Reconstruction in the Generalized Stochastic Block Model

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Motivation

- Community detection in social or biological networks in the sparse regime with a (not too large) average degree.

- Labels to characterize various interaction types, e.g. strong and weak ties in friendship network.
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- Labels to characterize various interaction types, e.g. strong and weak ties in friendship network.
A model: the stochastic block model
The sparse stochastic block model

A random graph model on $n$ nodes with two parameters, $a, b \geq 0$. 

$n = 10$
The sparse stochastic block model

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Assign each vertex spin $+1$ or $-1$ uniformly at random.

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A random graph model on $n$ nodes with two parameters, $a, b \geq 0$.

- Independently for each pair $(u, v)$:
  - if $\sigma_u = \sigma_v$, draw the edge w.p. $a/n$.
  - if $\sigma_u \neq \sigma_v$, draw the edge w.p. $b/n$.

$a = 4, b = 2$
Reconstruction problem

- Reconstruct the underlying spin configuration \( \sigma \) based on the observed labeled graph.

- **Sparse graph**: as \( n \to \infty \), the asymptotic degree distribution is Poisson with mean \( \frac{a+b}{2} \). With on average, \( \frac{a}{2} \) neighbors in the same community and \( \frac{b}{2} \) in the other community.

- Isolated nodes render exact reconstruction impossible. Focus on **positively correlated** reconstruction, i.e., \( \hat{\sigma} \) agrees with \( \sigma \) in more than 1/2 of all its entries.
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Phase transition

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\tau = \frac{(a - b)^2}{2(a + b)}. 
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Conjectured by Decelle, Krzakala, Moore, Zdeborova ’11 based on statistical physics arguments.  
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Proposition

Suppose that for sufficiently large $c$ and $c'$,

\[
\frac{(a - b)^2}{a + b} \geq c + c' \frac{a}{a + b} \ln \left( \frac{a + b}{2} \right),
\]

then 'trimming+spectral+greedy improvement' outputs a positively correlated partition a.a.s.

Coja-Oghlan '10
What if \( a, b \to \infty \) ?

**Proposition**

Assume \( a \geq \ln^5 n \) and \((a - b)^2 > 164(a + b)\), then the clustering problem is solvable by the simple spectral method.

Lelarge, Massoulié, Xu ’13

Lower bound (valid for any \( a \))

**Proposition**

For \( \alpha < 1/2 \), define \( \delta = \frac{1}{2} - \inf_{\hat{\sigma}} \mathbb{P}(d(\sigma, \hat{\sigma}) > \alpha) \). Then \( \delta > 0 \) implies

\[
\frac{(a - b)^2}{2(a + b)} > 1 - H(\alpha),
\]

with \( H(x) = -x \log x - (1 - x) \log(1 - x) \).

Proof: Fano’s inequality.
What if $a, b \to \infty$?

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Proof: Fano’s inequality.
Spectral analysis

Assume that $a > \ln^5 n$, and $a - b \approx \sqrt{a + b}$ so that $a \sim b$.

$$A = \frac{a + b}{2} \frac{1 1^T}{\sqrt{n} \sqrt{n}} + \frac{a - b}{2} \frac{\sigma \sigma^T}{\sqrt{n} \sqrt{n}} + A - \mathbb{E}[A]$$

$\frac{a+b}{2}$ is the mean degree and degrees in the graph are very concentrated in the regime $a > \ln^5 n$. We can construct

$$A - \frac{a + b}{2n} J = \frac{a - b}{2} \frac{\sigma \sigma^T}{\sqrt{n} \sqrt{n}} + A - \mathbb{E}[A]$$
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Spectrum of the noise matrix

The matrix $A - \mathbb{E}[A]$ is a symmetric random matrix with independent centered entries having variance $\sim \frac{a}{n}$. To have convergence to the Wigner semicircle law, we need to normalize the variance to $\frac{1}{n}$.

$$ESD \left( \frac{A - \mathbb{E}[A]}{\sqrt{a}} \right) \to \mu_{sc}(x) = \begin{cases} \frac{1}{2\pi} \sqrt{4 - x^2}, & \text{if } |x| \leq 2; \\ 0, & \text{otherwise}. \end{cases}$$
Naive spectral analysis

To sum up, we can construct:

\[
M = \frac{1}{\sqrt{a}} \left( A - \frac{a + b}{2n} J \right) \\
= \theta \frac{\sigma}{\sqrt{n}} \frac{\sigma^T}{\sqrt{n}} + \frac{A - \mathbb{E}[A]}{\sqrt{a}},
\]

with \( \theta = \frac{a-b}{2\sqrt{a}} \).

