Reconstruction in the Generalized Stochastic Block Model

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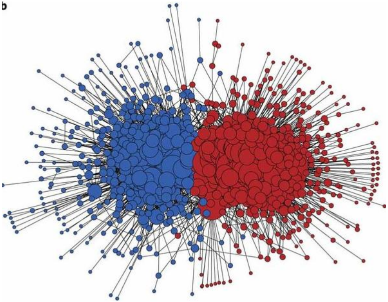
GDR ISIS-Phénix, Nov 25

Motivation

- Community detection in social or biological networks in the sparse regime with a (not too large) average degree.



- Labels to characterize various interaction types, e.g. strong and weak ties in friendship network.

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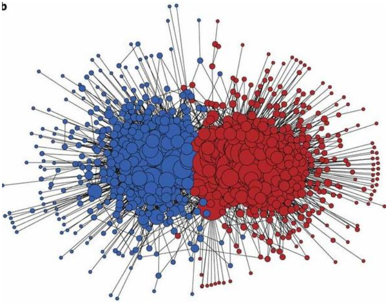

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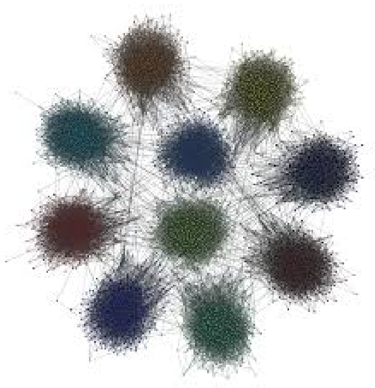
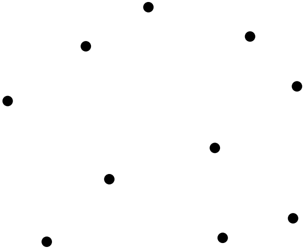
A model: the stochastic block model



The sparse stochastic block model

A random graph model on n nodes with two parameters, $a, b \geq 0$.

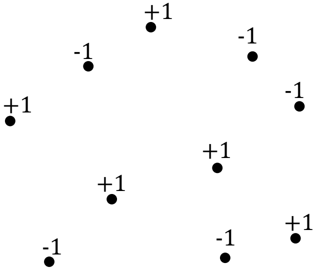


$n = 10$

The sparse stochastic block model

A random graph model on n nodes with two parameters, $a, b \geq 0$.

- Assign each vertex spin $+1$ or -1 uniformly at random.



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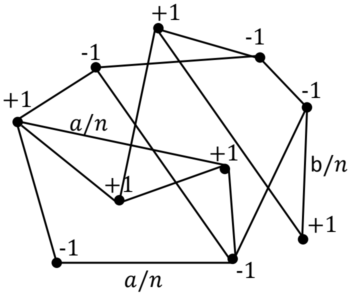

$a = 4, b = 2$

Reconstruction problem

- Reconstruct the underlying spin configuration σ based on the observed labeled graph.
- Sparse graph: as $n \to \infty$, the asymptotic degree distribution is Poisson with mean $\frac{a+b}{2}$. With on average, $\frac{a}{2}$ neighbors in the same community and $\frac{b}{2}$ in the other community.
- Isolated nodes render exact reconstruction impossible. Focus on positively correlated reconstruction, i.e., $\hat{\sigma}$ agrees with σ in more than $1/2$ of all its entries.

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Phase transition

Theorem

If $\tau > 1$, then positively correlated reconstruction is possible.
If $\tau < 1$, then positively correlated reconstruction is impossible.

$$\tau = \frac{(a-b)^2}{2(a+b)}.$$

Conjectured by Decelle, Krzakala, Moore, Zdeborova '11 based on statistical physics arguments.

- Non-reconstruction proved by Mossel, Neeman, Sly '12.
- Reconstruction proved by Massoulié '13 and Mossel, Neeman, Sly '13.

