# Susceptible-Infective-Susceptible Epidemics: impact of graph topology

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# SIS model

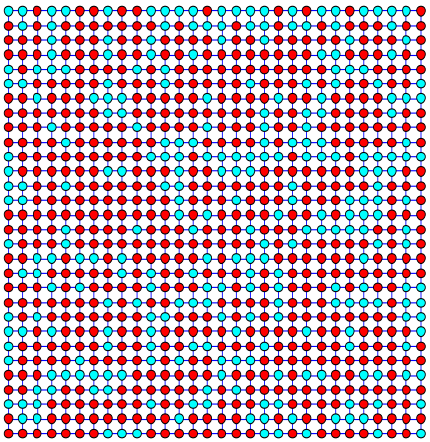
- Basic model: graph $G = (V, E)$
- Each infected node infects each of its neighbors at rate $\beta$ , and becomes healthy at rate $\delta$

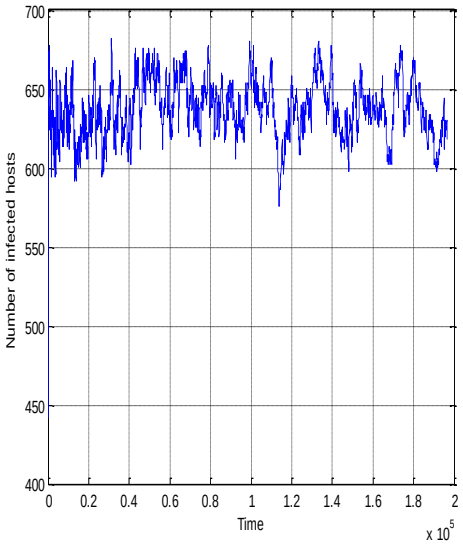
May model propagation of mutating virus, or replication of data in volatile memories
$\Rightarrow$ Markov jump process on $\{0,1\}^V$ with non-zero transition rates

$$\begin{aligned} q(x, x+e_i) &= \beta \textstyle\sum_{j\sim i} x_j,\ i \in V,\ x \in \{0,1\}^V,\ x_i = 0; \\ q(x, x-e_i) &= \delta,\ i \in V,\ x \in \{0,1\}^V,\ x_i = 1. \end{aligned}$$

Stationary regime: complete extinction (absorbing state)
Goal: understand impact of $\beta$, $\delta$ and topology of $G$ on time to extinction

# Example of a grid network






Behaviour characterized by [Durrett-Liu,Durrett-Schonmann,'88]: there is a critical threshold $c > 0$ such that:
$\beta/\delta > c \Rightarrow$ long survival (expected time to extinction: exponential in $n = |V|$),
$\beta/\delta < c \Rightarrow$ fast extinction (expected time to extinction logarithmic in $n = |V|$),

# outline

- Main results:
  Fast extinction and spectral radius
  Long survival and isoperimetric constant
- Markovian images of Markov processes
- Coupling of Generalized Birth-and-Death processes
- Proofs
- SIR and spectral radius; SI and isoperimetric constant

# Fast extinction and spectral radius

## Definition

The spectral radius $\rho(A)$ of matrix $A$ is the largest modulus of its eigenvalues.

## Theorem

*Let $\rho$ be the spectral radius of the adjacency matrix $A$ of graph $G = (V, E)$. The time to extinction $T$ verifies for all $t > 0$:*

$$\mathbb{P}(T \geq t) \leq n e^{(\beta\rho - \delta)t},$$

*where $n = |V|$.*

## Corollary

*If $\beta\rho < \delta$, then $\mathbb{E}(T) \leq \frac{\ln(n)+1}{\delta - \beta\rho}$*

