Susceptible-Infective-Susceptible Epidemics: impact of graph topology

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Basic model: graph $G = (V, E)$

Each infected node infects each of its neighbors at rate $\beta$, and becomes healthy at rate $\delta$.

May model propagation of mutating virus, or replication of data in volatile memories.

$\Rightarrow$ Markov jump process on $\{0, 1\}^V$ with non-zero transition rates:

$$q(x, x + e_i) = \beta \sum_{j \sim i} x_j, \ i \in V, \ x \in \{0, 1\}^V, \ x_i = 0;$$

$$q(x, x - e_i) = \delta, \ i \in V, \ x \in \{0, 1\}^V, \ x_i = 1.$$  

Stationary regime: complete extinction (absorbing state)

Goal: understand impact of $\beta$, $\delta$ and topology of $G$ on time to extinction.
Example of a grid network

Behaviour characterized by [Durrett-Liu, Durrett-Schonmann, ’88]:
there is a critical threshold \( c > 0 \) such that:
\[
\beta / \delta > c \Rightarrow \text{long survival (expected time to extinction: exponential in } n = |V|),
\]
\[
\beta / \delta < c \Rightarrow \text{fast extinction (expected time to extinction logarithmic in } n = |V|),
\]
Main results:
- Fast extinction and spectral radius
- Long survival and isoperimetric constant

Markovian images of Markov processes

Coupling of Generalized Birth-and-Death processes

Proofs

SIR and spectral radius; SI and isoperimetric constant
Fast extinction and spectral radius

Definition

The spectral radius $\rho(A)$ of matrix $A$ is the largest modulus of its eigenvalues.

Theorem

Let $\rho$ be the spectral radius of the adjacency matrix $A$ of graph $G = (V, E)$. The time to extinction $T$ verifies for all $t > 0$:

$$P(T \geq t) \leq ne^{(\beta \rho - \delta)t},$$

where $n = |V|$.

Corollary

If $\beta \rho < \delta$, then $E(T) \leq \frac{\ln(n) + 1}{\delta - \beta \rho}$.
Definition

For a graph $G = (V, E)$ and any $m < n$, the isoperimetric constant $\eta_m$ is defined as

$$\eta_m = \min_{S \subset V, |S| \leq m} \frac{|E(S, \overline{S})|}{|S|},$$

where $E(S, \overline{S})$ denotes the set of edges between $S$ and its complement $\overline{S} = V \setminus S$. 

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Theorem

Assume that for some $r \in ]0, 1[$ and some $m < n$, $\beta \eta_m \geq \frac{\delta}{r}$. Then there is a function $f : \mathbb{N} \to \mathbb{R}$ such that $\lim_{k \to \infty} f(k) = 0$ and for any $k \in \mathbb{N}$,

$$\mathbb{P}(T \geq \frac{k}{2\delta m}) \geq \frac{1 - r}{1 - r^m} \left( \frac{1 - r^{m-1}}{1 - r^m} \right)^k (1 - f(k))$$

Corollary

If for fixed $r \in ]0, 1[$ and a sequence of graphs $G_n$ each on $n$ nodes one has for some $m = m(n)$ with $\lim_{n \to \infty} m(n) = +\infty$:

$\beta \eta_m(G_n) \geq \frac{\delta}{r}$, then the time $T_n$ to extinction of the $(\beta, \delta)$-epidemic process on $G_n$ verifies:

$$\mathbb{E}[\delta T_n] \geq \frac{(1 - r)^2}{3m} \lfloor r^{-m+2} \rfloor = e^{\Omega(m)}.$$
Example: complete graph

For complete graph on $n$ nodes, $\rho = n - 1$ and $\eta_m = n - m$.