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Efficiency of Spectral Algorithms

Boppana '87, Condon, Karp '01, Carson, Impagliazzo '01, McSherry '01, Kannan, Vempala, Vetta '04...

Proposition

Suppose that for sufficiently large c and c',

$$\frac{(a-b)^2}{a+b} \geq c + c' \frac{a}{a+b} \ln\left(\frac{a+b}{2}\right),$$

then 'trimming+spectral+greedy improvement' outputs a positively correlated partition a.a.s.

Coja-Oghlan '10

What if $a, b \to \infty$?

Proposition

Assume $a \geq \ln^5 n$ and $(a-b)^2 > 164(a+b)$, then the clustering problem is solvable by the simple spectral method.

Lelarge, Massoulié, Xu '13

Lower bound (valid for any a)

Proposition

For $\alpha < 1/2$, define $\delta = \frac{1}{2} - \inf_{\hat{\sigma}} \mathbb{P}(d(\sigma, \hat{\sigma}) > \alpha)$. Then $\delta > 0$ implies

$$\frac{(a-b)^2}{2(a+b)} > 1 - H(\alpha),$$

with $H(x) = -x \log x - (1-x)\log(1-x)$.

Proof: Fano's inequality.

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Spectral analysis

Assume that $a > \ln^5 n$, and $a - b \approx \sqrt{a+b}$ so that $a \sim b$.

$$A = \frac{a+b}{2}\frac{\mathbf{1}}{\sqrt{n}}\frac{\mathbf{1}^T}{\sqrt{n}} + \frac{a-b}{2}\frac{\sigma}{\sqrt{n}}\frac{\sigma^T}{\sqrt{n}} + A - \mathbb{E}[A]$$

$\frac{a+b}{2}$ is the mean degree and degrees in the graph are very concentrated in the regime $a > \ln^5 n$. We can construct

$$A - \frac{a+b}{2n}J = \frac{a-b}{2}\frac{\sigma}{\sqrt{n}}\frac{\sigma^T}{\sqrt{n}} + A - \mathbb{E}[A]$$

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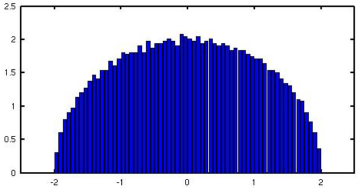
Spectrum of the noise matrix

The matrix $A - \mathbb{E}[A]$ is a symmetric random matrix with independent centered entries having variance $\sim \frac{a}{n}$.
To have convergence to the Wigner semicircle law, we need to normalize the variance to $\frac{1}{n}$.



$$ESD\left(\frac{A - \mathbb{E}[A]}{\sqrt{a}}\right) \to \mu_{sc}(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4 - x^2}, & \text{if } |x| \leq 2; \\ 0, & \text{otherwise.} \end{cases}$$

Naive spectral analysis

To sum up, we can construct:

$$\begin{aligned} M &= \frac{1}{\sqrt{a}}\left(A - \frac{a+b}{2n}J\right) \\ &= \theta\frac{\sigma}{\sqrt{n}}\frac{\sigma}{\sqrt{n}}^T + \frac{A - \mathbb{E}[A]}{\sqrt{a}}, \end{aligned}$$

with $\theta = \frac{a-b}{2\sqrt{a}}$.
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Some hope

A lower bound on the spectral radius of $M = \theta \frac{\sigma}{\sqrt{n}} \frac{\sigma^T}{\sqrt{n}} + W$:

$$\lambda_1(M) = \sup_{\|x\|=1} \|Mx\| \geq \|M \frac{\sigma}{\sqrt{n}}\|$$

But

$$\begin{aligned} \|M \frac{\sigma}{\sqrt{n}}\|^2 &= \theta^2 + \|W \frac{\sigma}{\sqrt{n}}\|^2 + 2\langle W, \frac{\sigma}{\sqrt{n}} \rangle \\ &\approx \theta^2 + \frac{1}{n} \sum_{i,j} W_{ij}^2 \\ &\approx \theta^2 + 1. \end{aligned}$$

