Réseaux: Fiche d’exercices 2
16 février 2015

**Exercice 1:** Coupling.
Let $X_n$ be an irreducible aperiodic Markov chain with invariant probability $\pi$ on state space $E$. We consider two independent realizations of the Markov chain $X_n$ and $Y_n$ with $X_0 = x$ and $Y_0$ following the distribution $\pi$. We define $W_n = (X_n, Y_n) \in E^2$.

(a) Show that $W_n$ is an irreducible aperiodic positive recurrent Markov chain on $E \times E$.

(b) Show that $T = \inf\{n \geq 0, X_n = Y_n\}$ is a stopping time. We define $Z_n = \begin{cases} X_n, & n < T \\ Y_n, & n \geq T \end{cases}$

Show that $Z_n$ is a Markov chain with the same distribution as $X_n$.

(c) Show that $\lim_{n \to \infty} \mathbb{P}(X_n = y) = \pi(y)$.

**Exercice 2:** Wald identity.
Consider a random walk $S_n = \sum_{i=1}^{n} X_i$ where $X_i$ is a sequence of i.i.d. random variables with $\mathbb{E}[|X_1|] < \infty$. We will show that for any stopping time $T$ with $\mathbb{E}[T] < \infty$, we have $\mathbb{E} [S_T] = \mathbb{E} [T] \mathbb{E} [X_1]$.

(a) Give a counter-example when $T$ is not a stopping time with $\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = 1/2$.

(b) Show that $S = \sum_{n=1}^{\infty} X_n 1(T \geq n)$.

(c) Conclude when $T$ is bounded by a constant $M$.

(d) Conclude in general.

**Exercice 3:** Reflected random walk.
We consider the reflected random walk defined by $S_0 \in \mathbb{N}$ and for $n \geq 0$, $S_{n+1} = \max(S_n + \Delta_{n+1}, 0)$ where the sequence $\Delta_n$ is a sequence of i.i.d. random variables in $\mathbb{Z}$ such that $\mathbb{E} [\Delta_1] < 0$ and $\mathbb{P}(\Delta_1 = 1) > 0$.

(a) Show that $S_n$ is an aperiodic irreducible Markov chain on $\mathbb{R}^+$.

(b) Assume first that $\Delta_n \geq -1$ for all $n$. Let $T = \inf\{n > 0, S_n = 0\}$. Prove that $\mathbb{E}[T] < \infty$.

(c) In general, we define $M = \max \left( 0, \max_{0 \geq \Delta \geq -1} \sum_{n=0}^{\Delta} \right)$.

Show that $\mathbb{P}(M < \infty) = 1$ and that if $S_0 = M$, then the law of $S_n$ is the same as the law of $M$. Conclude about the recurrence of $S_n$. 

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Exercice 4: Instability of Aloha

Let \( A_n \) be the number of messages arriving at the end of time slot \( n \). We are given a sequence of i.i.d Bernoulli random variables \( \{B_n\} \) with mean \( p \) independent of the \( A_n \). If at the beginning of time slot \( n \), there are \( k \) messages waiting to be transmitted, each of them try to use the channel with probability \( p \) and the transmission is successful if \( \sum_{i=1}^{k} B_n = 1 \). Hence the evolution of the number \( L_n \) of messages to be transmitted at the beginning of the \( n \)-th time slot is given by:

\[
L_{n+1} = L_n + A_n - 1 \left( \sum_{i=1}^{n} B_i = 1 \right)
\]

(a) Show that if \( 0 < P(A_n = 0) < 1 \) then \( L_n \) is an aperiodic irreducible Markov chain.

(b) From now on, we assume that \( A_1 \) is a Poisson random variable with mean \( \lambda > 0 \). Compute \( \mathbb{E}[z^{A_1}] \) and \( \mathbb{E}[A_1 z^{A_1-1}] \).

(c) We define \( S_1 = \sum_{i=1}^{\infty} A_i \). Compute \( P \left( \sum_{i=1}^{\infty} B_i = 1 \right) \).

(d) Show that \( P \left( \cap_{x=0}^{\infty} \left\{ \sum_{i=1}^{\infty} B_i \neq 1 \right\} \right) > 0 \) for \( x \) sufficiently large.

(e) Show that \( L_n \) is transient.

We will now show a stronger result: with probability one, there are only a finite number of successful transmissions. Before that, prove the following general result: given a sequence of events \( E_n \) with \( \sum_{n=1}^{\infty} P(E_n) < \infty \), show that the probability that infinitely many of the events occur is 0.

(f) Show that

\[
\sum P \left( \text{there is at least one successful transmission} \right) < \infty.
\]

Conclude.

Exercice 5: Birth-death process

Let \( X_n \) be a Markov chain defined by its transition probabilities

\[
\forall i \geq 0, \; p = \begin{cases} 
\lambda & \text{if } j = i + 1, \\
1 - \lambda =: \mu & \text{if } j = \max(i - 1, 0)
\end{cases}
\]

with \( \lambda_0 = 1 \) and \( 0 < \lambda < 1 \) for all \( i \geq 1 \). We define

\[
S_1 = 1 + \sum_{i=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{i-1}}{\mu_1 \cdots \mu_{i-1}} \quad \text{and} \quad S_2 = 1 + \sum_{i=1}^{\infty} \frac{\mu_1 \cdots \mu_{i-1}}{\lambda_1 \cdots \lambda_{i-1}}
\]

Show that \( X_n \) is positive recurrent iff \( S_1 < \infty \) and \( X_n \) is recurrent iff \( S_2 = \infty \).