Réseaux: Fiche d'exercices 7

18 novembre 2015

Exercice 1: Spectral radius of Erdős Rényi graphs

Let A denote the adjacency matrix of an Erdős Rényi graph $G(n,p)$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of A. Let d_{max} and d_{avg} denote the maximum and average degree of $G(n,p)$. We assume that $np >> \ln n$. In particular, the graph is connected with high probability (w.h.p.) and by Perron-Frobenius theorem, we know that $\lambda_1 > \max_{i\geq 2}\{|\lambda_i|\}$ and that there exists an eigenvector $v = (v_1, \ldots, v_n)$ associated to λ_1 with $v_i > 0$ for all i.

(a) Show that $\lambda_1 \leq d_{max}$ and $\lambda_1 \geq d_{avg}$.

(b) Show that $d_{avg} \geq np(1 + o(1)$ w.h.p.

(c) Let D_i denote the sum of the entries on the i-th row. Show that $\mathbb{P}(D_i \geq (1+\delta)\mathbb{E}[D_i]) \leq 2e^{-np\delta^2}$ and then that for $np >> \ln n$, $d_{max} = np(1 + o(1))$ w.h.p.

Exercice 2: SIR on general graphs and on Erdős Rényi graphs

The SIR process is described by the following discrete-time model: each node that is infected at the beginning of a time slot attempts to infect each of its neighbours; each infection attempt is successful with probability β independent of other infection attempts. Each infected node is removed at the end of the time slot.

For a graph G, we denote by A its adjacency matrix. Let λ_i be the real eignevalues of A with associated eigenvectors $\mathbf{x}_i$. We denote by $\rho = \max\{|\lambda_i|\}$ the spectral radius of A.

(a) Show that if $\beta \notin \{\lambda_1^{-1}, \ldots, \lambda_n^{-1}\}$ then we have $(I - \beta A)^{-1} = \sum_{i=1}^{n} \frac{1}{1-\beta\lambda_i}\mathbf{x}_i\mathbf{x}_i^T$.

(b) Show that if $\beta\rho < 1$, $\mathbf{y}^T(I - \beta A)^{-1}\mathbf{x} \leq \frac{\|\mathbf{y}\|\|\mathbf{x}\|}{1-\beta\rho}$

Consider a SIR process started from k initial infectives on a finite graph G and denote by Y the (random) total number of nodes that are eventually removed.

(c) Show that the probability that node v ever gets infected is bounded from above by:

$$\sum_{t=0}^{\infty}\sum_{u\in V}(\beta A)^t_{uv}1(u \text{ initially infected}).$$

(d) Show that if $\beta\rho < 1$, $\mathbb{E}[Y] \leq \frac{1}{1-\beta\rho}\sqrt{nk}$.

We can get a better bound for Erdős Rényi graphs. Consider an Erdős Rényi graph $G(n,p)$ with $np >> \ln n$ and define $c_n = \beta np$. Consider the SIR epidemic on such a graph with one node initially infected.

(e) Show that if $\limsup c_n < 1$, then $\mathbb{E}[Y]$ is bounded by a constant that does not depend on n for all n sufficiently large.

(f) Show that if $\liminf c_n > 1$, then $\mathbb{E}[Y] \geq \gamma^2 n + o(n)$ where $\gamma > 0$ solves $\gamma + e^{-\gamma c} = 1$.

Exercice 3: SIS on Erdős Rényi graphs

Recall that $\eta_m(G)$ denote the isoperimetric constant of G defined by:

$$\eta_m(G) = \inf_{|S|\leq m} \frac{|E(S,\overline{S})|}{|S|}.$$

We will prove that for $\alpha \in (0,1)$, $\eta_{\alpha n}(G(n,p)) = (1+o(1))(1-\alpha)np$ w.h.p. when $G(n,p)$ is an Erdős Rényi graph with $np >> \ln n$. We denote $m = \alpha n$, $d = np$ and always assume that $np >> \ln n$.

(a) Let $k > 1-\alpha$. Prove that

$$\mathbb{P}(\eta_m(G(n,p)) > kd) \leq \mathbb{P}(Bin(m(n-m),p) > km(n-1)p).$$

(b) Use Chernoff bound to show that $\eta_m(G(n,p)) \leq kd$ w.h.p.

(c) Prove that for $0 < k < 1-\alpha$,

$$\mathbb{P}(\eta_m(G(n,p)) < kd) \leq \sum_{i=1}^{m} \binom{n}{i} \mathbb{P}(Bin(i(n-i),p) < kdi).$$

(d) Prove that for any Binomial random variable B, we have

$$\mathbb{P}(B < (1-\delta)\mathbb{E}[B]) < e^{-\mathbb{E}[B]\delta^2/2}.$$

(e) Take $\delta = 1 - k\frac{n-1}{m-1}$ and show that

$$\mathbb{P}\eta_m(G(n,p)) < kd) \leq \sum_{i=1}^{m} \binom{n}{i} e^{-i(n-i)p\epsilon},$$

where $\epsilon = \frac{1}{2}\left(1 - \frac{k}{1-\alpha}\right)^2$.

(f) Conclude for $\eta_m(G(n,p))$.

(g) When does a SIS epidemic dies out quickly or slowly on an Erdős Rényi graph $G(n,p)$ with $np >> \ln n$?