Réseaux: Fiche d'exercices 2

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Exercice 1: Coupling.

Let X_n be an irreducible aperiodic Markov chain with invariant probability π on state space E. We consider two independent realizations of the Markov chain X_n and Y_n with $X_0 = x$ and Y_0 following the distribution π. We define $W_n = (X_n, Y_n) \in E^2$.

(a) Show that W_n is an irreducible aperiodic positive recurrent Markov chain on $E \times E$.

(b) Show that $T = \inf\{n \geq 0,\ X_n = Y_n\}$ is a stopping time. We define

$$Z_n = \begin{cases} X_n, & n < T \\ Y_n, & n \geq T \end{cases}$$

Show that Z_n is a Markov chain with the same distribution as X_n.

(c) Show that $\lim_{n\to\infty} \mathbb{P}_x(X_n = y) = \pi(y)$.

Exercice 2: Wald identity.

Consider a random walk $S_n = \sum_{i=1}^n X_i$ where X_i is a sequence of i.i.d. random variables with $\mathbb{E}[|X_1|] < \infty$. We will show that for any stopping time T with $\mathbb{E}[T] < \infty$, we have

$$\mathbb{E}[S_T] = \mathbb{E}[T]\mathbb{E}[X_1]$$

(a) Give a counter-example when T is not a stopping time with $\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = 1/2$.

(b) Show that $S_T = \sum_{n=1}^\infty X_n 1(T \geq n)$.

(c) Conclude when T is bounded by a constant M.

(d) Conclude in general.

Exercice 3: Reflected random walk.

We consider the reflected random walk defined by $S_0 \in \mathbb{N}$ and for $n \geq 0$, $S_{n+1} = \max(S_n + \Delta_{n+1}, 0)$ where the sequence Δ_n is a sequence of i.i.d. random variables in $\mathbb{Z}$ such that $\mathbb{E}[\Delta_1] < 0$ and $\mathbb{P}(\Delta_1 = 1) > 0$.

(a) Show that S_n is an aperiodic irreducible Markov chain on $\mathbb{R}^+$.

(b) Assume first that $\Delta_n \geq -1$ for all n. Let $T = \inf\{n > 0, S_n = 0\}$. Prove that $\mathbb{E}[T] < \infty$.

(c) In general, we define

$$M = \max\left(0, \max_{k\geq 0} \sum_{i=-k}^{0} \Delta_i\right).$$

Show that $\mathbb{P}(M < \infty) = 1$ and that if $S_0 = M$, then the law of S_n is the same as the law of M. Conclude about the recurrence of S_n.

Exercice 4: Instability of Aloha

Let A_n be the number of messages arriving at the end of time slot n. We are given a sequence of i.i.d Bernoulli random variables $\{B_{n,i}\}_{n,i\in\mathbb{N}}$ with mean p independent of the A_n. If at the beginning of time slot n, there are k messages waiting to be transmitted, each of them try to use the channel with probability p and the transmission is successful if $\sum_{i=1}^{k} B_{n,i} = 1$. Hence the evolution of the number L_n of messages to be transmitted at the beginning of the n-th time slot is given by:

$$L_{n+1} = L_n + A_n - 1\left(\sum_{i=1}^{L_n} B_{n,i} = 1\right)$$

(a) Show that if $0 < \mathbb{P}(A_n = 0) < 1$ then L_n is an aperiodic irreducible Markov chain.

(b) From now on, we assume that A_1 is a Poisson random variable with mean $\lambda > 0$. Compute $\mathbb{E}[z^{A_1}]$ and $\mathbb{E}[A_1 z^{A_1-1}]$.

(c) We define $S_t = \sum_{i=1}^{t} A_i$. Compute $\mathbb{P}\left(\sum_{i=1}^{x+S_t} B_{t,i} = 1\right)$.

(d) Show that $\mathbb{P}\left(\cap_{t=0}^{\infty}\left\{\sum_{i=1}^{x+S_t} B_{t,i} \neq 1\right\}\right) > 0$ for x sufficiently large.

(e) Show that L_n is transient.

We will now show a stronger result: with probability one, there are only a finite number of successful transmissions. Before that, prove the following general result: given a sequence of events E_n with $\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \infty$, show that the probability that infinitely many of the events occur is 0.

(f) Show that

$$\sum_x \mathbb{P}_x(\text{ there is at least one successful transmission}) < \infty.$$

Conclude.

Exercice 5: Birth-death process

Let X_n be a Markov chain defined by its transition probabilities

$$\forall i \geq 0,\ p_{ij} = \begin{cases} \lambda_i & \text{if } j = i+1, \\ 1-\lambda_i =: \mu_i & \text{if } j = \max(i-1, 0) \end{cases}$$

with $\lambda_0 = 1$ and $0 < \lambda_i < 1$ for all $i \geq 1$. We define

$$S_1 = 1 + \sum_{i=1}^{\infty} \frac{\lambda_0 \ldots \lambda_{i-1}}{\mu_1 \ldots \mu_i} \text{ and, } S_2 = 1 + \sum_{i=1}^{\infty} \frac{\mu_1 \ldots \mu_i}{\lambda_1 \ldots \lambda_i}$$

Show that X_n is positive recurrent iff $S_1 < \infty$ and X_n is recurrent iff $S_2 = \infty$.