{[G-o,v]}(z)}.$$

On a Galton-Watson tree (GWT), we obtain a Recursive Distributional Equation (RDE) and 'the mean of its solution is the limit of (1)', Aldous & Steele (03).

THE RDE AT ZERO TEMPERATURE

Fixed point equation for the distribution of $f(z) \in [0,1]$:

$$f(z) \stackrel{d}{=} \frac{z^2}{z^2 + \sum_{i=1}^{N} f_i(z)},$$

where $N \sim$ the standard size biased degree distribution of the random graph.

By iterating once

$$f(z) \stackrel{d}{=} \frac{1}{1 + \sum_{i=1}^{N} \frac{1}{z^2 + \sum_{j=1}^{N_i} f_{ij}(z)}}$$

and letting $z \longrightarrow 0$:

$$f(0) \stackrel{d}{=} \frac{1}{1 + \sum_{i=1}^{N} \left(\sum_{j=1}^{N_i} f_{ij}(0) \right)^{-1}}$$

SOLVING THE RDE AT ZERO TEMPERATURE

If φ is the generating function of the asymptotic degree distribution, let

$$F(x) = \varphi'(1)x\overline{x} + \varphi(1-x) + \varphi(1-\overline{x}) - 1,$$

where $\overline{x} = \varphi'(1-x)/\varphi'(1)$.

F admits an historical record at x if $x = \overline{\overline{x}}$ and $F(x) > F(y)$ for any $0 \leq y < x$.

Theorem 6. *If $p_1 < \ldots < p_r$ are the locations of the historical records of F, then the RDE admits exactly r solutions, say $0 \leq X_1 <_{st} \ldots <_{st} X_r \leq 1$, and for any $i \in \{1, \ldots, r\}$, $\mathbb{E}[X_i] = F(p_i)$.*

For Erdős-Rényi random graphs, $\varphi(x) = e^{\lambda(1-x)}$, F has a unique record and we recover the result of Karp and Sipser.

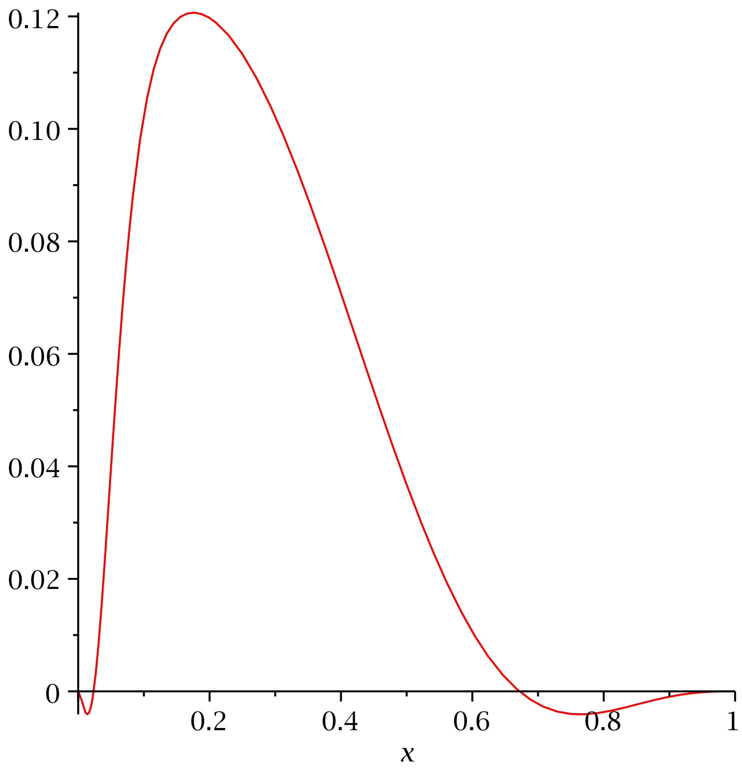
The log-concave condition of Bohman and Frieze ensures that F has a unique record.

OPEN PROBLEM

What happens when the RDE has more than one solution?

No correlation decay: influence of boundary conditions remains positive.

:-(



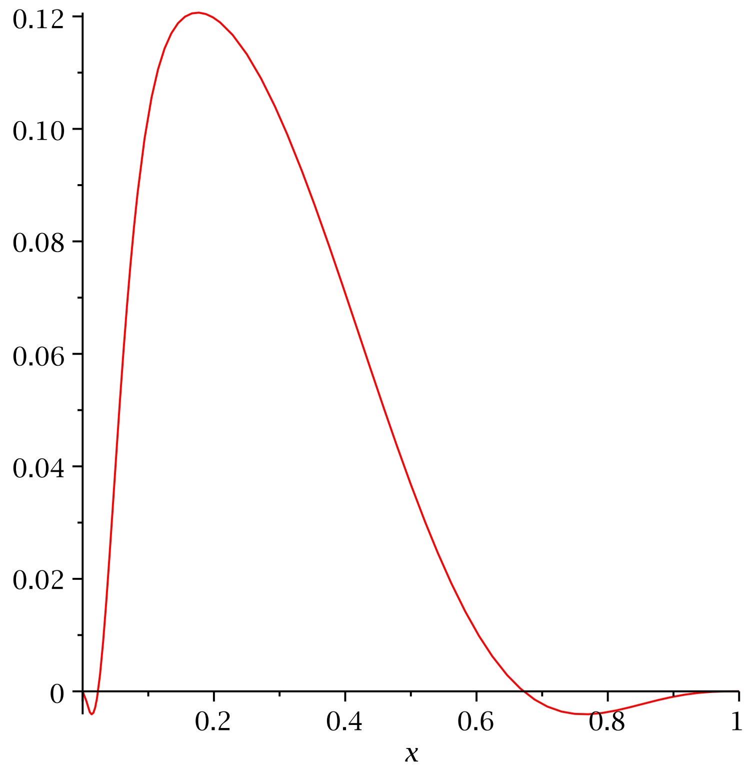
Function F for $\pi_d = \frac{d}{1+d}$ and $\pi_{d^3} = \frac{1}{1+d}$.