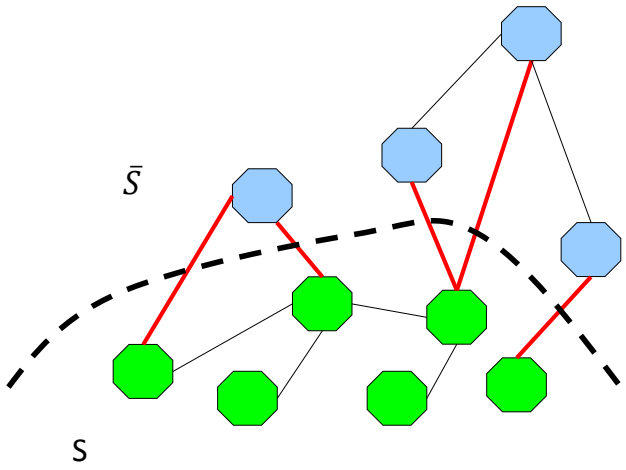
# Long survival and isoperimetric constants

## Definition

For a graph $G = (V, E)$ and any $m < n$, the isoperimetric constant $\eta_m$ is defined as

$$\eta_m = \min_{S \subset V, |S| \leq m} \frac{|E(S, \overline{S})|}{|S|},$$

where $E(S, \overline{S})$ denotes the set of edges between $S$ and its complement $\overline{S} = V \setminus S$.



# Long survival and isoperimetric constants

## Theorem

*Assume that for some* $r \in ]0,1[$ *and some* $m < n$, $\beta\eta_m \geq \frac{\delta}{r}$.
*Then there is a function* $f : \mathbb{N} \to \mathbb{R}$ *such that* $\lim_{k\to\infty} f(k) = 0$ *and for any* $k \in \mathbb{N}$,

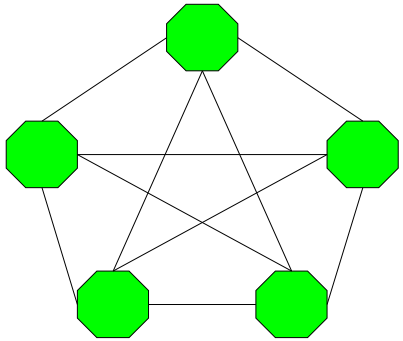
$$\mathbb{P}(T \geq \frac{k}{2\delta m}) \geq \frac{1-r}{1-r^m}\left(\frac{1-r^{m-1}}{1-r^m}\right)^k (1-f(k))$$

## Corollary

*If for fixed* $r \in ]0,1[$ *and a sequence of graphs* $G_n$ *each on* $n$ *nodes one has for some* $m = m(n)$ *with* $\lim_{n\to\infty} m(n) = +\infty$ *:* $\beta\eta_m(G_n) \geq \frac{\delta}{r}$, *then the time* $T_n$ *to extinction of the* $(\beta,\delta)$*-epidemic process on* $G_n$ *verifies:*

$$\mathbb{E}[\delta T_n] \geq \frac{(1-r)^2}{3m}\lfloor r^{-m+2}\rfloor = e^{\Omega(m)}.$$

# Example: complete graph



For complete graph on $n$ nodes, $\rho = n - 1$ and $\eta_m = n - m$.

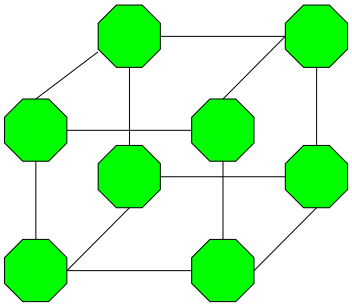
$\Rightarrow$ for fixed $\epsilon \in ]0, 1[$,

if $\beta(n-1) \leq \delta(1-\epsilon)$, $\mathbb{E}[\delta T_n] \leq \frac{\ln(n)+1}{\epsilon} = O(\ln(n))$;
if $\beta(n-1) \geq \delta(1+\epsilon)$, for $m = n\epsilon/2$ one has $\beta\eta_m \geq \delta/r$ with $r^{-1} = (1+\epsilon)(1-\epsilon/2) > 1$, so that

$\mathbb{E}[\delta T_n] \geq e^{\Omega(n)}$

A sharp transition with respect to $(\beta n/\delta)$ at 1.

