Susceptible-Infective-Removed Epidemics and Erdős-Rényi random graphs

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SIR epidemics: the Reed-Frost model

- Individuals $i \in [n]$ when infected, attempt to infect all neighbors independently with probability p in subsequent time slot and then die
- Focus on complete graph (everyone neighbor of everyone)
- Associated model: Erdős-Rényi random graph $\mathcal{G}(n, p)$: undirected graph on node set $[n]$. Edge (i, j) present iff $\xi_{ij} = 1$ where $\{\xi_{ij}\}_{i<j}$: i.i.d., Bernoulli (p)

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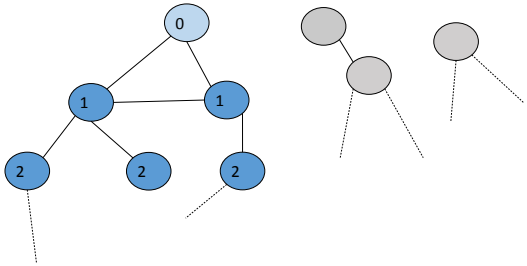

$\Rightarrow$ For initial set X_0 of infective nodes at time 0, i infected at time t iff $d_G(X_0, i) = t$

Set of nodes eventually infected: $\cup_{i \in X_0} \Gamma(i)$ where $\Gamma(i)$: graph's connected component including i

Outline

Seminal results by Erdős and Rényi (1959-1960)

- First phase transition: emergence of giant component
  [Tool: branching processes & Chernoff's inequality]
- Second phase transition: emergence of connectivity
  [Tools: 1st and 2nd moment methods; Poisson approximation]

Emergence of giant component

Analysis of graph's connected components: let $C(i)$: size of i-th largest connected component (in number of nodes) in $\mathcal{G}(n,p)$

Theorem

Let $p = \lambda/n$ for fixed $\lambda > 0$
Sub-critical case *($\lambda < 1$): there exists $f(\lambda)$ such that*

$$\lim_{n\to\infty} \mathbb{P}(C(1) \leq f(\lambda)\log(n)) = 1$$

Super-critical case *($\lambda > 1$): there exists $g(\lambda)$ such that for all $\delta > 0$,*

$$\lim_{n\to\infty} \mathbb{P}(|\frac{C(1)}{n} - (1 - p_{ext})| \leq \delta,\ C(2) \leq g(\lambda)\log(n)) = 1,$$

where p_{ext}: extinction probability of Poisson (λ) branching process, i.e. smallest root of $x = e^{\lambda(x-1)}$ in $[0,1]$

Interpretation

Sub-critical regime: Only logarithmically sized components i.e. no global outbreak

Super-critical regime: with probability $1 - p_{ext}$, epidemics started from randomly selected node reaches $n[1 - p_{ext} + o(1)]$ others, i.e. macroscopic outbreak

Note: only one giant component, others still logarithmic

Sub-critical regime

- Exploration of connected component $\Gamma(i_0)$: initialized with active set $\mathcal{A}_0 = \{i_0\}$ and killed set $\mathcal{B}_0 = \emptyset$
- At time t pick $j_t \in \mathcal{A}_{t-1}$, kill it and activate its neighbours not yet activated (set $\mathcal{D}_t$)
  $\Rightarrow \mathcal{A}_t = \mathcal{A}_{t-1} \setminus \{j_t\} \cup \mathcal{D}_t,\ \mathcal{B}_t = \mathcal{B}_{t-1} \cup \{j_t\}$
- Notation:
  $A_t = |\mathcal{A}_t|,\ D_t = |\mathcal{D}_t| \ \Rightarrow A_t = 1 - t + D_1 + \cdots + D_t$
- Conditionally on $\mathcal{F}_{t-1} = \sigma(A_0, \ldots, A_{t-1})$,
  $D_t \sim \mathrm{Bin}(p, n - 1 - D_0 - \cdots - D_{t-1})$
- Size C of connected component:

$$C = \inf\{t > 0 : A_t = 0\}$$

Sub-critical regime, continued

- Processes $\{A_t\}, \{D_t\}$ can be extended after end of component's exploration

- Upper bound:

$$\mathbb{P}(C > k) = \mathbb{P}(A_1, \ldots, A_k > 0) \leq \mathbb{P}(A_k > 0)$$

- Chernoff's bounding technique: $\mathbb{P}(A_k > 0) \leq e^{-kh(1)}$

  where $h(x) = \lambda h_1(x/\lambda),\ h_1(x) = x\log(x) - x + 1$: Chernoff's exponent for Poisson (λ) random variable

- Union bound allows to conclude

Super-critical regime $\lambda > 1$

Lemma

For any $k > 0, d_1, \ldots, d_k \in \mathbb{N}^k$,
$\lim_{n\to\infty} \mathbb{P}(D_1^k = d_1^k) = \prod_{s=1}^k e^{-\lambda} \frac{\lambda^{d_s}}{d_s!}$,
hence $\lim_{n\to\infty} \mathbb{P}(C \leq k) = \mathbb{P}(Z \leq k)$
where Z*: total population of Poisson* (λ) *branching process*