As a result, we get

$$\lambda_1(M) > 2 \Leftrightarrow \theta > 1 \Leftrightarrow \frac{(a-b)^2}{2(a+b)} > 1.$$

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Baik, Ben Arous, Péché phase transition

Rank one perturbation of a Wigner matrix:

$$\lambda_1(\theta\sigma\sigma^T + W) \overset{a.s}{\to} \begin{cases} \theta + \frac{1}{\theta} & \text{if } \theta > 1, \\ 2 & \text{otherwise.} \end{cases}$$

Let $\tilde{\sigma}$ be the eigenvector associated with $\lambda_1(\theta uu^T + W)$, then

$$|\langle \tilde{\sigma}, \sigma \rangle|^2 \overset{a.s}{\to} \begin{cases} 1 - \frac{1}{\theta^2} & \text{if } \theta > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Baik, Ben Arous, Péché '05

Rigorous proof of the phase transition for $a \geq \ln^5 n$

Proposition

Assume $a \geq \ln^5 n$. Then the clustering problem is solvable by the simple spectral method, provided

$$\frac{(a-b)^2}{2(a+b)} > 1.$$

Lelarge '13

Proof: control the spectral norm thanks to Vu '05 and adapt the argument in Benaych-Georges, Nadakuditi '11.

In agreement with Nadakuditi, Newman '12.

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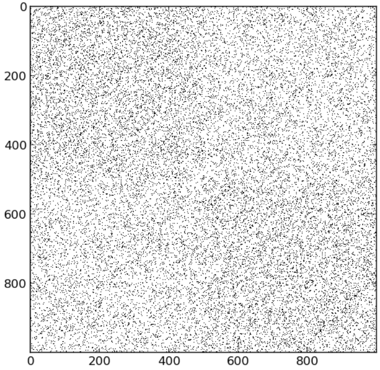
Spectral Algorithm

Original adjacency matrix with 2 communities. $a = 120$, $b = 92$, $\theta = \frac{a-b}{\sqrt{2(a+b)}} = 1.46385...$



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Spectrum of the original adjacency matrix. $a = 120$, $b = 92$,
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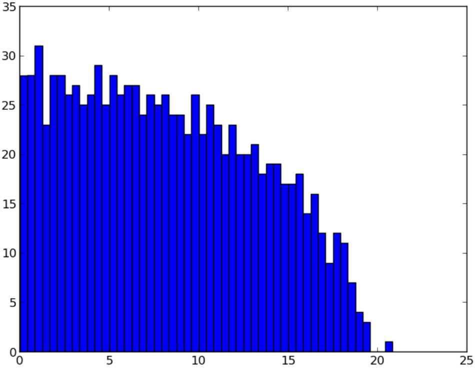

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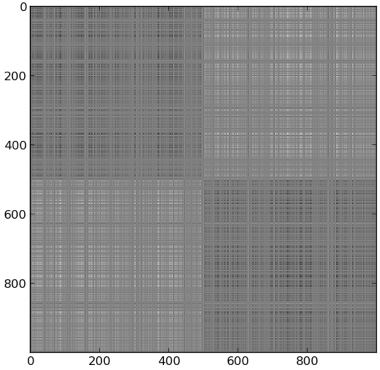
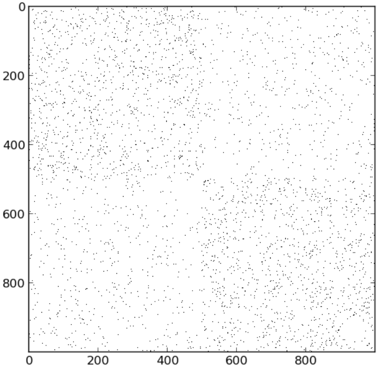
Rank-1 approximation of the adjacency matrix. $a = 120$, $b = 92$, $\theta = \frac{a-b}{\sqrt{2(a+b)}} = 1.46385...$



