Diffusions et cascades dans les graphes aléatoires

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Journées MAS 2010, Bordeaux.
theoretic diffusion model...
...on a network.

- Everybody start with ICQ.
- Total payoff = sum of the payoffs with each neighbor.
- A fraction of the population is forced to
- If $2(1-q) > 3q$, i.e. $2 > 5q$
Threshold Model

- The state of agent $i$ is represented by $X_i$.
- $X_i = \begin{cases} 0 & \text{if } \text{icq} \ 	ext{and talk} \\
1 & \text{if } \text{icq} \end{cases}$
- $\sum_{j \sim i} X_j \geq qd_i$
Does there exist a finite group of players such that their action under best response dynamics spreads contagiously everywhere?

Contagion threshold: $q_c$, largest $q$ for which contagious dynamics are possible.

Example: interaction on the line

$q_c = \frac{1}{2}$
Another example: d-regular trees

\[ q_c = \frac{1}{d} \]
What happens for random graphs?

• Random graphs with given degree sequence introduced by Molloy and Reed (1995).

• Examples:
  – Random regular graphs.
  – Erdös-Rényi graphs, $G(n, \frac{\lambda}{n})$.

• We are interested in large population asymptotics.

$\lambda$

\[ D \overset{d}{\Rightarrow} \text{Poi}(\lambda) \]
What happens for random graphs?

- Random graphs are locally tree-like.
- Random \( d \)-regular graph:

\[
q_c = \sup \left\{ q : \mathbb{E} \left[ D(D - 1) \mathbb{1} \left( D < q^{-1} \right) \right] > \mathbb{E}[D] \right\}
\]

- \( q_c = \frac{1}{d} \)
Erdös–Réyni graphs \( G(n, \lambda/n) \)

No cascade

Global cascades

\[ \lambda \]
Erdős-Rényi graphs $G(n, \frac{\lambda}{n})$

- Pivotal players: giant component of the subgraph with degrees
Phase transition continued

- What happens when \( q \) is bigger than the cascade capacity?
- If the seed plays \( B \) forever, the model is monotone.
- In a finite graph, there is only one possible final state for the epidemic.
Locally tree-like
Independent computations on trees
\[ Y_i = \begin{cases} 1 & \text{if infected from 'below'} \\ 0 & \text{otherwise.} \end{cases} \]

\[ Y_i = 1 - (1 - \sigma_i) \mathbb{1} \left( \sum_{\ell \rightarrow i} Y_{\ell} \leq q d_i \right) \]
\[ Y \overset{d}{=} 1 - (1 - \sigma) \mathbb{1} \left( \sum_{\ell=1}^{\hat{D}-1} Y_\ell \leq q\hat{D} \right) \]

\[ z = \mathbb{P}(Y = 0) \]

\[ \lambda z^2 = (1 - \alpha) h(z) \]

\[ h(z) = \sum_{s, r \geq s - \lfloor qs \rfloor} r p_s \binom{s}{r} z^r (1 - z)^{s-r} \]
\[ \max\{z \in [0,1], \lambda z^2 - (1 - \alpha) h(z) = 0\} z^* \]
• The locally tree-like structure gives the intuition for the solution...
• ... but not the proof!
• Configuration model + results of Janson and Luczak.
• Generic epidemic model which allows to retrieve basic results for random graphs...
• .... and new ones: contagion threshold, phase transitions.
- Diffusion and Cascading Behavior in Random Networks. Available at http://www.di.ens.fr/~lelarge