1. \(M/G/\infty/\infty\) queue and Poisson process

We consider a Poisson process \(N \leftrightarrow \{T_n\}_{n>0}\) on \(\mathbb{R}_+\) with intensity \(\lambda > 0\) and to each point \(T_n\) attach a service time \(\sigma_n\), where \(\{\sigma_n\}_{n>0}\) is i.i.d., independent of \(N\), with density \(f\) on \(\mathbb{R}_+\).

1.1 Show that the set of points \(\{(T_n, \sigma_n)\}_{n>0}\) constitutes a (generalized) Poisson process. Determine its intensity function.

1.2 The number \(X_t\) of customers present at time \(t > 0\) is given by \(X_t = \sum_{n>0} \mathbb{1}_{T_n \leq t < T_n + \sigma_n}\). Show that \(X_t\) admits a Poisson distribution. Determine its parameter.

1.3 We now assume that \(N\) extends to all of \(\mathbb{R}\) so that, letting \(X_t = \sum_{n \in \mathbb{Z}} \mathbb{1}_{T_n \leq t < T_n + \sigma_n}\), we obtain a stationary process. Determine the stationary covariance \(C(s) := \text{Cov}(X_t, X_{t+s})\).

1.4 Is the process \(\{X_t\}\) Markovian for general density \(f\) of service times \(\sigma_n\) ?

2. \(M/M/1/\infty\) queues with Processor Sharing discipline

We consider a single server queue with customer arrivals at instants of Poisson process \(N\) on \(\mathbb{R}_+\) with intensity \(\lambda > 0\), i.i.d. service times \(\sigma_n\) independent of \(N\) with Exponential(\(\mu\)) distribution. Service discipline is Processor Sharing, i.e. when there are \(k > 0\) customers present, each receives service at speed \(1/k\).

2.1 Show that the number of customers in the queue is Markovian. Determine its transition rates and a stationary measure. Is the process reversible? Under what condition is it ergodic?

2.2 Assume now there are \(K\) distinct customer types, customers of type \(i \in [K]\) arriving at instants of Poisson process \(N_i\) with intensity \(\lambda_i > 0\), the \(N_i\) being mutually independent. Assume that service times of all customers of all types are i.i.d. with Exponential(\(\mu\)) distribution (and independent of the \(N_i\)). Let \(X_i(t)\) be the number of type \(i\)-customers present at time \(t\). Answer same questions as in 2.1.
2.3 Assume now a network of $L$ stations indexed by $\ell \in [L]$, $K$ distinct customer types, $k \in [K]$. Assume a fixed network, with $n_k$ customers of type $k$, each following a fixed cyclic route $\ell(1,k), \ell(2,k), \ldots, \ell(d_k,k), \ell(1,k), \ldots$, each $\ell$ appearing at most once per cycle. Finally assume that service at station $\ell$ is Processor Sharing, with service times there with Exponential($\mu_\ell$) distribution.

Noting $X_{k\ell}$ the number of customers of type $k$ at station $\ell$, prove that a stationary measure for $\{X_{k\ell}\}_{k \in [K], \ell \in [L]}$ is given by, noting $y_\ell := \sum_{k \in [K]} \lambda_{k\ell} x_{k\ell}$.

$$\pi(x) = \left( \frac{X}{K} \right) = \left( \sum_{k \in [K]} x_{k\ell} = n_k \right) \left( \sum_{\ell \in [L]} \sum_{k \in [K]} \mu_\ell \right)^{-1}$$

Hint: Determine rates $q_{xx'}$ of generator, and associated rates $\tilde{q}_{xx'}$ such that

$$\pi(x) q_{xx'} = \pi(x') \tilde{q}_{xx'}, \ x \neq x' \quad (1)$$

then verify that $p_{x \neq x'} \tilde{q}_{xx'} = p_{x \neq x'} q_{xx'}$ to conclude.

2.4 We now set $\mu_\ell = AC \rho_{k\ell} n_k = Aw_k C_\ell$, for fixed $w_k$, $C_\ell$, and let $A \to \infty$. We also set $x_{k\ell} = A v_{k\ell}$ and $y_\ell = A w_k$ with $u_\ell = \sum_{\ell \in [L]} v_{k\ell}$. Show with a crude version of Stirling’s formula that for $A \to \infty$, the stationary distribution $\pi$ concentrates its mass on solutions of the optimization problem

$$\text{Max} \sum_{k \in [K]} \sum_{\ell \in [L]} v_{k\ell} \log \left( \frac{w_k}{c_{\ell \max}} \right)$$

$$\text{Over} \sum_{\ell \in [L]} v_{k\ell} = w_k, k \in [K].$$

2.5 The above scenario admits the following interpretation: service types correspond to individual transmissions; each transmission is regulated by a fixed window control with $Aw_k$ the window size in number of packets, and $C_\ell$ the capacity (in bytes/s) of server $\ell$. The limit $A \to \infty$ corresponds to a “small data packets / high transmission rates” regime.

We will admit that (2) is a concave maximization problem, whose optimum $\{v^*_{k\ell}\}$ is characterized as achieving the maximum of

$$L(\{v_{k\ell}\}, \{\beta_k\}) := \sum_{k \in [K]} \sum_{\ell \in [L]} v_{k\ell} \log \left( \frac{u_\ell}{C_\ell v_{k\ell}} \right) + \sum_{k \in [K]} \beta_k (w_k - v_{k\ell})$$

over $\{v_{k\ell}\} \geq 0$ for some suitable vector of multipliers $\{\beta_k\} \in \mathbb{R}^K$.

Argue from the corresponding solution, taking $C_\ell v_{k\ell}/u_\ell$ as the rate of transmission $k$ for any $\ell \in k$ with $u_\ell > 0$, that the resulting rates correspond to $\langle w, 1 \rangle$-fairness, or weighted proportional fairness.

3. Jackson networks and Kleinrock’s square root law

Consider a Jackson network with stations $i \in I$, routing probabilities $p_{ij}$, single server queues at each station, and service time distributions Exponential(1). Let $\lambda_i > 0$ be the solutions of the traffic equations. Assume that a total capacity $C$ is available, and to be distributed among the servers.

3.1 Write the stationary measure for a particular allocation $C_i$ of capacity to each server $i$, $C_i > 0$, $i \in I, \sum_{i \in I} C_i = C$. 

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3.2 Determine under which condition an allocation $C_i$ makes the system ergodic.

3.3 Assume the system can be made ergodic. Determine the allocation which minimizes the average number of customers in the system.