# Example: hypercube



Hypercube $G = \{0,1\}^d$ on $n = 2^d$ nodes:
$\rho = d$, $G = \eta_m \geq d - k$ for $m = 2^k, k < d$ (ref: [Harper'64])

Fix $\epsilon \in ]0,1[$.
If $\beta d \leq \delta(1-\epsilon)$, then $\mathbb{E}[\delta T_n] \leq \frac{\ln(n)+1}{\epsilon} = O(\ln(n))$;

If $\beta d \geq \delta(1+\epsilon)$, for $m = 2^{\epsilon d/2}$, $\eta_m \geq (1-\epsilon/2)d$.
Hence $\beta\eta_m \geq \delta/r$ with $r < 1$, so that
$\mathbb{E}[\delta T_n] \geq e^{\Omega(m)} = e^{\Omega(n^{\epsilon/2})}$.

A sharp transition with respect to $(\beta d/\delta)$ at 1.

# Example: Erdős-Rényi graph with super-logarithmic average degree

**proposition**

Let $G = \mathcal{G}(n, d/n)$ with $d >> \ln(n)$, and some fixed $\alpha \in ]0, 1[$. One then has the convergences in probability

$$\lim_{n\to\infty} \frac{\rho(A)}{d} = 1, \quad \lim_{n\to\infty} \frac{\eta_{\alpha n}}{(1-\alpha)d} = 1.$$

**Corollary**

*Let $\epsilon > 0$ be fixed. One has the following with high probability with respect to $G$:*
*If $\beta d \leq (1-\epsilon)\delta$, then $\mathbb{E}\frac{T_n}{\delta} \leq \frac{2\ln(n)}{\epsilon} = O(\ln(n))$.*
*If $\beta d \leq (1+\epsilon)\delta$, then $\mathbb{E}\frac{T_n}{\delta} \geq e^{\Omega(\epsilon n)} = e^{\Omega(n)}$.*

# Markovian transforms of Markov chains

Let $\{X_n\}_{n\in\mathbb{N}}$ be a Markov chain on countable set $E$ with transition matrix $(p_{ij})_{i,j\in E}$.
For countable set $F$ and $f : E \to F$, let $Y_n := f(X_n),\ n \in \mathbb{N}$.

**Theorem**

*If for some transition matrix $\hat{P} = (\hat{p}_{uv})_{u,v\in F}$, one has*

$$\forall x \in E,\ v \in F,\quad \sum_{y\in E: f(y)=v} p_{xy} = \hat{p}_{f(x),v},$$

*then $\{Y_n\}_{n\in\mathbb{N}}$ is a Markov chain on $F$ with transition matrix $\hat{P}$.*

**Proof:** by evaluating $\mathbb{P}(Y_0^k = y_0^k)$ for arbitrary $y_0^k \in F^{k+1}$...

**Remark:** In general, image of Markov chain fails to be Markovian.
Example: $X_0^\infty = \{0, 1, 2, 0, 1, 2, \ldots\}$,
$f(x) = \mathbb{I}_{x=2} \Rightarrow Y_0^\infty = \{0, 0, 1, 0, 0, 1, \ldots, \}$

# Markovian transforms of Markovian jump processes

Let $\{X(t)\}_{t\in\mathbb{R}_+}$ be a non-explosive Markov jump process on countable set $E$ with infinitesimal generator $(q_{ij})_{i,j\in E}$.
For countable set $F$ and $f : E \to F$, let $Y(t) := f(X(t)),\ t \in \mathbb{R}_+$.

**Theorem**

*If for some matrix $\hat{Q} = (\hat{q}_{uv})_{u,v\in F}$ such that $\forall u \in F, \hat{q}_{u,u} = -\sum_{v\neq u} q_{uv} =: -\hat{q}(u)$, one has*

$$\forall x \in E,\ v \in F : f(x) \neq v, \quad \sum_{y\in E: f(y)=v} q_{xy} = \hat{q}_{f(x),v},$$

*then $\{Y(t)\}_{t\in\mathbb{R}_+}$ is a Markov jump process on $F$ with infinitesimal generator $\hat{Q}$.*

# Markovian transforms of Markovian jump processes

**Proof:**

Let $\{X_n, \tau_n\}_{n\in\mathbb{N}}$ (resp. $\{Y_n, \hat{\tau}_n\}_{n\in\mathbb{N}}$): sequence of values and corresponding sojourn times taken by $\{X(t)\}_{t\in\mathbb{R}_+}$(resp. $\{Y(t)\}_{t\in\mathbb{R}_+}$).

