1. \( M/G/\infty/\infty \) queue and Poisson process

We consider a Poisson process \( N \leftrightarrow \{T_n\}_{n>0} \) on \( \mathbb{R}_+ \) with intensity \( \lambda > 0 \) and to each point \( T_n \) attach a service time \( \sigma_n \), where \( \{\sigma_n\}_{n>0} \) is i.i.d., independent of \( N \), with density \( f \) on \( \mathbb{R}_+ \).

1.1 Show that the set of points \( \{(T_n, \sigma_n)\}_{n>0} \) constitutes a (generalized) Poisson process. Determine its intensity function.

1.2 The number \( X_t \) of customers present at time \( t > 0 \) is given by
\[ X_t = \sum_{n>0} 1_{T_n \leq t < T_n + \sigma_n}. \]
Show that \( X_t \) admits a Poisson distribution. Determine its parameter.

1.3 We now assume that \( N \) extends to all of \( \mathbb{R} \) so that, letting
\[ X_t = \sum_{n \in \mathbb{Z}} 1_{T_n \leq t < T_n + \sigma_n}, \]
we obtain a stationary process. Determine the stationary covariance \( C(s) := \text{Cov}(X_t, X_{t+s}) \).

1.4 Is the process \( \{X_t\} \) Markovian for general density \( f \) of service times \( \sigma_n \)?

2. \( M/M/1/\infty \) queues with Processor Sharing discipline

We consider a single server queue with customer arrivals at instants of Poisson process \( N \) on \( \mathbb{R}_+ \) with intensity \( \lambda > 0 \). i.i.d. service times \( \sigma_n \) independent of \( N \) with Exponential(\( \mu \)) distribution. Service discipline is Processor Sharing, i.e. when there are \( k > 0 \) customers present, each receives service at speed \( 1/k \).

2.1 Show that the number of customers in the queue is Markovian. Determine its transition rates and a stationary measure. Is the process reversible? Under what condition is it ergodic?

2.2 Assume now there are \( K \) distinct customer types, customers of type \( i \in [K] \) arriving at instants of Poisson process \( N_i \) with intensity \( \lambda_i > 0 \), the \( N_i \) being mutually independent. Assume that service times of all customers of all types are i.i.d. with Exponential(\( \mu \)) distribution (and independent of the \( N_i \)). Let \( X_i(t) \) be the number of type \( i \)-customers present at time \( t \). Answer same questions as in 2.1.
2.3 Assume now a network of $L$ stations indexed by $\ell \in [L]$, $K$ distinct customer types, $k \in [K]$. Assume a fixed network, with $n_k$ customers of type $k$, each following a fixed cyclic route $\ell(1,k), \ell(2,k), \ldots, \ell(d_k,k)$, each $\ell$ appearing at most once per cycle. Finally assume that service at station $\ell$ is Processor Sharing, with service times there with Exponential($\mu_\ell$) distribution.

Noting $X_{k\ell}$ the number of customers of type $k$ at station $\ell$, prove that a stationary measure for $\{X_{k\ell}\}_{k \in [K], \ell \in k}$ is given by, noting $y_\ell := \sum_{k \in \ell} x_{k\ell}$,

$$\pi(x) = \left( \prod_{k \in [K]} \mathbf{1}_{\sum_{\ell \in [L]} x_{k\ell} = n_k} \right) \prod_{\ell \in [L]} \left( \frac{y_\ell y_\ell^{x_{k\ell}}}{\prod_{k \in \ell} x_{k\ell}!} \right)$$

Hint: Determine rates $q_{xx'}$ of generator, and associated rates $\tilde{q}_{xx'}$ such that

$$\pi(x) q_{xx'} = \pi(x') \tilde{q}_{xx'}, \quad x \neq x', \quad (1)$$

then verify that $\sum_{x \neq x'} \tilde{q}_{xx'} = \sum_{x \neq x'} q_{xx'}$ to conclude.

2.4 We now set $\mu_\ell = AC_\ell$, $n_k = Aw_\ell$, for fixed $w_\ell$, $C_\ell$, and let $A \to \infty$. We also set $x_{k\ell} = Av_\ell$ and $y_\ell = Aw$ with $u_\ell = \sum_{k \in \ell} v_\ell$. Show with a crude version of Stirling’s formula that for $A \to \infty$, the stationary distribution $\pi$ concentrates its mass on solutions of the optimization problem

$$\text{Maximize} \quad \sum_{k \in [K]} \sum_{\ell \in k} v_\ell \log\left( \frac{w_\ell}{C_\ell v_\ell} \right)$$

$$\text{over} \quad v_\ell \geq 0, \quad k \in [K], \quad \ell \in k,$$

$$\text{such that} \quad \sum_{\ell \in k} v_\ell = w_\ell, \quad k \in [K]. \quad (2)$$

2.5 The above scenario admits the following interpretation: service types correspond to individual transmissions; each transmission is regulated by a fixed window control with $Aw_\ell$ the window size in number of packets, and $C_\ell$ the capacity (in bytes/s) of server $\ell$. The limit $A \to \infty$ corresponds to a “small data packets / high transmission rates” regime.

We will admit that (2) is a concave maximization problem, whose optimum $\{v_\ell^*\}$ is characterized as achieving the maximum of

$$L(\{v_\ell\}, \{\beta_k\}) := \sum_{k \in [K]} \sum_{\ell \in k} v_\ell \log\left( \frac{u_\ell}{C_\ell v_\ell} \right) + \sum_{k \in [K]} \beta_k (w_k - \sum_{\ell \in k} v_\ell)$$

over $\{v_\ell\} \geq 0$ for some suitable vector of multipliers $\{\beta_k\} \in \mathbb{R}^K$.

Argue from the corresponding solution, taking $C_\ell v_\ell / u_\ell$ as the rate of transmission $\ell$ for any $\ell \in k$ with $u_\ell > 0$, that the resulting rates correspond to $(w,1)$-fairness, or weighted proportional fairness.

3. Jackson networks and Kleinrock’s square root law

Consider a Jackson network with stations $i \in I$, routing probabilities $p_{ij}$, single server queues at each station, and service time distributions Exponential($1$). Let $\lambda_i > 0$ be the solutions of the traffic equations. Assume that a total capacity $C$ is available, and to be distributed among the servers.

3.1 Write the stationary measure for a particular allocation $C_i$ of capacity to each server $i$, $C_i > 0$, $\sum_{i \in I} C_i = C$. 

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3.2 Determine under which condition an allocation $C_i$ makes the system ergodic.

3.3 Assume the system can be made ergodic. Determine the allocation which minimizes the average number of customers in the system.