ACN 915: Algorithms for networked information
Homework 2

all homeworks need to be returned on February 6

Exercise 1

Let $A$ be an $n \times d$ matrix with singular values $\sigma_1, \ldots, \sigma_r$ and right singular vectors $v_1, \ldots, v_r$. We define the Frobenius norm of $A$ by $\|A\|_F = \sqrt{\sum_{j,k} a_{jk}^2}$.

1. For a row $a_j$ of $A$, show that $\sum_{i=1}^r (a_j^T v_i)^2 = ka_j k^2$.
2. Show that $kA_k = \sum_{i=1}^k \sigma_i u_i v_i^T$, where $u_i$ are the left singular vectors.
3. Show that $A_k$ has rank $k$.

Let $V_k = \text{Vect}(v_1, \ldots, v_k)$.
4. Show that the rows of $A_k$ are the projections of the rows of $A$ onto $V_k$.

Consider $B$ with rank $k \leq r$ minimizing $kA - Bk_F$ and let $V$ be the space spanned by the rows of $B$.
5. Show that $\text{dim}(V) \leq k$ and that each row of $B$ is the projection of the corresponding row of $A$ onto $V$.
6. Show that $kA - A_k k_F \leq kA - Bk_F$.

Exercise 2

We consider a set $V$ of candidates. The extended Condorcet criterion requires: if $X, Y$ are a partition of the set $V$ of all candidates and for all $x \in X$, $y \in Y$, $x$ beats $y$ in direct comparison (i.e. more voters prefer $x$ to $y$ than vice versa), then all of $X$ should precede all of $Y$.

Recall that a tournament graph $G$ on $V$ is a complete oriented graph on $V$.

1. Show that each tournament graph $G$ contains a Hamiltonian path (i.e. a path going through each node exactly once).
2. Show that for any profile of voters, there is a rank aggregation (i.e. a consensus ordering) satisfying the extended Condorcet criterion.