Spectral Algorithm: low degree

Original adjacency matrix with 2 communities. $a = 20$, $b = 9$, $\theta = \frac{a-b}{\sqrt{2(a+b)}} = 1.44437...$



Spectral Algorithm: low degree

Spectrum of the original adjacency matrix (after trimming).
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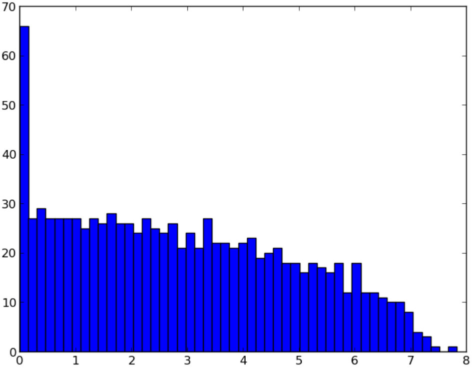

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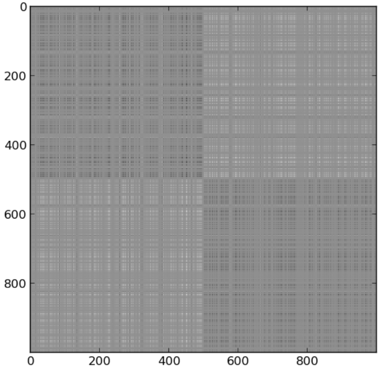
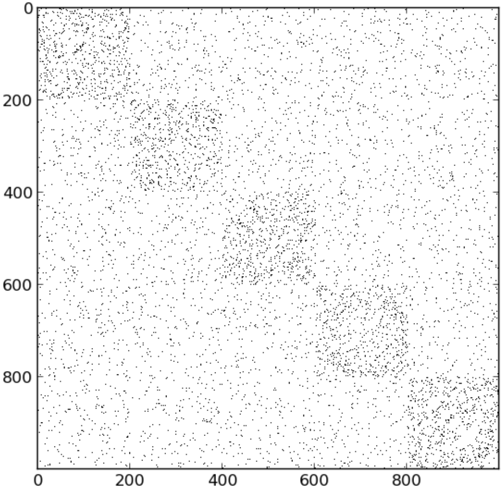


Spectral Algorithm: more communities

Original adjacency matrix with 5 communities.



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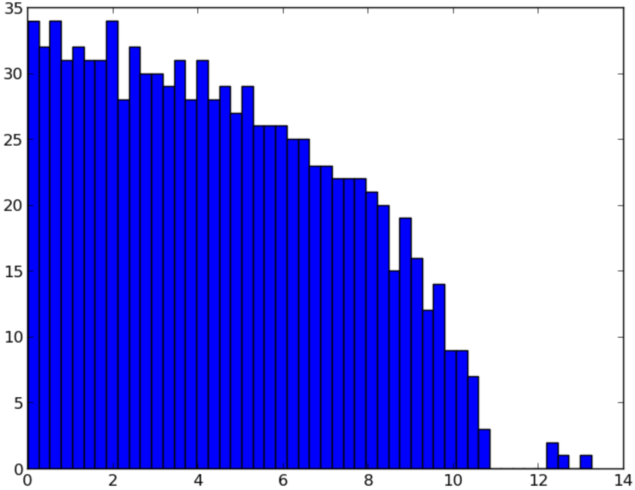
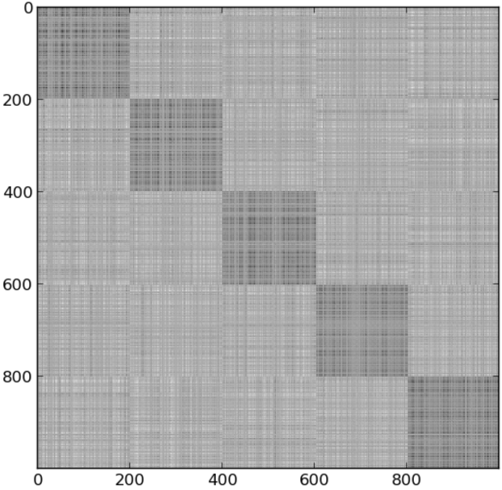
Spectrum of the original adjacency matrix.



