Diffusions et cascades dans les graphes aléatoires

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Game-theoretic diffusion model...

- Both receive payoff $q$.
- Both receive payoff $1-q>q$.
- Both receive nothing.

Morris (2000)
...on a network.

- Everybody start with ICQ.
- Total payoff = sum of the payoffs with each neighbor.
- A fraction of the population is forced to talk.
- If $2(1-q) > 3q$, i.e. $2 > 5q$
Threshold Model

• State of agent $i$ is represented by
  $$X_i = \begin{cases} 
  0 & \text{if } \text{icq} \\
  1 & \text{if } \text{talk}
  \end{cases}$$

• Switch from $\text{icq}$ to $\text{talk}$ if:
  $$\sum_{j \sim i} X_j \geq q d_i$$
Morris, Contagion (2000)

• Does there exist a finite groupe of players such that their action under best response dynamics spreads contagiously everywhere?

• Contagion threshold: $q_c = \text{largest } q \text{ for which contagious dynamics are possible.}$

• Example: interaction on the line

\[ q_c = \frac{1}{2} \]
Another example: d-regular trees

\[ q_c = \frac{1}{d} \]
What happens for random graphs?

- Random graphs with given degree sequence introduced by Molloy and Reed (1995).

- Examples:
  - Random regular graphs.
  - Erdös-Rényi graphs, $G(n, \lambda/n)$.

- We are interested in large population asymptotics.

- $D$ is the asymptotic degree distribution.

\[
D \xrightarrow{d} \text{Poi}(\lambda)
\]
What happens for random graphs?

- Random graphs are locally tree-like.

\[ q_c = \sup \left\{ q : \mathbb{E} \left[ D(D - 1) \mathbb{1} \left( D < q^{-1} \right) \right] > \mathbb{E}[D] \right\} \]

- Random d-regular graph: \( q_c = \frac{1}{d} \)
Erdös-Rényi graphs $G(n, \lambda/n)$

No cascade

Global cascades
Erdös-Rényi graphs $G(n, \lambda/n)$

- **Pivotal players**: giant component of the subgraph with degrees $\leq q^{-1}$
Phase transition continued

- What happens when \( q \) is bigger than the cascade capacity?
- If the seed plays \( B \) forever, the model is monotone.
- In a finite graph, there is only one possible final state for the epidemic.
Locally tree-like

Independent computations on trees
Branching Process Approximation

- Local structure of $G$ = random tree
- Recursive Distributional Equation (RDE) or:

$$Y_i = \begin{cases} 
1 & \text{if infected from 'below'} \\
0 & \text{otherwise.}
\end{cases}$$

$$Y_i = 1 - (1 - \sigma_i) \mathbb{1} \left( \sum_{\ell \rightarrow i} Y_\ell \leq qd_i \right)$$
Solving the RDE

\[ Y \overset{d}{=} 1 - (1 - \sigma) \mathbb{I} \left( \sum_{\ell=1}^{\hat{D}-1} Y_{\ell} \leq q \hat{D} \right) \]

\[ z = \mathbb{P}(Y = 0) \]

\[ \lambda z^2 = (1 - \alpha) h(z) \]

\[ h(z) = \sum_{s, r \geq s - \lfloor qs \rfloor} r \rho_s \binom{s}{r} z^r (1 - z)^{s-r} \]
Phase transition in one picture

\[ z^* = \max \{ z \in [0, 1], \, \lambda z^2 - (1 - \alpha)h(z) = 0 \} \]
Conclusion

• The locally tree-like structure gives the intuition for the solution...
• ... but not the proof!
• Configuration model + results of Janson and Luczak.
• Generic epidemic model which allows to retrieve basic results for random graphs...
• .... and new ones: contagion threshold, phase transitions.
Merci!

- Diffusion and Cascading Behavior in Random Networks. Available at http://www.di.ens.fr/~lelarge