$\Rightarrow$ for fixed $\epsilon \in ]0, 1[$,

if $\beta(n - 1) \leq \delta(1 - \epsilon)$, $\mathbb{E}[\delta T_n] \leq \frac{\ln(n)+1}{\epsilon} = O(\ln(n))$;
if $\beta(n - 1) \geq \delta(1 + \epsilon)$, for $m = n\epsilon/2$ one has $\beta \eta_m \geq \delta/r$ with $r^{-1} = (1 + \epsilon)(1 - \epsilon/2) > 1$, so that

$\mathbb{E}[\delta T_n] \geq e^{\Omega(n)}$

A sharp transition with respect to $(\beta n/\delta)$ at 1.
Example: hypercube

Hypercube $G = \{0, 1\}^d$ on $n = 2^d$ nodes:

$\rho = d$, $G = \eta_m \geq d - k$ for $m = 2^k$, $k < d$ (ref: [Harper’64])

Fix $\epsilon \in \]0, 1[$.

If $\beta d \leq \delta (1 - \epsilon)$, then $\mathbb{E}[\delta T_n] \leq \frac{\ln(n)+1}{\epsilon} = O(\ln(n))$;

If $\beta d \geq \delta (1 + \epsilon)$, for $m = 2^{\epsilon d/2}$, $\eta_m \geq (1 - \epsilon/2)d$.

Hence $\beta \eta_m \geq \delta / r$ with $r < 1$, so that

$\mathbb{E}[\delta T_n] \geq e^{\Omega(m)} = e^{\Omega(n^{\epsilon/2})}$.

A sharp transition with respect to $(\beta d / \delta)$ at 1.
Example: Erdős-Rényi graph with super-logarithmic average degree

**Proposition**

Let $G = G(n, d/n)$ with $d \gg \ln(n)$, and some fixed $\alpha \in ]0, 1[$. One then has the convergences in probability

$$\lim_{n \to \infty} \frac{\rho(A)}{d} = 1, \quad \lim_{n \to \infty} \frac{\eta_{\alpha n}}{(1 - \alpha)d} = 1.$$

**Corollary**

Let $\epsilon > 0$ be fixed. One has the following with high probability with respect to $G$:

If $\beta d \leq (1 - \epsilon)\delta$, then $\mathbb{E} \frac{T_n}{\delta} \leq \frac{2\ln(n)}{\epsilon} = O(\ln(n))$.

If $\beta d \leq (1 + \epsilon)\delta$, then $\mathbb{E} \frac{T_n}{\delta} \geq e^{\Omega(\epsilon n)} = e^{\Omega(n)}$. 
Markovian transforms of Markov chains

Let \( \{X_n\}_{n \in \mathbb{N}} \) be a Markov chain on countable set \( E \) with transition matrix \( (p_{ij})_{i,j \in E} \).
For countable set \( F \) and \( f : E \to F \), let \( Y_n := f(X_n), \ n \in \mathbb{N} \).

**Theorem**

If for some transition matrix \( \hat{P} = (\hat{p}_{uv})_{u,v \in F} \), one has
\[
\forall x \in E, \ v \in F, \ \sum_{y \in E : f(y) = v} p_{xy} = \hat{p}_{f(x),v},
\]
then \( \{Y_n\}_{n \in \mathbb{N}} \) is a Markov chain on \( F \) with transition matrix \( \hat{P} \).

**Proof:** by evaluating \( \mathbb{P}(Y_0^k = y_0^k) \) for arbitrary \( y_0^k \in F^{k+1}\).

**Remark:** In general, image of Markov chain fails to be Markovian.
**Example:** \( X_\infty = \{0, 1, 2, 0, 1, 2, \ldots\} \), \( f(x) = \mathbb{I}_{x=2} \Rightarrow Y_\infty = \{0, 0, 1, 0, 0, 1, \ldots\} \).
Let $\{X(t)\}_{t \in \mathbb{R}^+}$ be a non-explosive Markov jump process on countable set $E$ with infinitesimal generator $(q_{ij})_{i,j \in E}$.

For countable set $F$ and $f : E \rightarrow F$, let $Y(t) := f(X(t))$, $t \in \mathbb{R}^+$.