Spectral Algorithm: more communities

Rank-4 approximation of the adjacency matrix.



Extension: r symmetric communities

Proposition

Assume $a \geq \ln^5 n$ and $r \geq 2$ symmetric communities. Then the clustering problem is solvable by the simple spectral method, provided

$$\frac{(a-b)^2}{r(a+(r-1)b)} > 1.$$

Lelarge '13

The sparse labeled stochastic block model

A random graph model on n nodes with two parameters, $a, b \geq 0$ and two discrete prob. distributions, μ, ν.

- Independently for each pair (u, v):
  - if $\sigma_u = \sigma_v$, draw the edge w.p. a/n.
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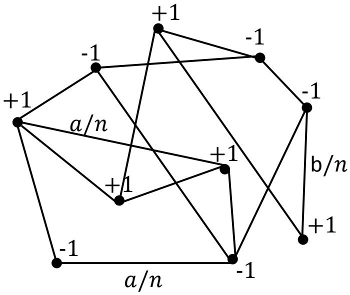

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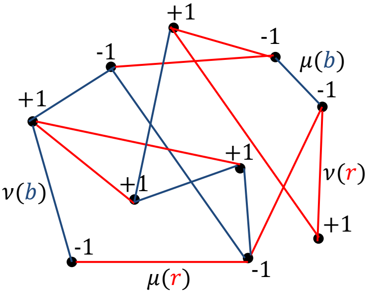
- Independently for each edge (u, v):
  - if $\sigma_u = \sigma_v$, label the edge with $L_{uv} \sim \mu$.
  - if $\sigma_u \neq \sigma_v$, label the edge with $L_{uv} \sim \nu$.



$$\mu(r) = 0.6, \mu(b) = 0.4$$
$$\nu(r) = 0.4, \nu(b) = 0.6$$

How to use labels?

- Maximum log likelihood estimation:

$$\max_{\sigma} \sum_{(u,v)\in E(G)} \sigma_u \sigma_v \log \frac{a\mu(L_{uv})}{b\nu(L_{uv})}$$

$$\text{s.t.} \sum_{u} \sigma_u = 0,\ \sigma_u \in \{-1, 1\}$$

- Minimum bisection with edge weights $w(\ell) = \log \frac{a\mu(\ell)}{b\nu(\ell)}$.
- Minimum bisection is NP-hard. Let's try some statistical physics!

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Phase transition (with labels)

Conjecture

If $\tau_L > 1$, then positively correlated reconstruction is possible.
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$$\tau_L = \frac{1}{2} \sum_{\ell \in \mathcal{L}} \frac{(a\mu(\ell) - b\nu(\ell))^2}{a\mu(\ell) + b\nu(\ell)}.$$

Heimlicher, Lelarge, Massoulié '12

- Generalize the result for (standard) stochastic block model and $\tau_L \geq \tau$.
- τ_L comes from the local stability analysis of a fixed point of Belief Propagation.

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Non-reconstruction

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If $\tau_L < 1$, then for any fixed vertices u and v, conditional on the spin of v, the spin of u is asymptotically uniformly distributed.

Lelarge, Massoulié, Xu, 13

- It further implies that it is impossible to reconstruct a positively correlated partition.
- Proof: similar to Mossel, Neeman, Sly '12, uses local tree argument, conditional independence property and the Ising spin model on labeled tree.