Lemma

For any $k \geq 1$, $A_k + k - 1 \sim Bin(1 - (1-p)^k, n-1)$,
hence $\mathbb{P}(C = k) \leq \mathbb{P}(Bin(1 - (1-p)^k, n-1) = k-1)$

Corollary

For any fixed $\epsilon, \delta > 0$, *there exists* $A > 0$ *such that*

$$|\mathbb{P}(C \leq A) - p_{ext}| \leq \epsilon, \ \mathbb{P}(|C/n - (1 - p_{ext})| \leq \delta) \geq 1 - p_{ext} - \epsilon$$

Super-critical regime, continued

- Fix $\epsilon, \delta > 0$. Call connected component **small** if $C \leq A$, **gigantic** if $|C/n - (1 - p_{ext})| \leq \delta$, **failed** otherwise.
- Repeatedly extract connected component until either failure or giant component found
- Probability of "finding giant", i.e. success, in at most M steps: at least

$$\sum_{m=0}^{M-1} (p_{ext} - \epsilon)^m [1 - p_{ext} - \epsilon] = \frac{1 - p_{ext} - \epsilon}{1 - p_{ext} + \epsilon} [1 - (p_{ext} - \epsilon)^M]$$

- Hence probability of success $1 - O(\epsilon)$ for $M \geq \Omega(\log(1/\epsilon))$
- Given success, remaining graph: $\mathcal{G}(n', p)$ with $n' \leq n - n(1 - p_{ext} - \delta) = n(p_{ext} + \delta)$
  $\Rightarrow$ a sub-critical Erdős-Rényi graph, as $\lambda p_{ext} < 1$
  (see notes on branching processes)

Connectivity

By previous result: for fixed $\lambda > 1$, giant component of size $\sim n(1 - p_{ext})$
For fixed λ, graph disconnected $\Rightarrow$ Under what regime is graph connected?

Theorem

For fixed $c \in \mathbb{R}$, assume $np = \log(n) + c$.
Then $\lim_{n\to\infty} \mathbb{P}(\mathcal{G}(n,p)$ connected$) = e^{-e^{-c}}$

Corollary

If $np - \log(n) \to +\infty$, then $\lim_{n\to\infty} \mathbb{P}(\mathcal{G}(n,p)$ connected$) = 1$
If $np - \log(n) \to -\infty$, then $\lim_{n\to\infty} \mathbb{P}(\mathcal{G}(n,p)$ connected$) = 0$

Proof strategy

- Show that number of **isolated nodes** (i.e. nodes of degree 0) admits asymptotically Poisson (e^{-c}) distribution [Poisson approximation method],

  hence $\lim_{n\to\infty} \mathbb{P}(\mathcal{A}) = e^{-e^{-c}}$ where
  $\mathcal{A} = \{\text{no isolated vertices in } \mathcal{G}(n,p)\}$

- Show that $\lim_{n\to\infty} \mathbb{P}(\mathcal{B}) = 0$ where
  $\mathcal{B} = \{\exists \text{ connected component of size } k \in \{2,\ldots,n/2\}\}$

- Use bounds

$$\mathbb{P}(\mathcal{A}) - \mathbb{P}(\mathcal{B}) \leq \mathbb{P}(\mathcal{G}(n,p) \text{ connected}) = \mathbb{P}(\mathcal{A} \cap \overline{\mathcal{B}}) \leq \mathbb{P}(\mathcal{A})$$

Basic tools: the first and second moment methods

Let $Z_u,\ u \in V$ be indicators of events and $X = \sum_{u \in V} Z_u$.

First moment method:
$\mathbb{P}(\exists u \in V : Z_u = 1) \leq \sum_{u \in V} \mathbb{E}(Z_u) = \mathbb{E}(X)$, hence "with high probability" none of these events occurs if $\lim_{n \to \infty} \mathbb{E}(X) = 0$.

Application: with high probability no isolated node in $\mathcal{G}(n, p)$ if $\lim_{n \to \infty} [np - \log(n)] = +\infty$.

Second moment method:
$\mathbb{P}(\forall u \in V, Z_u = 0) = \mathbb{P}(X = 0) \leq \frac{\mathrm{Var}(X)}{\mathbb{E}(X)^2}$. Hence if $\mathrm{Var}(X) = o(\mathbb{E}(X)^2)$, then with high probability some event occurs.

Application: with high probability there is some isolated node in $\mathcal{G}(n, p)$ if $\lim_{n \to \infty} [np - \log(n)] = -\infty$.

