Diffusions et cascades dans les graphes aléatoires

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Game-theoretic diffusion model...

- Both receive payoff $q$.
- Both receive payoff $1-q>q$.
- Both receive nothing.

Morris (2000)
...on a network.

- Everybody start with ICQ.
- Total payoff = sum of the payoffs with each neighbor.
- A fraction of the population is forced to talk.
- If $2(1-q) > 3q$, i.e. $2 > 5q$. 
Threshold Model

• State of agent $i$ is represented by

$$X_i = \begin{cases} 
0 & \text{if}\ i \text{ is } i_{ca} \\
1 & \text{if}\ i \text{ is } \text{talk} 
\end{cases}$$

• Switch from $i_{ca}$ to $\text{talk}$ if:

$$\sum_{j \sim i} X_j \geq qd_i$$

- Does there exist a finite groupe of players such that their action under best response dynamics spreads **contagiously** everywhere?
- **Contagion threshold**: $q_c = \text{largest } q \text{ for which contagious dynamics are possible.}$
- Example: interaction on the line

$$q_c = \frac{1}{2}$$
Another example: d-regular trees

\[ q_c = \frac{1}{d} \]
What happens for random graphs?

- Random graphs with given degree sequence introduced by Molloy and Reed (1995).
- Examples:
  - Random regular graphs.
  - Erdös-Rényi graphs, $G(n, \lambda/n)$.
- We are interested in large population asymptotics.
- $D$ is the asymptotic degree distribution.
What happens for random graphs?

• Random graphs are locally tree-like.

\[ a = \sup \left\{ a \cdot \mathbb{P} \left[ D(D - 1) \mid \frac{D}{n} < a^{-1} \right] \bigg\} > \mathbb{F} \left[ \bar{d} \right] \]

• Random d-regular graph: \( q_c = \frac{1}{d} \)
Erdös-Rényi graphs $G(n, \lambda/n)$

No cascade

Global cascades
Erdös-Rényi graphs $G(n, \lambda/n)$

- **Pivotal players**: giant component of the subgraph with degrees $\leq q^{-1}$
Phase transition continued

• What happens when $q$ is bigger than the cascade capacity?

• If the seed plays B forever, the model is monotone.

• In a finite graph, there is only one possible final state for the epidemic.
Locally tree-like

Independent computations on trees
Branching Process Approximation

- Local structure of G = random tree
- **Recursive Distributional Equation (RDE)** or:

\[
Y_i = \begin{cases} 
1 & \text{if infected from 'below'} \\
0 & \text{otherwise.}
\end{cases}
\]

\[
Y_i = 1 - (1 - \sigma_i) \mathbb{1} \left( \sum_{\ell \rightarrow i} Y_\ell \leq q d_i \right)
\]
Solving the RDE

\[ Y \overset{d}{=} 1 - (1 - \sigma) \mathbb{I} \left( \sum_{\ell=1}^{\hat{D}-1} Y_\ell \leq q \hat{D} \right) \]

\[ z = \mathbb{P}(Y = 0) \]

\[ h(z) = \sum_{s,r \geq s - \lfloor qs \rfloor} r \rho_s \binom{s}{r} z^r (1 - z)^{s-r} \]
Phase transition in one picture

\[ z^* = \max \{ z \in [0, 1], \lambda z^2 - (1 - \alpha) h(z) = 0 \} \]
Conclusion

• The locally tree-like structure gives the intuition for the solution...
• ... but not the proof!
• Configuration model + results of Janson and Luczak.
• Generic epidemic model which allows to retrieve basic results for random graphs...
• .... and new ones: contagion threshold, phase transitions.
Merci!

- Diffusion and Cascading Behavior in Random Networks. Available at http://www.di.ens.fr/~lelarge