We should be able to detect signal as soon as

\[
\theta > 2 \iff \frac{(a - b)^2}{2(a + b)} > 4
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Some hope

A lower bound on the spectral radius of \( M = \theta \frac{\sigma}{\sqrt{n}} \frac{\sigma^T}{\sqrt{n}} + W \):

\[
\lambda_1(M) = \sup_{\|x\|=1} \| Mx \| \geq \| M \frac{\sigma}{\sqrt{n}} \|
\]

But

\[
\| M \frac{\sigma}{\sqrt{n}} \|^2 = \theta^2 + \| W \frac{\sigma}{\sqrt{n}} \|^2 + 2 \langle W, \frac{\sigma}{\sqrt{n}} \rangle \\
\approx \theta^2 + \frac{1}{n} \sum_{i,j} W_{ij}^2 \\
\approx \theta^2 + 1.
\]

As a result, we get

\[
\lambda_1(M) > 2 \Leftrightarrow \theta > 1 \Leftrightarrow \frac{(a - b)^2}{2(a + b)} > 1.
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$$\approx \theta^2 + 1.$$

As a result, we get

$$\lambda_1(M) > 2 \iff \theta > 1 \iff \frac{(a - b)^2}{2(a + b)} > 1.$$
Rank one perturbation of a Wigner matrix:

\[ \lambda_1(\theta\sigma\sigma^T + W) \xrightarrow{a.s.} \begin{cases} \theta + \frac{1}{\theta} & \text{if } \theta > 1, \\ 2 & \text{otherwise}. \end{cases} \]

Let \( \tilde{\sigma} \) be the eigenvector associated with \( \lambda_1(\theta uu^T + W) \), then

\[ |\langle \tilde{\sigma}, \sigma \rangle|^2 \xrightarrow{a.s.} \begin{cases} 1 - \frac{1}{\theta^2} & \text{if } \theta > 1, \\ 0 & \text{otherwise}. \end{cases} \]

Baik, Ben Arous, Péché ’05
Rigorous proof of the phase transition for $a \geq \ln^5 n$

**Proposition**

Assume $a \geq \ln^5 n$. Then the clustering problem is solvable by the simple spectral method, provided

$$\frac{(a - b)^2}{2(a + b)} > 1.$$ 

**Lelarge ’13**

Proof: control the spectral norm thanks to Vu ’05 and adapt the argument in Benaych-Georges, Nadakuditi ’11. In agreement with Nadakuditi, Newman ’12.
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Spectral Algorithm

Original adjacency matrix with 2 communities. \( a = 120, \ b = 92, \)
\[ \theta = \frac{a-b}{\sqrt{2(a+b)}} = 1.46385... \]
Spectral Algorithm

Spectrum of the original adjacency matrix. $a = 120$, $b = 92$, 
\[ \theta = \frac{a-b}{\sqrt{2(a+b)}} = 1.46385\ldots \]
Rank-1 approximation of the adjacency matrix. \( a = 120, b = 92, \theta = \frac{a-b}{\sqrt{2(a+b)}} = 1.46385... \)
Spectral Algorithm: low degree

Original adjacency matrix with 2 communities. $a = 20$, $b = 9$, 
\[
\theta = \frac{a-b}{\sqrt{2(a+b)}} = 1.44437\ldots
\]
Spectral Algorithm: low degree

Spectrum of the original adjacency matrix (after trimming).

\[ a = 20, \ b = 9, \ \theta = \frac{a-b}{\sqrt{2(a+b)}} = 1.44437\ldots \]
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Rank-1 approximation of the adjacency matrix. $a = 20$, $b = 9$, 
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Spectral Algorithm: more communities

Original adjacency matrix with 5 communities.
Spectral Algorithm: more communities

Spectrum of the original adjacency matrix.
Spectral Algorithm: more communities

Rank-4 approximation of the adjacency matrix.
Extension: $r$ symmetric communities

### Proposition

Assume $a \geq \ln^5 n$ and $r \geq 2$ symmetric communities. Then the clustering problem is solvable by the simple spectral method, provided

$$\frac{(a - b)^2}{r(a + (r - 1)b)} > 1.$$
The sparse labeled stochastic block model

A random graph model on $n$ nodes with two parameters, $a, b \geq 0$ and two discrete prob. distributions, $\mu, \nu$.

- Independently for each pair $(u, v)$:
  - if $\sigma_u = \sigma_v$, draw the edge w.p. $a/n$.
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A random graph model on $n$ nodes with two parameters, $a, b \geq 0$ and two discrete prob. distributions, $\mu, \nu$.

- Independently for each edge $(u, v)$:
  - if $\sigma_u = \sigma_v$, label the edge with $L_{uv} \sim \mu$.
  - if $\sigma_u \neq \sigma_v$, label the edge with $L_{uv} \sim \nu$.

\[
\begin{align*}
\mu(r) &= 0.6, \mu(b) = 0.4 \\
\nu(r) &= 0.4, \nu(b) = 0.6
\end{align*}
\]
How to use labels?