**Theorem**

If for some matrix $\hat{Q} = (\hat{q}_{uv})_{u,v \in F}$ such that
\[ \forall u \in F, \hat{q}_{u,u} = -\sum_{v \neq u} q_{uv} =: -\hat{q}(u), \] one has
\[ \forall x \in E, v \in F : f(x) \neq v, \sum_{y \in E : f(y) = v} q_{xy} = \hat{q}_{f(x),v}, \]

then $\{Y(t)\}_{t \in \mathbb{R}^+}$ is a Markov jump process on $F$ with infinitesimal generator $\hat{Q}$. 
Markovian transforms of Markovian jump processes

Proof:

Let \( \{X_n, \tau_n\}_{n \in \mathbb{N}} \) (resp. \( \{Y_n, \hat{\tau}_n\}_{n \in \mathbb{N}} \)): sequence of values and corresponding sojourn times taken by \( \{X(t)\}_{t \in \mathbb{R}^+} \) (resp. \( \{Y(t)\}_{t \in \mathbb{R}^+} \)).

Write for \( \alpha > 0, x_0 \in E, y_1 \in F : f(x_0) \neq y_1 \), with sums over \( x_1^n \in E^n \) such that \( f(x_i) = f(x_0), i < n, f(x_n) = y_1 \):

\[
\begin{align*}
g(x_0) &:= \mathbb{E}(e^{-\alpha \hat{\tau}_0} \mathbb{I}_{Y_1 = y_1} | X_0 = x_0) \\
&= \sum_{n \geq 1} \sum x_1^n \mathbb{E}\left[ \prod_{i=1}^{n} \mathbb{I}_{X_i = x_i} e^{-\alpha \tau_{i-1}} | X_0 = x_0 \right] \\
&= \sum_{n \geq 1} \sum x_1^n \prod_{i=1}^{n} \frac{q_{x_{i-1}, x_i}}{q(x_{i-1}) + \alpha} \\
&= h(x_0) + \sum_{x_1 \neq x_0 : f(x_1) = f(x_0)} \frac{q_{x_0, x_1}}{q(x_0) + \alpha} g(x_1),
\end{align*}
\]

where \( h(x_0) := \sum_{x_1 : f(x_1) = y_1} \frac{q_{x_0, x_1}}{q(x_0) + \alpha} \).

By theorem’s assumption, \( g(x) = g^0(x) = \frac{\hat{q}_{f(x), y_1}}{\hat{q}(f(x)) + \alpha} \) is a solution of previous equation \( g = h + \mathcal{L}g \).
Markovian transforms of Markovian jump processes

Bounded solution of \( g = h + \mathcal{L}g \) is unique by non-explosiveness of \( \{X(t)\}_{t \in \mathbb{R}_+} \): indeed, by induction

\[
\mathcal{L}^n g(x_0) = \mathbb{E}[e^{-\alpha(\tau_0+\ldots+\tau_{n-1})} \prod_{i=1}^{n} \mathbb{I}_{f(X_i)=f(x_0)} g(X_n) | X_0 = x_0]
\]

hence for two solutions \( g, g^0 \) and any \( n > 0 \),

\[
g(x) - g^0(x) = \mathcal{L}^n g(x) - \mathcal{L}^n g^0(x) = \mathbb{E}[e^{-\alpha(\tau_0+\ldots+\tau_{n-1})} \prod_{i=1}^{n} \mathbb{I}_{f(X_i)=f(x_0)} (g(X_n) - g^0(X_n)) | X_0 = x] \to 0 \text{ as } n \to \infty
\]

(by non-explosiveness: \( \tau_0 + \cdots + \tau_n \to \infty \) almost surely; dominated convergence applies by boundedness of \( g - g^0 \) )

Strong Markov property:

\[
\mathbb{E}(\prod_{i=0}^{n-1} \mathbb{I}_{Y_{i+1}=Y_i} e^{-\alpha_i \hat{\tau}_i} | Y_0 = y_0) = \prod_{i=0}^{n-1} \frac{\hat{q}(y_{i}, y_{i+1})}{\hat{q}(y_i) + \alpha_i}
\]
**Definition:** Jump processes on $E = \mathbb{N}^k$ with only non-zero transition rates $q_{x,x+e_i} = \beta_i(x)$, $q_{x,x-e_i} = \delta_i(x)$, $x \in \mathbb{N}^k$, $i \in [k]$.

