1. $M/G/\infty/\infty$ queue and Poisson process

We consider a Poisson process $N \leftrightarrow \{T_n\}_{n>0}$ on $\mathbb{R}_+$ with intensity $\lambda > 0$ and to each point $T_n$ attach a service time $\sigma_n$, where $\{\sigma_n\}_{n>0}$ is i.i.d., independent of $N$, with density $f$ on $\mathbb{R}_+$.

1.1 Show that the set of points $\{(T_n, \sigma_n)\}_{n>0}$ constitutes a (generalized) Poisson process. Determine its intensity function.

1.2 The number $X_t$ of customers present at time $t > 0$ is given by $X_t = \sum_{n>0} 1_{T_n \leq t < T_n + \sigma_n}$. Show that $X_t$ admits a Poisson distribution. Determine its parameter.

1.3 We now assume that $N$ extends to all of $\mathbb{R}$ so that, letting $X_t = \sum_{n \in \mathbb{Z}} 1_{T_n \leq t < T_n + \sigma_n}$, we obtain a stationary process. Determine the stationary covariance $C(t) := \text{Cov}(X_t, X_{t+s})$.

1.4 Is the process $\{X_t\}$ Markovian for general density $f$ of service times $\sigma_n$?

2. $M/M/1/\infty$ queues with Processor Sharing discipline

We consider a single server queue with customer arrivals at instants of Poisson process $N$ on $\mathbb{R}_+$ with intensity $\lambda > 0$, i.i.d. service times $\sigma_n$ independent of $N$ with Exponential$(\mu)$ distribution. Service discipline is Processor Sharing, i.e. when there are $k > 0$ customers present, each receives service at speed $1/k$.

2.1 Show that the number of customers in the queue is Markovian. Determine its transition rates and a stationary measure. Is the process reversible? Under what condition is it ergodic?

2.2 Assume now there are $K$ distinct customer types, customers of type $i \in [K]$ arriving at instants of Poisson process $N_i$ with intensity $\lambda_i > 0$, the $N_i$ being mutually independent. Assume that service times of all customers of all types are i.i.d. with Exponential$(\mu)$ distribution (and independent of the $N_i$). Let $X_i(t)$ be the number of type $i$-customers present at time $t$. Answer same questions as in 2.1.
2.3 Assume now a network of \( L \) stations indexed by \( \ell \in [L] \), \( K \) distinct customer types, \( k \in [K] \). Assume a fixed network, with \( n_k \) customers of type \( k \), each following a fixed cyclic route \( \ell(1,k), \ell(2,k), \ldots, \ell(d_k, k), \ell(1,k), \ldots \), each \( \ell \) appearing at most once per cycle. Finally assume that service at station \( \ell \) is Processor Sharing, with service times there with Exponential(\( \mu_k \)) distribution.

Noting \( X_{k\ell} \) the number of customers of type \( k \) at station \( \ell \), prove that a stationary measure for \( \{X_{k\ell}\}_{k \in [K], \ell \in K} \) is given by, noting \( y_{\ell} := \sum_{k \geq \ell} x_{k\ell} \),

\[
\pi(x) = \left( \prod_{k \in [K]} 1^{\sum_{\ell \in [L]} x_{k\ell} = n_k} \right) \prod_{\ell \in [L]} \left( \frac{y_{\ell}\mu^{-\omega}_{\ell}}{\prod_{k \geq \ell}(x_{k\ell})!} \right)
\]

**Hint:** Determine rates \( q_{xx'} \) of generator, and associated rates \( \tilde{q}_{xx'} \) such that

\[
\pi(x)q_{xx'} = \pi(x')\tilde{q}_{x'x}, \quad x \neq x',
\]

then verify that \( \sum_{x \neq x'} \tilde{q}_{x'x} = \sum_{x \neq x'} q_{xx'} \) to conclude.

2.4 We now set \( \mu_{\ell} = AC_{\ell}, n_k = Aw_{k}, \) for fixed \( w_k, C_{\ell} \), and let \( A \to \infty \). We also set \( x_{k\ell} = Aw_{k\ell} \) and \( y_{\ell} = Aw \) with \( u_{\ell} = \sum_{k \in [K]} v_{k\ell} \). Show with a crude version of Stirling’s formula that for \( A \to \infty \), the stationary distribution \( \pi \) concentrates its mass on solutions of the optimization problem

\[
\begin{align*}
\text{Max} & \quad \sum_{k \in [K]} \sum_{\ell \in [L]} v_{k\ell} \log\left( \frac{u_{\ell}}{C_{\ell}v_{k\ell}} \right) \\
\text{Over} & \quad v_{k\ell} \geq 0, \; k \in [K], \ell \in k, \\
& \quad \sum_{\ell \in k} v_{k\ell} = w_k, \; k \in [K].
\end{align*}
\]

2.5 The above scenario admits the following interpretation: service types correspond to individual transmissions; each transmission is regulated by a fixed window control with \( Aw_k \) the window size in number of packets, and \( C_{\ell} \) the capacity (in bytes/s) of server \( \ell \). The limit \( A \to \infty \) corresponds to a “small data packets / high transmission rates” regime.

We will admit that (2) is a concave maximization problem, whose optimum \( \{v_{k\ell}^*\} \) is characterized as achieving the maximum of

\[
L(\{v_{k\ell}\}, \{\beta_k\}) := \sum_{k \in [K]} \sum_{\ell \in [K]} v_{k\ell} \log\left( \frac{u_{\ell}}{C_{\ell}v_{k\ell}} \right) + \sum_{k \in [K]} \beta_k (w_k - \sum_{\ell \in k} v_{k\ell})
\]

over \( \{v_{k\ell}\} \geq 0 \) for some suitable vector of multipliers \( \{\beta_k\} \in \mathbb{R}^K \).

Argue from the corresponding solution, taking \( C_{\ell}v_{k\ell}/u_{\ell} \) as the rate of transmission \( k \) for any \( \ell \in k \) with \( u_{\ell} > 0 \), that the resulting rates correspond to \((w,1)\)-fairness, or weighted proportional fairness.

3. Jackson networks and Kleinrock’s square root law

Consider a Jackson network with stations \( i \in I \), routing probabilities \( p_{ij} \), single server queues at each station, and service time distributions Exponential(1). Let \( \lambda_i > 0 \) be the solutions of the traffic equations. Assume that a total capacity \( C \) is available, and to be distributed among the servers.

3.1 Write the stationary measure for a particular allocation \( C_i \) of capacity to each server \( i \), \( C_i > 0, \sum_{i \in I} C_i = C \).
3.2 Determine under which condition an allocation $C_i$ makes the system ergodic.

3.3 Assume the system can be made ergodic. Determine the allocation which minimizes the average number of customers in the system.