Variation distance

Definition

Variation distance between two probability measures μ, ν on $(\Omega, \mathcal{F})$: $d_{var}(\mu, \nu) = 2 \sup_{\mathcal{A} \in \mathcal{F}} |\mu(\mathcal{A}) - \nu(\mathcal{A})|$

Alternative characterization: if μ, ν admit densities $\frac{d\mu}{d\pi}, \frac{d\nu}{d\pi}$ with respect to measure π (e.g., $\pi = \mu + \nu$) then

$d_{var}(\mu, \nu) = \int_\Omega \left| \frac{d\mu}{d\pi} - \frac{d\nu}{d\pi} \right| d\pi$

In particular for $\Omega = \mathbb{N}$ and $\pi = \sum_{n \in \mathbb{N}} \delta_n$, $d_{var}(\mu, \nu) = \sum_{n \in \mathbb{N}} |\mu_n - \nu_n|$

Definition

$\{\mu^{(n)}\}_{n \in \mathbb{N}}$ converges in variation to μ iff $\lim_{n \to \infty} d_{var}(\mu^{(n)}, \mu) = 0$

A strong form of convergence (implies convergence in distribution)

Poisson approximation: the Stein-Chen method

Theorem

Let $Z_u \in \{0,1\},\ u \in V$, $X = \sum_{u \in V} Z_u$.
Denote $\pi_u = \mathbb{E}(Z_u), \lambda = \mathbb{E}(X) = \sum_{u \in V} \pi_u$.
Assume $\exists \{Z_{uv}\}_{u,v \in V, v \neq u}$ *such that*

$$\forall u \in V,\ \mathbb{P}(\{Z_{uv}\}_{v \neq u} \in \cdot) = \mathbb{P}(\{Z_v\}_{v \neq u} \in \cdot | Z_u = 1).$$

Then:

$$d_{var}(X, Poisson(\lambda)) \leq 2 \min(1, 1/\lambda) \sum_{u \in V} \pi_u \left[\pi_u + \sum_{v \neq u} \mathbb{E}|Z_{uv} - Z_v| \right]$$

Applications

Proposition (Binomial approximation)

One has for all $n, \lambda \leq n$:
$d_{var}(\mathrm{Bin}(n, \lambda/n), \mathrm{Poisson}(\lambda)) \leq 2\min(1, \lambda)\frac{\lambda}{n}$

Proposition (Isolated nodes)

In $\mathcal{G}(n, p)$ with $np = \log(n) + c$, noting $\lambda = n(1-p)^{n-1} \sim e^{-c}$ and X: number of isolated nodes, then

$$d_{var}(X, \mathrm{Poisson}(\lambda)) \leq 2\lambda[1/n + p/(1-p)] = O(\log(n)/n)$$

Hence, $\lim_{n\to\infty} \mathbb{P}(X = 0) = e^{-e^{-c}}$

Stein-Chen method – proof arguments

Fact: for each $\lambda > 0, A \subset \mathbb{N}$, function $f : \mathbb{N} \to \mathbb{R}$ defined by

$$f(0) = 0,\ \lambda f(j+1) - j \cdot f(j) = \mathbb{I}_A(j) - \mathrm{Poi}_\lambda(A),\ j \in \mathbb{N}$$

is $\min(1, \lambda^{-1})$–Lipschitz

Write

$$\begin{aligned}
&|\mathbb{P}(X \in A) - \mathrm{Poi}_\lambda(A)| = |\mathbb{E}[\lambda f(X+1) - Xf(X)]| \\
&= \left| \sum_{u \in V} \pi_u \mathbb{E}\left[f(X+1) - f(1 + \sum_{v \neq u} Z_{uv}) \right] \right| \\
&\leq \sum_{u \in V} \pi_u \min(1, \lambda^{-1}) \mathbb{E} \left| \sum_{v \in V} Z_v - \sum_{v \neq u} Z_{uv} \right| \\
&\leq \sum_{u \in V} \pi_u \left[\pi_u + \sum_{v \neq u} \mathbb{E}|Z_v - Z_{uv}| \right]
\end{aligned}$$

Connectivity – final arguments

Let $\mathcal{A}_k = \{\exists \text{ connected component of size } k\}$.
By union bound, for $p = \Theta(\log(n)/n)$,
$\mathbb{P}(\mathcal{A}_2) \leq \binom{n}{2} p(1-p)^{2(n-2)} \leq O(p) = o(1)$

Similarly for $k \leq n/2$, $\mathbb{P}(\mathcal{A}_k) \leq \binom{n}{k} T_k p^{k-1}(1-p)^{k(n-k)}$
where T_k: number of trees on $[k]$

Cayley's theorem: $T_k = k^{k-2}$.
Hence

$$\begin{aligned}
\mathbb{P}(\mathcal{A}_k) &\leq \binom{n}{k} k^{k-2} p^{k-1}(1-p)^{k(n-k)} \\
&\leq \frac{n^k}{k!} k^{k-2} p^{k-1} e^{-pkn/2} \\
&\leq \frac{1}{p} \frac{1}{k^2\sqrt{k}} e^{k(1+\log(np)-np/2)}
\end{aligned}$$

Conclusion $\mathbb{P}(\cup_{2\leq k\leq n/2}\mathcal{A}_k) \leq \sum_{2\leq k\leq n/2} \mathbb{P}(\mathcal{A}_k) \to 0$ as $n \to \infty$

follows.

Takeaway messages

- connectivity of Erdős-Rényi graphs informs behaviour of SIR epidemics on complete graph
- Emergence of giant component of size $n(1 - p_{ext})$ as average degree crosses critical value 1
- Full connectivity for average degree $\log(n) + O(1)$
- Proof techniques: branching process approximation, Chernoff bounds; First and second moment methods; Poisson approximation via Stein-Chen method