# Réseaux: Fiche d'exercices 5

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**Exercice 1: Push-Pull Gossip**

We consider a variant of the propagation process SI: $n$ individuals are connected by a complete graph and the propagation starts with one infected node. Each node has his own Poisson clock with intensity $\lambda$ independent of the others. When the Poisson clock of an individual rings, then this individual contacts a neighbor uniformly at random. If any one of the two selected individuals is infected, then the two individuals become infected.

(a) Show that this process is twice as fast as the standard SI process.

(b) Compare with the process where each infected node contacts (when its clock rings) a non-infected neighbor uniformly at random.

**Exercice 2: Branching process**

We denote by $Z_n$ the number of individuals in the $n$-th generation of a branching process where, by convention, we let $Z_0 = 1$. Then $Z_n$ satisfies the recursion relation:

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_{n,i},$$

where $(X_{n,i})_{n,i\geq 1}$ is a doubly infinite array of i.i.d. random variables with distribution $X$. We define the extinction probability:

$$\eta = \mathbb{P}(\exists n,\ Z_n = 0).$$

(a) Give the value of $\eta$ when $\mathbb{P}(X \leq 1) = 1$.

(b) Let $\eta_n = \mathbb{P}(Z_n = 0)$. Show that $\eta_n \to \eta$.

(c) Show that $\mathbb{E}[Z_n] = \mathbb{E}[X]^n$. Give the value of $\eta$ when $\mathbb{E}[X] < 1$.

(d) Show that the probability generating function of $X$, $G_X(s) = \mathbb{E}[s^X]$ is strictly increasing and strictly convex for $s > 0$ when $\mathbb{P}(X \leq 1) < 1$.

(e) Let $G_n(s) = \mathbb{E}[s^{Z_n}]$. Show that $G_n(s) = G_X(G_{n-1}(s))$ and that $\eta$ is the smallest solution to $s = G_X(s)$.

(f) Show that $\eta < 1$ when $\mathbb{E}[X] > 1$. What is the value of $\eta$ when $\mathbb{E}[X] = 1$?

The total progeny $T$ of the branching process is $T = \sum_{n=0}^{\infty} Z_n$. We denote by $G_T(s)$ the probability generating function of $T$, i.e. $G_T(s) = \mathbb{E}[s^T]$.

(g) Show that for all $s \in [0,1)$,

$$G_T(s) = sG_X(G_T(s)) \text{ and, } G_T(1) = \eta.$$

Show that $\mathbb{E}[T] = (1 - \mathbb{E}[X])^{-1}$ when $\mathbb{E}[X] < 1$.

**Exercice 3: Subcritical inhomogeneous random graphs**

Let $K$ be a fixed integer and $a_1, \dots, a_K$ be positive real numbers with $\sum_{k=1}^K a_k = 1$. Let $B = (b_{k\ell})_{1\le k,\ell\le K}$ be a symmetric matrix with non-negative entries. We consider the random graph $G$ constructed on $n$ vertices as follows: nodes are divided into $K$ classes, where class $k$ has $a_k n$ nodes. For any two nodes $u$ and $v$ in classes $k$ and $\ell$, the edge $(uv)$ is present with probability $b_{k\ell}/n$ independently of everything else. Let $C_n$ be the size of the largest connected component of $G_n$.

(a) Let $M = (m_{k\ell})$ be the matrix defined by $m_{k\ell} = a_k b_{k\ell}$. Show that the eigenvalues of $M$ are real.

Let $\rho$ be the spectral radius of $M$. We will show that if $\rho < 1$ then there exists $A > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}(C_n \geq A \log n) = 0.$$

We consider an enumeration process of a connected component of $G$ which at each step deactivate simultaneously all active nodes and activate all the neighbors of these nodes which have not been activated so far. We denote by $Z_k(t)$ the number of active nodes of class $k$ at time $t$.

(b) Show that conditionally on $(Z_k(t-1))_{k=1,\dots,K}$, we have for $\ell \in [K]$

$$Z_\ell(t) \leq \sum_{k=1}^{K} \sum_{s=1}^{Z_k(t-1)} X_{k,\ell}(s,t),$$

where the random variables $X_{k,\ell}(s,t)$ are independent from the exploration until time $t-1$ and mutually independent with $X_{k,\ell}(s,t)$ following a Binomial distribution with parameters $a_\ell n$ and $b_{k\ell}/n$.

(c) Show that for $\theta_\ell > 0$, we have

$$\mathbb{E}\left[e^{\sum_{\ell=1}^K \theta_\ell Z_\ell(t)} | Z(t-1)\right] \leq \exp\left(\sum_{k,\ell=1}^{K} (e^{\theta_\ell} - 1) a_\ell b_{k\ell} Z_k(t-1)\right)$$

(d) Show that if $\rho < 1$, there exists $\theta > 0$ and $B < \infty$ such that

$$\mathbb{E}\left[e^{\theta \sum_{\ell=1}^K \sum_{t\geq 0} Z_\ell(t)}\right] \leq B < \infty.$$

(e) Conclude.