**Theorem**

Let $X, X'$ be two non-explosive GBD processes on $\mathbb{N}^k$ with respective birth and death rate functions $(\beta, \delta)$, $(\beta', \delta')$. Assume that $\forall x, x' \in \mathbb{N}^k, x \leq x'$, one has

$$x_i = x_i' \Rightarrow \beta_i(x) \leq \beta_i'(x'), \ \delta_i(x) \geq \delta_i'(x').$$

Then for ordered initial conditions $x(0) \leq x'(0)$ there is a coupled construction of processes $X, X'$ such that with probability 1, $\forall t \in \mathbb{R}_+, X(t) \leq X'(t)$. 
Consider Markov jump process on \( \{ x, x' \in \mathbb{N}^k : x \leq x' \} \) with only non-zero transition rates:

\[
\begin{align*}
    x_i < x_i' & \Rightarrow q(x,x'),(x+e_i,x') = \beta_i(x), \quad q(x,x'),(x-e_i,x') = \delta_i(x), \\
    q(x,x'),(x+,x'+e_i) = \beta'_i(x'), \quad q(x,x'),(x,x'-e_i) = \delta'_i(x'); \\
    x_i = x_i' & \Rightarrow q(x,x'),(x+e_i,x'+e_i) = \beta_i(x), \\
    q(x,x'),(x,x'+e_i) = \beta'_i(x') - \beta_i(x), \\
    q(x,x'),(x-e_i,x'-e_i) = \delta'_i(x'), \\
    q(x,x'),(x-e_i,x') = \delta_i(x) - \delta'_i(x').
\end{align*}
\]

Apply result on Markovian images of Markov jump processes to show that component processes \( X, X' \) are GBD with the desired birth and death rate functions.
Define branching random walk on graph $G = (V,E)$ as GBD process $X'$ on $\mathbb{N}^V$ with $\beta'(x) = \beta \sum_{j \sim i} x_j$ and $\delta'(x) = \delta x_i$.

View SIS process as GBD on $\mathbb{N}^V$ with $\beta_i(x) = \beta \sum_{j \sim i} x_j$, $\delta_i(x) = \delta x_i$.

Use previous Theorem to couple two processes $X, X'$ with initial conditions $x(0) \in \{0,1\}^V$ so that $\forall t \in \mathbb{R}_+, X(t) \leq X'(t)$

Bound probability of SIS survival:

$$P(T > t) \leq \mathbb{E}(\sum_i X_i(t)) \leq \mathbb{E} \sum_i X'_i(t).$$

Linearity of rate functions $\beta'_i, \delta'_i$:

$$\frac{d}{dt} \mathbb{E}(X'(t)) = \beta A \mathbb{E}(X'(t)) - \delta \mathbb{E}(X'(t))$$

$$\Rightarrow \mathbb{E}(X'(t)) = e^{t(\beta A - \delta I)} x(0).$$
Assumption: $\beta \eta_m \geq \delta / r$.
Consider jump process $Z$ on $\{0, \ldots, m\}$ with non-zero transition rates $q_{z, z+1} = \frac{\delta z}{r} \mathbb{1}_{z < m}$, $q_{z, z-1} = \delta z$.

Couple SIS process $X$ and $Z$ such that with probability 1, $\forall t \in \mathbb{R}_+, Z(t) \leq \sum_{i \in V} X_i(t)$ by choosing as non-zero transition rates:

$$
\sum_i x_i > z \Rightarrow q_{x, z},(x+e_i,z) = \beta(1-x_i) \sum_{j \sim i} x_j, q_{x, z},(x-e_i,z) = \delta x_i,
$$

$$
\sum_i x_i = z \Rightarrow q_{x, z},(x+e_i,z+1) = \frac{1}{M(x)} \beta(1-x_i) \sum_{j \sim i} x_j,
q_{x, z},(x+e_i,z) = [1 - \frac{1}{M(x)}] \beta(1-x_i) \sum_{j \sim i} x_j,
q_{x, z},(x-e_i,z-1) = x_i \delta,
$$

where $M(x) := \mathbb{P} \left[ \frac{\beta(1-x_i)}{\delta} \sum_{j \sim i} x_j \right] = \frac{\beta |E(S, \overline{S})|}{\delta |S|/r} \geq 1$ with $S = \{i : x_i = 1\}$.
Lower bound probability of survival: \( \mathbb{P}(T \geq t) \geq \mathbb{P}(Z(t) > 0) \).