Write for $\alpha > 0, x_0 \in E, y_1 \in F : f(x_0) \neq y_1$, with sums over $x_1^n \in E^n$ such that $f(x_i) = f(x_0), i < n, f(x_n) = y_1$ :

$$\begin{aligned} g(x_0) &:= \mathbb{E}(e^{-\alpha\hat{\tau}_0}\mathbb{I}_{Y_1=y_1}|X_0 = x_0) \\ &= \sum_{n\geq 1}\sum_{x_1^n}\mathbb{E}[\prod_{i=1}^n \mathbb{I}_{X_i=x_i}e^{-\alpha\tau_{i-1}}|X_0 = x_0] \\ &= \sum_{n\geq 1}\sum_{x_1^n}\prod_{i=1}^n \frac{q_{x_{i-1},x_i}}{q(x_{i-1})+\alpha} \\ &= h(x_0) + \sum_{x_1\neq x_0 : f(x_1)=f(x_0)} \frac{q_{x_0,x_1}}{q(x_0)+\alpha} g(x_1), \end{aligned}$$

where $h(x_0) := \sum_{x_1 : f(x_1)=y_1} \frac{q_{x_0,x_1}}{q(x_0)+\alpha}$.

By theorem's assumption, $g(x) = g^0(x) = \frac{\hat{q}_{f(x),y_1}}{\hat{q}(f(x))+\alpha}$ is a solution of previous equation $g = h + \mathcal{L}g$.

# Markovian transforms of Markovian jump processes

Bounded solution of $g = h + \mathcal{L}g$ is unique by non-explosiveness of $\{X(t)\}_{t\in\mathbb{R}_+}$: indeed, by induction

$$\mathcal{L}^n g(x_0) = \mathbb{E}[e^{-\alpha(\tau_0+\ldots+\tau_{n-1})} \prod_{i=1}^{n} \mathbb{I}_{f(X_i)=f(X_0)} g(X_n) | X_0 = x_0]$$

hence for two solutions $g, g^0$ and any $n > 0$,

$$\begin{aligned} & g(x) - g^0(x) = \mathcal{L}^n g(x) - \mathcal{L}^n g^0(x) \\ & = \mathbb{E}[e^{-\alpha(\tau_0+\ldots+\tau_{n-1})} \textstyle\prod_{i=1}^{n} \mathbb{I}_{f(X_i)=f(X_0)} (g(X_n) - g^0(X_n)) | X_0 = x] \\ & \to 0 \text{ as } n \to \infty \end{aligned}$$

(by non-explosiveness: $\tau_0 + \cdots + \tau_n \to \infty$ almost surely; dominated convergence applies by boundedness of $g - g^0$ )
Strong Markov property:

$$\mathbb{E}(\mathbb{I}_{Y_1^n = y_1^n} e^{-\sum_{i=0}^{n-1} \alpha_i \hat{\tau}_i} | Y_0 = y_0) = \prod_{i=0}^{n-1} \frac{\hat{q}_{y_i, y_{i+1}}}{\hat{q}(y_i) + \alpha_i}$$

# Generalized birth & death (GBD) processes

**Definition**: Jump processes on $E = \mathbb{N}^k$ with only non-zero transition rates $q_{x,x+e_i} = \beta_i(x)$, $q_{x,x-e_i} = \delta_i(x)$, $x \in \mathbb{N}^k, i \in [k]$.

