Réseaux: Fiche d'exercices 1

9 février 2015

Exercice 1:

(a) Let f be a concave function defined on a convex set Ω such that ∇f exists and is continuous on Ω. Show that

$$x^* \in \arg\max_{x\in\Omega} f(x) \Leftrightarrow \forall x \in \Omega,\ \nabla f(x^*)(x - x^*) \leq 0.$$

(b) A linear network has L links and $L+1$ routes such that route 0 uses all links whereas route $\ell \in \{1, \dots, L\}$ uses only link ℓ. For $i \in \{0, \dots, L\}$ there are $n_i > 0$ users on route i. For weights $w = (w_i, i = 0, \dots, L)$, and $\alpha > 0$, $\alpha \neq 1$, compute the (w, α)-fair allocation.

(c) Consider the limits $\alpha \to \infty$ and $\alpha \to 1$ with equal weights.

Exercice 2: the Nash bargaining solution.

We consider the following two players game: given a convex set $F \subset \mathbb{R}^2$ and a disagreement point $v \in \mathbb{R}^2$, the two players need to agree on a point $x \in F$ so that player 1 will get x_1 and player 2 will get x_2. If the players do not reach an agreement, then player 1 will get v_1 and player 2 will get v_2.

(a) Draw your favorite convex set in $\mathbb{R}^2$ and determine a max-min fair allocation. Now, choose an arbitrary point v and play with your neighbor!

In order to formalize a bit the problem, we introduce the axioms for Nash's bargaining solution denoted $\phi(F, v) \in \mathbb{R}^2$:

A1 (Strong efficiency). $\phi(F, v)$ is an allocation in F and for any $x \in F$, if $x \geq \phi(F, v)$, then $x = \phi(F, v)$.

A2 (Individual rationality). $\phi(F, v) \geq v$.

A3 (Scale covariance). For any numbers $\lambda_1, \lambda_2, \gamma_1, \gamma_2$ such that $\lambda_1\lambda_2 > 0$ if

$$G = \{(\lambda_1 x_2 + \gamma_1, \lambda_2 x_2 + \gamma_2),\ (x_1, x_2) \in F\}$$

and $w = (\lambda_1 v_1 + \gamma_1, \lambda_2 v_2 + \gamma_2)$, then $\phi(G, w) = (\lambda_1\phi_1(F, v) + \gamma_1, \lambda_2\phi(F, v) + \gamma_2)$.

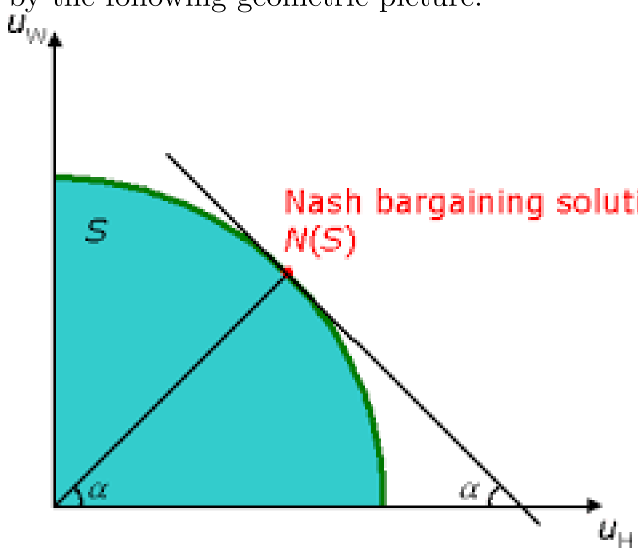
A4 (Independence of irrelevant alternatives). For any convex set G, if $G \subset F$ and $\phi(F, v) \in G$ then $\phi(G, v) = \phi(F, v)$.

A5 (Symmetry). If $v_1 = v_2$ and $\{(x_2, x_1),\ (x_1, x_2) \in F\} = F$, then $\phi_1(F, v) = \phi_2(F, v)$.

We will show that there is a unique solution function $\phi(.,.)$ that satisfies all the axioms and

$$\phi(F, v) \in \arg \max_{x \in F,\, x \geq v} (x_1 - v_1)(x_2 - v_2).$$

(b) Did your solution in (a) satisfy the axioms? Show that the Nash's bargaining solution is given by the following geometric picture:



(c) First conclude in the case where there is no point $y \in F$ such that $y > v$. For the rest of this exercice, we consider that there exists $y \in F$ with $y > v$.

(d) Show that we need only to prove the statement for $v = 0$ and F such that the maximum of $x_1 x_2$ over F is located at $(1, 1)$.

(e) In this case, show that F is below the line $x_1 + x_2 = 2$.

(f) Conclude by considering a suitable symmetric set.

Exercice 3: Modelling MulTCP.

We describe a weighted version of TCP, called MulTCP. Let w be a weight parameter and suppose:

- the rate of additive increase is multiplied by w, so that each acknowledgement increases $cwnd$ by $w/cwnd$ and;
- the multiplicative decrease factor becomes $1 - 1/(2w)$, so that after a congestion indication the window size becomes $(1 - 1/(2w))cwnd$.

(a) If the rate x_r is approximated by $cwnd/T_r$, where T_r is the round-trip time for route r, show that the evolution of x_r can be approximated by:

$$\frac{d}{dt}x_r(t) = \frac{w_r}{T_r^2} - \left(\frac{w_r}{T_r^2} + \frac{x_r^2}{2w_r}\right)\pi_r(t),$$

where $\pi_r(t) = \sum_{\ell\in r} p_\ell(y_\ell(t))$ with $y_\ell(t) = \sum_{r,\,\ell\in r} x_r(t)$.

(b) Recognize a primal algorithm and show that $x(t)$ converges to the stable point:

$$x_r = \frac{w_r}{T_r}\left(2\frac{1-\pi_r}{\pi_r}\right)^{1/2}.$$

Hint: $\arctan'(z) = \frac{1}{1+z^2}$.

(c) Show that when π_r is small or x_r is large, the rate is inversely proportional to the round-trip time T_r and to the square root of the packet loss probability π_r. Compare to a proportionally fair allocation.

(d) Consider a network with three links and two routes: two of the links have round-trip time T_1 and the last one has round-trip time $T_2 = 100T_1$. Moreover the two short links are congested with a loss probability p_1 whereas the long link is not $p_2 \approx 0$. The short route uses the two short congested links and the long route uses a short link and a long one. If you had the choice to locate a cache at the end of one of this route, which one would you choose? What changes under proportional fairness?

Exercice 4: Multi-path routing.
We consider now a scenario where each source s may use several routes. We then define two matrices: $A_{sr} = 1$ if route r is possible for source s and $B_{r\ell} = 1$ if link ℓ is on route r. We also denote by $s(r)$ the unique source of route r. The network utility maximization problem becomes:

$$\max_{x\geq 0} \sum_s U_s\left(\sum_r A_{sr}x_r\right) - \sum_\ell C_\ell\left(\sum_r B_{r\ell}x_r\right).$$

(a) Show that even if the U_s are strictly concave and the C_ℓ are strictly convex, unicity of the maximizing problem is lost in general.

(b) Propose a primal algorithm and modify it to ensure its convergence.