Diffusion and Cascading Behavior in Random Networks

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(1) Diffusion Model
inspired from game theory
and statistical physics.

(2) Results
from a mathematical analysis.

(3) Heuristic
(0) Context

Crossing the Chasm
(Moore 1991)

The Tipping Point
How Little Things Can Make a Big Difference
Malcolm Gladwell
(1) Diffusion Model

(2) Results

(3) Heuristic
1) Coordination game...

- Both receive payoff $q$.
- Both receive payoff $1-q>q$.
- Both receive nothing.
(1)...on a network.

- Everybody start with **ICQ**.
- Total payoff = sum of the payoffs with each neighbor.
- A seed of nodes switches to **Talk**.

(Morris 2000)
(1) Threshold Model

- State of agent i is represented by

\[ X_i = \begin{cases} 
0 & \text{if } \text{icq} \\
1 & \text{if } \text{talk}
\end{cases} \]

- Switch from \text{icq} to \text{talk} if:

\[
\sum_{j \sim i} X_j \geq qd_i
\]
(1) Model for the network?

Statistical physics: bootstrap percolation.
(1) Model for the network?
(1) Random Graphs

• Random graphs with given degree sequence introduced by Molloy and Reed (1995).

• Examples:
  – Erdös-Rényi graphs, $G(n, \lambda/n)$.
  – Graphs with power law degree distribution.

• We are interested in large population asymptotics.

• Average degree is $\lambda$. 
(1) Diffusion Model
\[ q = \text{relative threshold} \]
\[ \lambda = \text{average degree} \]

(2) Results

(3) Heuristic
(1) Diffusion Model

$q = \text{relative threshold}$

$\lambda = \text{average degree}$

(2) Results

(3) Heuristic
(2) Contagion (Morris 2000)

• Does there exist a finite groupe of players such that their action under best response dynamics spreads contagiously everywhere?

• Contagion threshold: \( q_c = \) largest \( q \) for which contagious dynamics are possible.

• Example: interaction on the line

\[ q_c = \frac{1}{2} \]
(2) Another example: d-regular trees

\[ q_c = \frac{1}{d} \]
(2) Some experiments

Seed = one node, $\lambda=3$ and $q=0.24$
(source: the Technoverse blog)
(2) Some experiments

Seed = one node, $\lambda=3$ and $1/q>4$

(source: the Technoverse blog)
(2) Some experiments

Seed = one node, $\lambda=3$ and $q=0.24$ (or $1/q>4$) (source: the Technooverse blog)
(2) Contagion threshold

In accordance with (Watts 2002)
(2) A new Phase Transition
(2) Pivotal players

- Giant component of players requiring only one neighbor to switch.

Tipping point: Diffusion like standard epidemic

Chasm: Pivotal players = Early adopters
(2) q above contagion threshold

- New parameter: size of the seed as a fraction of the total population $0 < \alpha < 1$.
- Monotone dynamic $\rightarrow$ only one final state.
(2) Minimal size of the seed, $q > 1/4$

Tipping point: Connectivity helps

Chasm: Connectivity hurts
(2) $q > 1/4$, low connectivity

Connectivity helps the diffusion.
(2) $q > 1/4$, high connectivity

Connectivity inhibits the global cascade, but once it occurs, it facilitates its diffusion.
(2) Equilibria for $q < q_c$

- Trivial equilibria: all A / all B
- Initial seed applies best-response, hence can switches back. If the dynamic converges, it is an equilibrium.
- Robustness of all A equilibrium?
- Initial seed = 2 pivotal neighbors

$\rightarrow$ pivotal equilibrium
(2) Strength of Equilibria for $q < q_c$

Mean number of trials to switch from all A to pivotal equilibrium
(2) Coexistence for $q < q_c$

Players A

Players B

Coexistence
(1) Diffusion Model

(2) Results

(3) Heuristic
(3) Locally tree-like

Independent computations on trees
(3) Branching Process Approximation

- Local structure of $G = \text{random tree}$
- **Recursive Distributional Equation (RDE)** or:

$$Y_i = \begin{cases} 1 & \text{if infected from 'below'} \\ 0 & \text{otherwise.} \end{cases}$$

$$Y_i = 1 - (1 - \sigma_i) \mathbb{1} \left( \sum_{\ell \to i} Y_\ell \leq q d_i \right)$$
(3) Solving the RDE

\[
\ell \leq q \hat{D}
\]

\[
\mathbb{1} = 1 - (1 - \sigma) \prod_{\ell=1}^{D-1} Y
\]

\[
z = \mathbb{P}(Y = 0)
\]

\[
\lambda z^2 = (1 - \alpha) h(z)
\]

\[
h(z) = \sum_{s,r \geq s-\lfloor qs \rfloor} r p_s \binom{s}{r} z^r (1 - z)^{s-r}
\]
(3) Phase transition in one picture

\[ z^* = \max\{z \in [0, 1], \lambda z^2 - (1 - \alpha)h(z) = 0\} \]
Conclusion

• Simple tractable model:
  – Threshold rule introduces local dependencies
  – Random network: heterogeneity of population

• 2 regimes:
  – Low connectivity: tipping point
  – High connectivity: chasm

• More results in the paper:
  – Heterogeneity of thresholds, active/inactive links, rigorous proof.
Thank you!

- Diffusion and Cascading Behavior in Random Networks. Available at http://www.di.ens.fr/~lelarge