**Theorem**

*Let $X, X'$ be two non-explosive GBD processes on $\mathbb{N}^k$ with respective birth and death rate functions $(\beta, \delta)$, $(\beta', \delta')$. Assume that $\forall x, x' \in \mathbb{N}^k, x \leq x'$, one has*

$$x_i = x'_i \Rightarrow \beta_i(x) \leq \beta'_i(x'),\ \delta_i(x) \geq \delta'_i(x').$$

*Then for ordered initial conditions $x(0) \leq x'(0)$ there is a coupled construction of processes $X, X'$ such that with probability 1, $\forall t \in \mathbb{R}_+$, $X(t) \leq X'(t)$.*

# Proof elements

Consider Markov jump process on $\{x, x' \in \mathbb{N}^k : x \leq x'\}$ with only non-zero transition rates:

$$\begin{aligned}
x_i < x'_i \quad &\Rightarrow \quad q_{(x,x'),(x+e_i,x')} = \beta_i(x),\ q_{(x,x'),(x-e_i,x')} = \delta_i(x), \\
&\qquad\quad q_{(x,x'),(x+,x'+e_i)} = \beta'_i(x'),\ q_{(x,x'),(x,x'-e_i)} = \delta'_i(x'); \\
x_i = x'_i \quad &\Rightarrow \quad q_{(x,x'),(x+e_i,x'+e_i)} = \beta_i(x), \\
&\qquad\quad q_{(x,x'),(x,x'+e_i)} = \beta'_i(x') - \beta_i(x), \\
&\qquad\quad q_{(x,x'),(x-e_i,x'-e_i)} = \delta'_i(x'), \\
&\qquad\quad q_{(x,x'),(x-e_i,x')} = \delta_i(x) - \delta'_i(x').
\end{aligned}$$

Apply result on Markovian images of Markov jump processes to show that component processes $X, X'$ are GBD with the desired birth and death rate functions.

# Fast extinction and spectral radius: proof

Define branching random walk on graph $G = (V, E)$ as GBD process $X'$ on $\mathbb{N}^V$ with $\beta'_i(x) = \beta \sum_{j \sim i} x_j$ and $\delta'_i(x) = \delta x_i$.

View SIS process as GBD on $\mathbb{N}^V$ with $\beta_i(x) = \beta \mathbb{I}_{x_i=0} \sum_{j \sim i} x_j$, $\delta_i(x) = \delta x_i$.
Use previous Theorem to couple two processes $X, X'$ with initial conditions $x(0) \in \{0,1\}^V$ so that $\forall t \in \mathbb{R}_+,\ X(t) \leq X'(t)$
Bound probability of SIS survival:

$$\mathbb{P}(T > t) \leq \mathbb{E}(\sum_i X_i(t)) \leq \mathbb{E} \sum_i X'_i(t).$$

Linearity of rate functions $\beta'_i, \delta'_i$:

$$\begin{aligned} \frac{d}{dt}\mathbb{E}(X'(t)) &= \beta A \mathbb{E}(X'(t)) - \delta \mathbb{E}(X'(t)) \\ \Rightarrow \mathbb{E}(X'(t)) &= e^{t(\beta A - \delta I)} x(0). \end{aligned}$$

# Long survival and isoperimetric constants: proof

Assumption: $\beta\eta_m \geq \delta/r$.
Consider jump process $Z$ on $\{0,\ldots,m\}$ with non-zero transition rates $q_{z,z+1} = \frac{\delta z}{r}\mathbb{I}_{z<m}$, $q_{z,z-1} = \delta z$.

Couple SIS process $X$ and $Z$ such that with probability 1, $\forall t \in \mathbb{R}_+, Z(t) \leq \sum_{i\in V} X_i(t)$ by choosing as non-zero transition rates:

$$\begin{array}{lll}
\sum_i x_i > z & \Rightarrow & q_{(x,z),(x+e_i,z)} = \beta(1-x_i)\sum_{j\sim i} x_j,\ q_{(x,z),(x-e_i,z)} = \delta x_i, \\
 & & q_{(x,z),(x,z+1)} = \frac{\delta z}{r}\mathbb{I}_{z<m}, q_{(x,z),(x,z-1)} = \delta z, \\
\sum_i x_i = z & \Rightarrow & q_{(x,z),(x+e_i,z+1)} = \frac{1}{M(x)}\beta(1-x_i)\sum_{j\sim i} x_j, \\
 & & q_{(x,z),(x+e_i,z)} = [1-\frac{1}{M(x)}]\beta(1-x_i)\sum_{j\sim i} x_j, \\
 & & q_{(x,z),(x-e_i,z-1)} = x_i\delta,
\end{array}$$

where $M(x) := \frac{\sum_i \beta(1-x_i)\sum_{j\sim i} x_j}{\delta\sum_i x_i/r} = \frac{\beta|E(S,\overline{S})|}{\delta|S|/r} \geq 1$ with $S = \{i : x_i = 1\}$

# Long survival and isoperimetric constants: proof

Lower bound probability of survival: $\mathbb{P}(T \geq t) \geq \mathbb{P}(Z(t) > 0)$.