Maximum log likelihood estimation:

$$\max_{\sigma} \sum_{(u,v) \in E(G)} \sigma_u \sigma_v \log \frac{a_\mu(L_{uv})}{b_\nu(L_{uv})}$$

s.t. $$\sum_u \sigma_u = 0, \sigma_u \in \{-1, 1\}$$

Minimum bisection with edge weights $$w(\ell) = \log \frac{a_\mu(\ell)}{b_\nu(\ell)}$$.

Minimum bisection is NP-hard. Let’s try some statistical physics!
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**Conjecture**

If $\tau_L > 1$, then positively correlated reconstruction is possible. If $\tau_L < 1$, then positively correlated reconstruction is impossible.

$$
\tau_L = \frac{1}{2} \sum_{\ell \in \mathcal{L}} \frac{(a_\mu(\ell) - b_\nu(\ell))^2}{a_\mu(\ell) + b_\nu(\ell)}.
$$

Heimlicher, Lelarge, Massoulié '12

- Generalize the result for (standard) stochastic block model and $\tau_L \geq \tau$.
- $\tau_L$ comes from the local stability analysis of a fixed point of Belief Propagation.
Phase transition (with labels)

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Conjecture

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If $\tau_L < 1$, then positively correlated reconstruction is impossible.

$$\tau_L = \frac{1}{2} \sum_{\ell \in L} \frac{(a_{\mu}(\ell) - b_{\nu}(\ell))^2}{a_{\mu}(\ell) + b_{\nu}(\ell)}.$$
Non-reconstruction

**Theorem**

If $\tau_L < 1$, then for any fixed vertices $u$ and $v$, conditional on the spin of $v$, the spin of $u$ is asymptotically uniformly distributed.

*Lelarge, Massoulié, Xu, 13*

- It further implies that it is impossible to reconstruct a positively correlated partition.
- Proof: similar to Mossel, Neeman, Sly ’12, uses local tree argument, conditional independence property and the Ising spin model on labeled tree.
# Non-reconstruction

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Lelarge, Massoulié, Xu, 13

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A is the weighted adjacency matrix:
\[ A_{uv} = 1((u, v) \in E(G))w(L_{uv}). \]

Spectral method as a relaxation of the minimum bisection:
\[
\max \sum_{(u,v)} \sigma_u A_{uv} \sigma_v
\]
\[
s.t. \sum_i \sigma_i = 0, \|\sigma\|_2 = 1.
\]

Perturbed low-rank matrix \( A \)
\[
E[A|\sigma] = \frac{a\mu + b\nu}{2n}11^\top + \frac{a\mu - b\nu}{2n}\sigma\sigma^\top.
\]

Curse from vertices of high degrees \( \Omega(\frac{\log n}{\log \log n}) \).
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Spectral method with labels

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Spectral method with labels

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- Curse from vertices of high degrees $\Omega(\frac{\log n}{\log\log n})$. 
Spectral Algorithm: rigorous results

Remove nodes of degree greater than $\frac{3}{2} \frac{a+b}{2}$.

'Optimal' weight function:

$$w(\ell) = \frac{a_\mu(\ell) - b_\nu(\ell)}{a_\mu(\ell) + b_\nu(\ell)}$$

Theorem

If $\tau_L > C\sqrt{a+b}$, then w.h.p. the spectral algorithm gives a positively correlated partition.

- Proof: spectrum of truncated ER random graph, extension of Feige Ofek ’05
Spectral Algorithm: empirical results

\[ n=1000 \]

\[ \varepsilon \]

Overlap

\[ a=12,b=8 \]

\[ a=6,b=4 \]

\[ a=3,b=1 \]
Rigorous results for $a \geq \ln^5 n$

**Proposition**

Assume $a \geq \ln^5 n$ and $r \geq 2$ symmetric communities. Then the clustering problem is solvable by the simple spectral method, provided

$$\frac{1}{r} \sum_{\ell} \frac{(a_\mu(\ell) - b_\nu(\ell))^2}{a_\mu(\ell) + (r - 1)b_\nu(\ell)} > 1$$

Lelarge ’13
Proved using more Random Matrix Theory.
Some results for models with latent space allowing to relax the low-rank assumption and overlapping communities. If the signal strength is at least $\log n$, then consistent estimation of the edge label distribution is possible.

For the planted clique problem, clique of size larger than $\sqrt{n}$ are detectable by a simple spectral algorithm. Deshpande Montanari '13 message passing algorithm works for sizes $\sqrt{n/e} = 0.60653...\sqrt{n}$. However cliques of size $2\log_2 n$ can be found by exhaustive search...
Extensions

- Some results for models with latent space allowing to relax the low-rank assumption and overlapping communities. If the signal strength is at least $\log n$, then consistent estimation of the edge label distribution is possible.

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How well does the spectral algorithm performs in term of ’overlap’? What if parameters are unknown?

Is there a computational threshold for $r \geq 5$?

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Summary

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