- Analysis of \( Z \): embedded chain with transition probabilities
  \( p_{z,z+1} = \frac{1}{1+r} \mathbb{I}_{z<m}, \quad p_{z,z-1} = \frac{r}{1+r} \).

- "Gambler's ruin problem": probability of reaching \( m \) before 0 when at \( z \):
  \( \frac{1-r^z}{1-r^m} \).

- Probability of at least \( k \) visits to state \( m \) before absorption at 0:
  \( \geq \frac{1-r}{1-r^m} \left( \frac{1-r^{m-1}}{1-r^m} \right)^{k-1} \).

- Sojourn times in state \( m \): \( \text{Exp}(\delta m) \). Hence
  \[
  \mathbb{P}(T \geq \frac{k}{2\delta m}) \geq \frac{1-r}{1-r^m} \left( \frac{1-r^{m-1}}{1-r^m} \right)^{k-1} \mathbb{P}(S_1 + \ldots + S_k \geq k/2)
  \]

  with \( S_i \): i.i.d., \( \text{Exp}(1) \).

Announced result follows by writing
\( f(k) = \mathbb{P}(S_1 + \ldots + S_k < k/2) \): \( \lim_{k \to \infty} f(k) = 0 \) follows by law of large numbers.
Consider Reed-Frost process with neighbor infection parameter $\beta$ on graph $G = (V, E)$. Then:

**Theorem**

Suppose $\beta \rho < 1$. Then the total number of nodes eventually removed verifies

$$
\mathbb{E} \sum_{i \in V} Y_i(\infty) \leq \frac{1}{1 - \beta \rho} \sqrt{n \sum_{i \in V} X_i(0)}.
$$

If moreover $G$ is $d$-regular, then

$$
\mathbb{E} \sum_{i \in V} Y_i(\infty) \leq \frac{1}{1 - \beta \rho} \sum_{i \in V} X_i(0).
$$
By union bound,

\[ \mathbb{P}(Y_u(\infty) = 1) \leq \sum_{t \geq 0} \sum_{u_0, \ldots, u_t} \beta^t X_{u_0}(0) \]

\[ = \sum_{t \geq 0} \sum_{v \in V} (\beta A)^t_{uv} X_v(0) \]

where \( u_0, \ldots, u_t \): graph path with \( u_t = u \).

Hence

\[ \mathbb{E} \sum_u Y_u(\infty) \leq \sum_{t \geq 0} e^{T(\beta A)^t} X(0) \]

\[ = e^{T(I - \beta A)^{-1}} X(0) \]

\[ = \sum_i < x_i, e > \frac{1}{1 - \beta \lambda_i} < x_i, X(0) > \]
Consider SI process on $G = (V, E)$ with infection rate $\lambda$ along each edge.

Then propagation is at least as fast as SI process on complete graph with infection rate $\lambda \eta n/2$.

**Proof:** Couple SI infection process $X$ with jump process on $\{1, \ldots, n\}$ with only non-zero transition rates $q_{z,z+1} = \lambda \eta n/2 \frac{z(n-z)}{n-1}$ so that $\forall t \in \mathbb{R}_+, \sum_{i \in V} X_i(t) \geq Z(t)$. 
Takeaway messages

- Behaviour of SIS epidemics undergoes phase transitions as ratio $\beta/\delta$ crosses thresholds.

- Graph topology determines thresholds; in several scenarios (complete graph, hypercube, E-R graphs), spectral radius and isoperimetric constants are close, hence a single threshold.

- Coupling constructions allow control of complex process by simpler ones.