- Analysis of $Z$: embedded chain with transition probabilities $p_{z,z+1} = \frac{1}{1+r}\mathbb{I}_{z<m}$, $p_{z,z-1} = \frac{r}{1+r}$.
- "Gambler's ruin problem": probability of reaching $m$ before $0$ when at $z$: $\frac{1-r^z}{1-r^m}$.
- Probability of at least $k$ visits to state $m$ before absorption at 0: $\geq \frac{1-r}{1-r^m}(\frac{1-r^{m-1}}{1-r^m})^{k-1}$.
- Sojourn times in state $m$: Exp($\delta m$). Hence

$$\mathbb{P}(T \geq \frac{k}{2\delta m}) \geq \frac{1-r}{1-r^m}(\frac{1-r^{m-1}}{1-r^m})^{k-1}\mathbb{P}(S_1 + \ldots + S_k \geq k/2)$$

with $S_i$: i.i.d., Exp(1).

Announced result follows by writing $f(k) = \mathbb{P}(S_1 + \ldots + S_k < k/2)$: $\lim_{k\to\infty} f(k) = 0$ follows by law of large numbers.

# SIR epidemics and spectral radius

Consider Reed-Frost process with neighbor infection parameter $\beta$ on graph $G = (V, E)$. Then:

**Theorem**

*Suppose $\beta\rho < 1$. Then the total number of nodes eventually removed verifies*

$$\mathbb{E} \sum_{i \in V} Y_i(\infty) \leq \frac{1}{1 - \beta\rho} \sqrt{n \sum_{i \in V} X_i(0)}.$$

*If moreover $G$ is $d$- regular, then*

$$\mathbb{E} \sum_{i \in V} Y_i(\infty) \leq \frac{1}{1 - \beta\rho} \sum_{i \in V} X_i(0).$$

# SIR epidemics: proof

By union bound,

$$\begin{aligned}\mathbb{P}(Y_u(\infty)=1) &\leq \sum_{t\geq 0}\sum_{u_0,\ldots,u_t}\beta^t X_{u_0}(0)\\ &= \sum_{t\geq 0}\sum_{v\in V}(\beta A)^t_{uv}X_v(0)\end{aligned}$$

where $u_0,\ldots,u_t$: graph path with $u_t=u$.
Hence

$$\begin{aligned}\mathbb{E}\sum_u Y_u(\infty) &\leq \sum_{t\geq 0} e^T(\beta A)^t X(0)\\ &= e^T(I-\beta A)^{-1}X(0)\\ &= \sum_i < x_i, e > \frac{1}{1-\beta\lambda_i} < x_i, X(0) >\end{aligned}$$

# SI epidemics and isoperimetric constant

Consider SI process on $G = (V, E)$ with infection rate $\lambda$ along each edge.

Then propagation is at least as fast as SI process on complete graph with infection rate $\lambda \eta_{n/2}$.

**Proof:** Couple SI infection process $X$ with jump process on $\{1, \ldots, n\}$ with only non-zero transition rates $q_{z,z+1} = \lambda \eta_{n/2} \frac{z(n-z)}{n-1}$ so that $\forall t \in \mathbb{R}_+, \sum_{i \in V} X_i(t) \geq Z(t)$.

# Takeaway messages

- Behaviour of SIS epidemics undergoes phase transitions as ratio $\beta/\delta$ crosses thresholds
- Graph topology determines thresholds; in several scenarios (complete graph, hypercube, E-R graphs), spectral radius and isoperimetric constants are close, hence a single threshold
- Coupling constructions allow control of complex process by simpler ones