Algorithms for Networked Information
TD4

October 12, 2015

Exercise 1 Cheeger Inequality

In this exercise, let G be a simple d-regular graph. We define the connectivity of a cut by:

$$\phi(S) = \frac{e(S, V-S)}{\frac{d}{|V|}|S||V-S|},$$

and the graph's isoperimetric constant by: $\phi(G) = \min_{S \subset V, S \neq \emptyset, V} \phi(S)$.

The expansion of a cut is defined by:

$$h(S) = \frac{e(S, V-S)}{d \min\{|S|, |V-S|\}},$$

and the graph's expansion rate by: $h(G) = \min_{S \subset V, S \neq \emptyset, V} h(S)$. The computation of $h(G)$ is NP-hard and the best algorithm by Arora, Rao & Vazirani (2009) gives an $O(\sqrt{\log n})$ approximation.

1. Show that

$$h(G) \leq \phi(G) \leq 2h(G).$$

We consider the normalized adjacency matrix $M = \frac{1}{d}A$ of graph G with eigenvalues $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The goal of this exercise is to show

$$\frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)}.$$

2. Show that

$$\begin{aligned} \lambda_2 &= \sup_{\mathbf{x} \in \mathbb{R}^n,\, \|\mathbf{x}\|=1,\, \mathbf{x} \perp \mathbf{1}} \mathbf{x}^t M \mathbf{x}, \\ \lambda_n &= \inf_{\mathbf{x} \in \mathbb{R}^n,\, \|\mathbf{x}\|=1} \mathbf{x}^t M \mathbf{x} \end{aligned}$$

3. Show that, for every vector $\mathbf{x}$,

$$\sum_{i,j} M_{ij}(x_i - x_j)^2 = 2\mathbf{x}^t\mathbf{x} - 2\mathbf{x}^t M \mathbf{x}.$$

   Deduce that

$$1 - \lambda_2 = \inf_{\mathbf{x} \in \mathbb{R}^n,\, \|\mathbf{x}\|=1,\, \mathbf{x} \perp \mathbf{1}} \frac{1}{2} \sum_{ij} M_{ij}(x_i - x_j)^2,$$

   and that $\lambda_2 = 1$ if and only if G is not connected.

4. Show that, for every vector $\mathbf{x}$,

$$\sum_{ij} M_{ij}(x_i + x_j)^2 = 2\mathbf{x}^t\mathbf{x} + 2\mathbf{x}^t M\mathbf{x}.$$

Deduce that $\lambda_n = -1$ if and only if one connected component of G is bipartite.

5. Show that

$$\phi(G) = \min_{\mathbf{x}\in\{0,1\}^V - \{\mathbf{0},\mathbf{1}\}} \frac{\sum_{ij} M_{ij}|x_i - x_j|^2}{\frac{1}{n}\sum_{ij}|x_i - x_j|^2}$$

6. Show that

$$1 - \lambda_2 = \min_{\mathbf{x}\in\mathbb{R}^V - \mathbb{R}\mathbf{1}} \frac{\sum_{ij} M_{ij}|x_i - x_j|^2}{\frac{1}{n}\sum_{ij}|x_i - x_j|^2}.$$

7. Deduce that $1 - \lambda_2 \le \phi(G) \le 2h(G)$.

To show the converse inequality, we introduce the following algorithm:
Spectral Partition

- Input: graph $G = (V, E)$ and a vector $\mathbf{x} \in \mathbb{R}^V$.
- Order the nodes by decreasing order of entries in $\mathbf{x}$, i.e., $V = \{v_1, \ldots, v_n\}$ with $x_{v_1} \le x_{v_2} \le \cdots \le x_{v_n}$.
- Let $i \in \{1, \ldots, n-1\}$ such that $h(\{v_1, \ldots, v_i\})$ is minimal.
- Output: $S = \{v_1, \ldots, v_i\}$.

Given a graph G and a vector $\mathbf{x} \in \mathbb{R}$, we define:

$$\delta = \frac{\sum_{i,j} M_{ij}|x_i - x_j|^2}{\frac{1}{n}\sum_{i,j}|x_i - x_j|^2},$$

where M is the normalized adjacency matrix. We will show that, if S is the algorithm's output, then $h(S) \le \sqrt{2\delta}$.

To simplify notation, we assume that $V = \{1, \ldots, n\}$ and that $x_1 \le x_2 \le \cdots \le x_n$, so that our goal is to show that there is an i such that $h(\{1, \ldots, i\}) \le \sqrt{2\delta}$. For that, we will use the probabilistic method.

8. Show that we can assume $x_{\lfloor n/2 \rfloor} = 0$ and $x_1^2 + x_n^2 = 1$ without loss of generality.
9. Let T be a random variable with values in $[x_1, x_n]$ such that $\mathbb{P}(a \le t \le b) = \int_a^b 2|t|dt$ for $x_1 \le a \le b \le x_n$. Let $S_T = \{i, x_i \le T\}$. Show that

$$\mathbb{E}[\min\{|S_T|, |V - S_T|\}] = \sum_i x_i^2.$$

10. Show that

$$\mathbb{P}\big((i,j) \text{ is cut by } (S_T, V - S_T)\big) \le |x_i - x_j|\left(|x_i| + |x_j|\right).$$

11. Deduce that

$$\frac{1}{d}\mathbb{E}[e(S_T, V - S_T)] \le \frac{1}{2}\sqrt{\sum_{i,j} M_{ij}(x_i - x_j)^2}\sqrt{\sum_{i,j} M_{ij}\left(|x_i| + |x_j|\right)^2}$$

12. Show that $\sum_{i,j} M_{ij}(x_i - x_j)^2 \le 2\delta \sum_i x_i^2$ and that $\sum_{i,j} M_{ij}\left(|x_i| + |x_j|\right)^2 \le 4\sum_i x_i^2$. Deduce that $\frac{1}{d}\mathbb{E}[e(S_T, V - S_T)] \le \sqrt{2\delta}\sum_i x_i^2$.
13. Deduce that there is an S of the form $\{1, \ldots, i\}$ such that $h(S) \le \sqrt{2\delta}$.
14. Deduce that $h(G) \le \sqrt{2(1 - \lambda_2)}$.