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Spectral method with labels

- A is the weighted adjacency matrix: $A_{uv} = 1((u,v) \in E(G))w(L_{uv})$.
- Spectral method as a relaxation of the minimum bisection:

$$\max \sum_{(u,v)} \sigma_u A_{uv} \sigma_v$$
$$\text{s.t. } \sum_i \sigma_i = 0, \; \|\sigma\|_2 = 1.$$

- Perturbed low-rank matrix A

$$\mathbb{E}[A|\sigma] = \frac{a\overline{\mu} + b\overline{\nu}}{2n}\mathbf{1}\mathbf{1}^\top + \frac{a\overline{\mu} - b\overline{\nu}}{2n}\sigma\sigma^\top.$$

- Curse from vertices of high degrees $\Omega(\frac{\log n}{\log \log n})$.

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Spectral Algorithm: rigorous results

Remove nodes of degree greater than $\frac{3}{2}\frac{a+b}{2}$.
'Optimal' weight function:

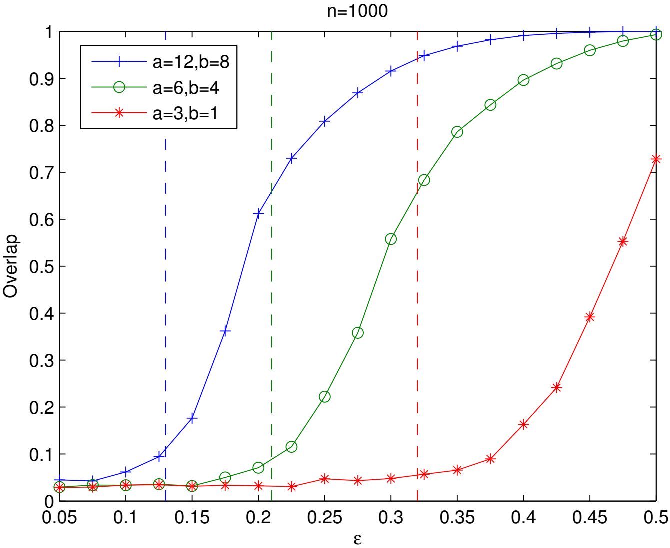
$$w(\ell) = \frac{a\mu(\ell) - b\nu(\ell)}{a\mu(\ell) + b\nu(\ell)}$$

Theorem

If $\tau_L > C\sqrt{a+b}$, then w.h.p. the spectral algorithm gives a positively correlated partition.

- Proof: spectrum of truncated ER random graph, extension of Feige Ofek '05

Spectral Algorithm: empirical results



Rigorous results for $a \geq \ln^5 n$

Proposition

Assume $a \geq \ln^5 n$ and $r \geq 2$ symmetric communities. Then the clustering problem is solvable by the simple spectral method, provided

$$\frac{1}{r}\sum_{\ell}\frac{(a\mu(\ell)-b\nu(\ell))^2}{a\mu(\ell)+(r-1)b\nu(\ell)} > 1$$

Lelarge '13

Proved using more Random Matrix Theory.

Extensions

- Some results for models with latent space allowing to relax the low-rank assumption and overlapping communities. If the signal strength is at least $\log n$, then consistent estimation of the edge label distribution is possible.
- For the planted clique problem, clique of size larger than $\sqrt{n}$ are detectable by a simple spectral algorithm. Deshpande Montanari '13 message passing algorithm works for sizes $\sqrt{n/e} = 0.60653...\sqrt{n}$. However cliques of size $2\log_2 n$ can be found by exhaustive search...

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Summary

- For the labeled (not too sparse) stochastic block model, there is a phase transition between an impossible regime and an easy regime where the simple spectral algorithm is successful.
- How well does the spectral algorithm performs in term of 'overlap'? What if parameters are unknown?
- Is there a computational threshold for $r \geq 5$?

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