Exercise 1

Let $A$ be an $n \times d$ matrix with singular values $\sigma_1, \ldots, \sigma_r$ and right singular vectors $v_1, \ldots, v_r$. We define the Frobenius norm of $A$ by

$$\|A\|_F = \sqrt{\sum_{j,k} a_{jk}^2}.$$  

1. For $a_j$ a row of $A$, show that $\sum_{i=1}^r (a_j^T v_i)^2 = \|a_j\|^2$.

2. Show that $\|A\|_F = \sum_{i=1}^r \sigma_i^2$.

For $1 \leq k \leq r$, we define $A_k = \sum_{i=1}^k \sigma_i u_i v_i^T$, where $u_i$ are the left singular vectors.

3. Show that $A_k$ has rank $k$.

Let $V_k = V e c t(v_1, \ldots, v_k)$.

4. Show that the rows of $A_k$ are the projections of the rows of $A$ onto $V_k$.

Consider $B$ with rank $k \leq r$ minimizing $\|A - B\|_F$ and let $V$ be the space spanned by the rows of $B$.

5. Show that $\dim(V) \leq k$ and that each row of $B$ is the projection of the corresponding row of $A$ onto $V$.

6. Show that $\|A - A_k\|_F \leq \|A - B\|_F$.

Exercise 2

We consider a set $V$ of candidates. The extended Condorcet criterion requires: if $X, Y$ are a partition of the set $V$ of all candidates and for all $x \in X, y \in Y$, $x$ beats $y$ in direct comparison (i.e. more voters prefer $x$ to $y$ than vice versa), then all of $X$ should precede all of $Y$.

Recall that a tournament graph $G$ on $V$ is a complete oriented graph on $V$.

1. Show that each tournament graph $G$ contains a Hamiltonian path (i.e. a path going through each node exactly once).

2. Show that for any profile of voters, there is a rank aggregation (i.e. a consensus ordering) satisfying the extended Condorcet criterion.