Algorithms for Networked Information
DM1

to be handed in by October 26, 2015

Exercise 1 AdWords auctions (*12 points*)

We consider the problem of assigning bidders $a = 1,\dots,A$ to a number of slots $s = 1,\dots,S$. We assume that the slots are ordered according to their click-through rates (CTR): $x_1 > x_2 > \cdots > x_S$. Every bidder has a value v_a equal to the mean profit by click and $u_{as} = v_a x_s$ is the mean profit for bidder a if they get slot s. We also assume that the bidders are ordered according to their values, i.e., $v_1 > v_2 > \cdot > v_A$.

The slots are sold by means of an auction. Each bidder submits a bid b_a and the slot with the best CTR is assigned to the bidder with the highest bid, the slot with the second best CTR is assigned to the bidder with the second highest bid, etc. For every click, the bidder having been assigned slot s has to pay the price p_s equal to the bid just below their own. We denote by $a(s)$ the bidder having s^{th} highest bid. We thus have $p_s = b_{a(s+1)}$ for $s \in S$. We also suppose that the number A of bidders is larger than the number S of slots, i.e., $S < A$ (so that p_s is well-defined for all $s \le S$). We adopt the convention $x_s = 0$ and $p_s = 0$ for $s > S$.

Finally, the profit a given bidder receives at slot s is equal to $(v_{a(s)} - p_s)x_s = (v_{a(s)} - b_{a(s+1)})x_s$. We model the auction by a full-information game. All bidders simultaneously submit their bids b_a. The bids are then ordered and the prices determined as described above. The strategy for bidder a is hence a real number $b_a \ge 0$.

1. Show that every Nash equilibrium satisfies:

$$\begin{aligned}
(v_{a(s)} - p_s)x_s &\ge (v_{a(s)} - p_t)x_t, \quad \text{for } t > s\\
(v_{a(s)} - p_s)x_s &\ge (v_{a(s)} - p_{t-1})x_t, \quad \text{for } t < s,
\end{aligned}$$

   where $p_t = b_{a(t+1)}$.

We define a symmetric Nash equilibrium (SNE) by: for all t, s,

$$(v_{a(s)} - p_s)x_s \ge (v_{a(s)} - p_t)x_t, \text{ where } p_t = b_{t+1}. \tag{1}$$

2. Show that $v_{a(s)} \ge p_s$ in every SNE.
3. Show that $v_{a(s-1)} \ge v_{a(s)}$ for all s in every SNE. In particular, $a(s) = s$.
4. Show that $p_{s-1} \ge p_s$ for all $s \le S$ in every SNE. If $v_s > p_s$, then $p_{s-1} > p_s$.
5. Show that every SNE is a Nash equilibrium.
6. Show that, if a set of bids $(b_1, b_2, \dots, b_A)$ satisfies (1) for $t = s+1$ and $t = s-1$, then all the inequalities for all s are satisfied.
7. Show that $b_s^L \le b_s \le b_s^U$ where

$$\begin{aligned}
b_s^U x_{s-1} &= \sum_{t \ge s} v_{t-1}(x_{t-1} - x_t),\\
b_s^L x_{s-1} &= \sum_{t \ge s} v_t(x_{t-1} - x_t).
\end{aligned}$$

By summing these equations, we obtain a lower bound R^L_{SNE} and an upper bound R^U_{SNE} on the seller's revenue:

$$\begin{aligned} R^U_{SNE} &= \sum_{s=1}^{S} s v_s (x_s - x_{s+1}), \\ R^L_{SNE} &= \sum_{s=1}^{S} s v_{s+1} (x_s - x_{s+1}). \end{aligned}$$

However, these bounds are valid only for SNEs. Let R^L_{NE} and R^U_{NE} be the minimum and maximum seller's revenue for all possible Nash equilibria, respectively.

8. Show that $R^U_{NE} = R^U_{SNE} \geq R^L_{SNE} \geq R^L_{NE}$.

We now turn to the case where the game is no longer a full-information game: the values v_a are private, whereas the CTRs x_s are public.

9. For this question, we consider the particular case of three bidders and two slots with the following values: $v_1 = 10$, $v_2 = 4$, $v_3 = 2$ and $x_1 = 200$, $x_2 = 199$. Show that being honest is not a dominant strategy.
10. We modify the auctioning system. The assignment of slots is done as before, but the payment structure is modified as follows:

$$p_{s-1} = \sum_{t \geq s} b_{a(t)} (x_{t-1} - x_t)$$

    To what kind of auction does the case $S = 1$ correspond? For every S, show that being honest is now a dominant strategy.
11. Show that the revenue is R^L_{SNE}.
12. For this question, we consider the particular case of three bidders and two slots with the following values: $v_1 = 12$, $v_2 = 8$, $v_3 = 4$ and $x_1 = 400$, $x_2 = 200$. Find the seller's revenue for the original auctioning system if all players are honest. Is it a Nash equilibrium?

Exercise 2 Spectral Partition algorithm (*10 points*)

The goal of this exercise is to implement the Spectral Partition algorithm and to verify the implementation on real networks. Use the programming language of your choice and the input format of the SNAP network data base (https://snap.stanford.edu/data/index.html). That is, the network's edges are of the form "`node1 node2`", one line at a time. For example, the complete network with 4 nodes is entered as follows:

```
1 2
1 3
1 4
2 3
2 4
3 4
```

Verify your implementation with at least one network of the SNAP data base. Specify which networks you used.