Reseaux: Fiche d’exercices 1
9 février 2015

Exercice 1:
(a) Let \( f \) be a concave function defined on a convex set \( \Omega \) such that \( \nabla f \) exists and is continuous on \( \Omega \). Show that
\[ x^* \in \arg \max_{x \in \Omega} f(x) \iff \forall x \in \Omega, \nabla f(x^*)(x - x^*) \leq 0. \]

(b) A linear network has \( L \) links and \( L + 1 \) routes such that route 0 uses all links whereas route \( \ell \in \{1, \ldots, L\} \) uses only link \( \ell \). For \( i \in \{0, \ldots, L\} \) there are \( n_i > 0 \) users on route \( i \). For weights \( w = (w_i, i = 0, \ldots, L) \), and \( \alpha > 0, \alpha \neq 1 \), compute the \((w, \alpha)\)-fair allocation.

(c) Consider the limits \( \alpha \to \infty \) and \( \alpha \to 1 \) with equal weights.

Exercice 2: the Nash bargaining solution.
We consider the following two players game: given a convex set \( F \subset \mathbb{R}^2 \) and a disagreement point \( v \in \mathbb{R}^2 \), the two players need to agree on a point \( x \in F \) so that player 1 will get \( x_1 \) and player 2 will get \( x_2 \). If the players do not reach an agreement, then player 1 will get \( v_1 \) and player 2 will get \( v_2 \).

(a) Draw your favorite convex set in \( \mathbb{R}^2 \) and determine a max-min fair allocation. Now, choose an arbitrary point \( v \) and play with your neighbor!

In order to formalize a bit the problem, we introduce the axioms for Nash’s bargaining solution denoted \( \phi(F, v) \in \mathbb{R}^2 \):

A1 (Strong efficiency). \( \phi(F, v) \) is an allocation in \( F \) and for any \( x \in F \), if \( x \geq \phi(F, v) \), then \( x = \phi(F, v) \).

A2 (Individual rationality). \( \phi(F, v) \geq v \).

A3 (Scale covariance). For any numbers \( \lambda_1, \lambda_2, \gamma_1, \gamma_2 \) such that \( \lambda_1 \lambda_2 > 0 \) if
\[ G = \{(\lambda_1 x_2 + \gamma_1, \lambda_2 x_2 + \gamma_2), (x_1, x_2) \in F \} \]
and \( w = (\lambda_1 v_1 + \gamma_1, \lambda_2 v_2 + \gamma_2) \), then \( \phi(G, w) = (\lambda_1 \phi(F, v) + \gamma_1, \lambda_2 \phi(F, v) + \gamma_2) \).

A4 (Independence of irrelevant alternatives). For any convex set \( G \), if \( G \subset F \) and \( \phi(F, v) \in G \) then \( \phi(G, v) = \phi(F, v) \).
A5 (Symmetry). If \( v_1 = v_2 \) and \( \{(x_2, x_1), (x_1, x_2) \in F\} = F \), then \( \phi_1(F, v) = \phi_2(F, v) \).

We will show that there is a unique solution function \( \phi(.,.) \) that satisfies all the axioms and

\[
\phi(F, v) \in \arg \max_{x \in F, x \geq v} (x_1 - v_1)(x_2 - v_2).
\]

(b) Did your solution in (a) satisfy the axioms? Show that the Nash’s bargaining solution is given by the following geometric picture:

(c) First conclude in the case where there is no point \( y \in F \) such that \( y > v \). For the rest of this exercise, we consider that there exists \( y \in F \) with \( y > v \).

(d) Show that we need only to prove the statement for \( v = 0 \) and \( F \) such that the maximum of \( x_1x_2 \) over \( F \) is located at \((1, 1)\).

(e) In this case, show that \( F \) is below the line \( x_1 + x_2 = 2 \).

(f) Conclude by considering a suitable symmetric set.

**Exercice 3: Modelling MulTCP.**

We describe a weighted version of TCP, called MulTCP. Let \( w \) be a weight parameter and suppose:

- the rate of additive increase is multiplied by \( w \), so that each acknowledgement increases \( cwnd \) by \( w/cwnd \) and;

- the multiplicative decrease factor becomes \( 1 - 1/(2w) \), so that after a congestion indication the window size becomes \( (1 - 1/(2w))cwnd \).
a) If the rate $x_r$ is approximated by $cwnd/T_r$, where $T_r$ is the round-trip time for route $r$, show that the evolution of $x_r$ can be approximated by:

$$\frac{d}{dt}x_r(t) = \frac{w_r}{T_r^2} - \left( \frac{w_r}{T_r^2} + \frac{x_r^2}{2w_r} \right) \pi_r(t),$$

where $\pi_r(t) = \sum_{\ell \in r} p_l(y_\ell(t))$ with $y_\ell(t) = \sum_{r, \ell \in r} x_r(t)$.

b) Recognize a primal algorithm and show that $x(t)$ converges to the stable point:

$$x_r = \frac{w_r}{T_r} \left( 2 \frac{1 - \pi_r}{\pi_r} \right)^{1/2}.$$

Hint: $\arctan'(z) = \frac{1}{1+z^2}$.

c) Show that when $\pi_r$ is small or $x_r$ is large, the rate is inversely proportional to the round-trip time $T_r$ and to the square root of the packet loss probability $\pi_r$. Compare to a proportionally fair allocation.

d) Consider a network with three links and two routes: two of the links have round-trip time $T_1$ and the last one has round-trip time $T_2 = 100T_1$. Moreover the two short links are congested with a loss probability $p_1$ whereas the long link is not $p_2 \approx 0$. The short route uses the two short congested links and the long route uses a short link and a long one. If you had the choice to locate a cache at the end of one of this route, which one would you choose? What changes under proportional fairness?

**Exercice 4:** Multi-path routing.

We consider now a scenario where each source $s$ may use several routes. We then define two matrices: $A_{sr} = 1$ if route $r$ is possible for source $s$ and $B_{r\ell} = 1$ if link $\ell$ is on route $r$. We also denote by $s(r)$ the unique source of route $r$. The network utility maximization problem becomes:

$$\max_{x \geq 0} \sum_s U_s \left( \sum_r A_{sr}x_r \right) - \sum_\ell C_\ell \left( \sum_r B_{r\ell}x_r \right).$$

a) Show that even if the $U_s$ are strictly concave and the $C_\ell$ are strictly convex, unicity of the maximizing problem is lost in general.

b) Propose a primal algorithm and modify it to ensure its convergence.