PROBLEM

What happens when the RDE has more than one solution?

Look at positive temperature:

:-)



Function F for $\pi_d = \frac{d}{1+d}$ and $\pi_{d^3} = \frac{1}{1+d}$.

THE RDE AT POSITIVE TEMPERATURE

A contracting property:

$$\begin{aligned} |f(z) - g(z)| &= \left| \frac{z^2}{z^2 + \sum_{i=1}^N f_i(z)} - \frac{z^2}{z^2 + \sum_{i=1}^N g_i(z)} \right| \\ &\leq \frac{\sum_{i=1}^N |f_i(z) - g_i(z)|}{z^2}. \end{aligned}$$

Hence with the Wasserstein distance:

$$W(f(z), g(z)) \leq \frac{\mathbb{E}[N]}{z^2} W(f(z), g(z)),$$

so that for $z > \sqrt{\mathbb{E}[N]}$, the RDE has a unique solution by the Banach fixed point Theorem.

HEILMANN-LIEB THEOREM

We denote by $\mathcal{H}$ the space of holomorphic functions on the half-plane $\mathbb{H}_+ = \{z \in \mathbb{C}, \Re(z) > 0\}$ taking real non-negative values on $(0, \infty)$.

Theorem 7 (Heilmann-Lieb (72)). $z \in \mathbb{H}_+ \implies P_G(z) \in \mathbb{H}_+$.

Hence, the marginal probability of exposure $f_{[G,o]}(z) = \mu_G^z(o \text{ is uncovered}) = \frac{zP_{G-o}(z)}{P_G(z)}$ belongs to $\mathcal{H}$.

Corollary 1. *The RDE on $\mathcal{H}$:*

$$f(.) \stackrel{d}{=} \frac{(.)^2}{(.)^2 + \sum_{i=1}^N f_i(.)},$$

has a unique solution and the energy density converges to:

$$\frac{1}{n}\langle H(M)\rangle_{z,G_n} = \frac{1}{n}\sum_{o\in V} f_{[G_n,o]}(z) \to \mathbb{E}\left[\frac{z^2}{z^2 + \sum_{i=1}^{N'} f_i(z)}\right],$$

where f_i are iid solutions of the RDE.

EXCHANGE OF LIMITS

Let $\mathcal{U}_G^z$ be the (random) set of vertices uncovered in a Boltzmann matching of G at temperature z.

$$\begin{tikzcd}
\mathcal{U}_{G_n}^z \arrow[r, "n\to\infty"] \arrow[d, "z\to 0"'] & \mathcal{U}_T^z \arrow[d, "z\to 0"] \\
\mathcal{U}_{G_n}^0 \arrow[r, dotted, "?", "n\to\infty"'] & \mathcal{U}_T^0
\end{tikzcd}$$

A priori,

$$\mathbb{E}[f_T(0)] = \max_{x\in[0,1]} F(x) \geq \limsup \frac{1}{n}\langle H(M)\rangle_{0,G_n} = \limsup \frac{1}{n}\sum_{o\in V} f_{[G_n,o]}(0)$$

This bound is tight!

A SIMPLE TRICK

The entropy of the law μ_G^z is defined by:

$$S_G(z) = -\sum_M \mu_G^z(M) \ln \mu_G^z(M).$$

A simple rewriting of the matching defect polynomial gives:

$$\frac{\ln P_G(z)}{\ln z} = \langle H(M)\rangle_{z,G} + \frac{S_G(z)}{\ln z} \leq \langle H(M)\rangle_{0,G},$$

and

$$S_G(z) \leq \ln(\text{number of matchings}).$$

Hence dividing by n, we get

$$\lim_{z\to 0}\lim_{n\to\infty} \frac{1}{n}\langle H(M)\rangle_{z,G_n} + \frac{1}{n}\frac{S_{G_n}(z)}{\ln z} = \mathbb{E}[f_T(0)] \leq \liminf \frac{1}{n}\langle H(M)\rangle_{0,G_n}$$

SUMMARY OF STORY (iii)

- If G_n converges weakly under uniform rooting to a rooted Galton-Watson tree (GWT) $\mathcal{T}$ (with a finite first moment for the offspring distribution), then its matching number converges to the largest solution of the zero temperature RDE associated with $\mathcal{T}$.

- If there is only one solution to the zero temperature RDE, either there is no core or there exists a (almost) perfect matching on the core.

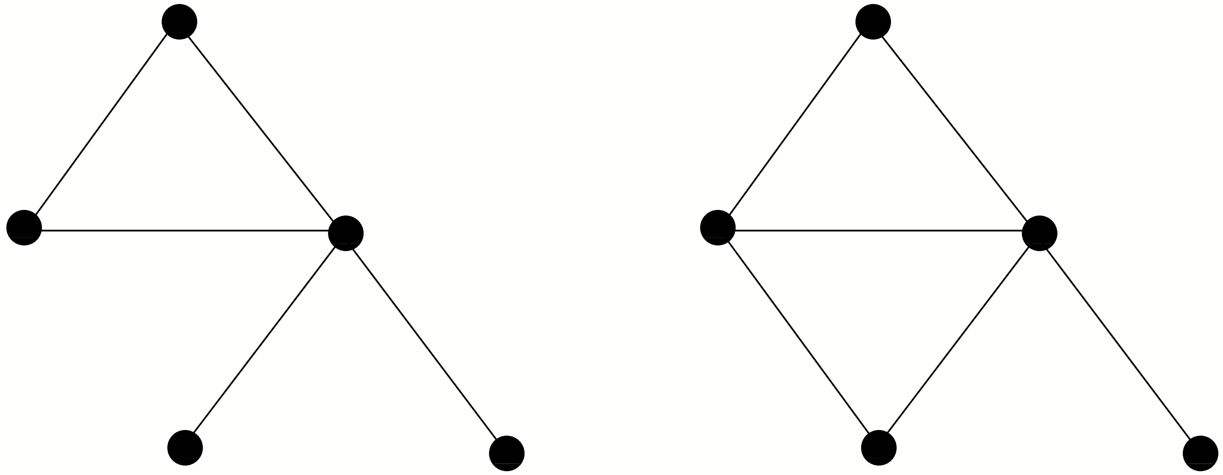
- If there are more than one solution to the zero temperature RDE, a positive fraction of the vertices of the core are uncovered in any maximum matching.
  $\longrightarrow$ disprove a conjecture of Wormald on Hamiltonicity of random graphs.

SUMMARY OF STORY (ii)

Our proof for the convergence of the rank works only when the ’zero temperature’ RDE has a unique solution, i.e. when there is a (almost) perfect matching on the core.

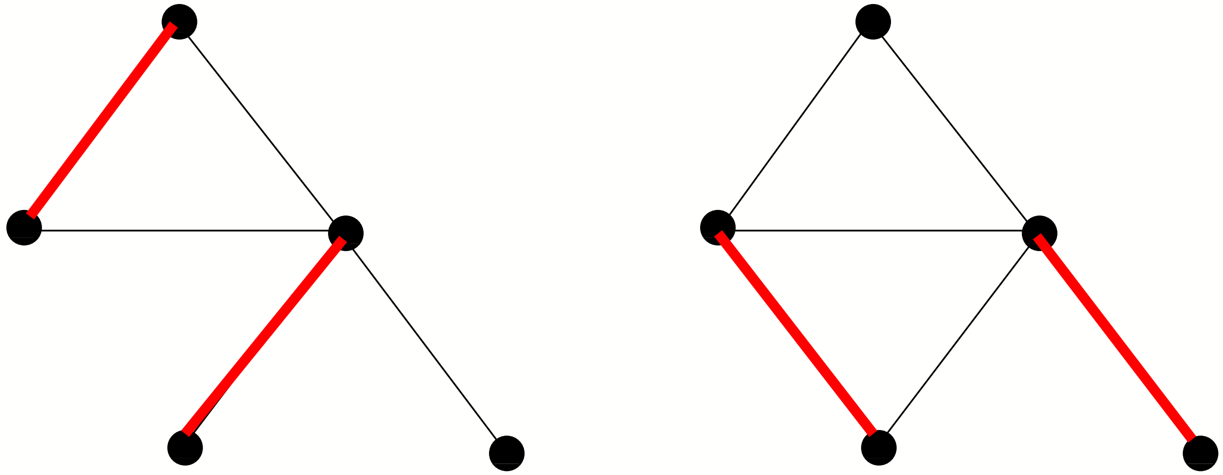
SUMMARY OF STORY (i)

Theorem 8 (Wästlund)**.** *On a finite graph, Alice wins if and only if every maximum matching covers the starting point.*



SUMMARY OF STORY (i)

Theorem 9 (Wästlund)**.** *On a finite graph, Alice wins if and only if every maximum matching covers the starting point.*



SUMMARY OF STORY (i)

Undirected Vertex Geography on an Erdős-Rényi graph:

- if $\lambda \leq e$, the core is asymptotically negligible so the average probability of loss is given by the Lambert function.

- if $\lambda > e$, ??

END OF THE STORIES

THANK YOU!