Signal-to-interference ratio in wireless communication networks

Paul Keeler,
Weierstrass Institute, Berlin

January 18, 2016
Communication networks have long inspired stochastic models

1909: Erlang discovered the Poisson process in a teletraffic context

1961: Gilbert proposed a stochastic and purely geometric model for wireless networks, considered birth of continuum percolation

1980s: Engineers developed mostly geometric models based on Poisson process for certain wireless networks.

1990s: Zozi and Pupolin did some pioneering (but often forgotten or overlooked) work on a signal-to-interference ratio (SIR) model.

2000s: Engineers returned to developing geometric models for ad hoc networks such as sensor networks and mobile ad hoc networks

2001: Baccelli and Błaszczyszyn, motivated by information theory, introduced a SIR coverage model based on stochastic geometry

2005/6: Dousse and friends introduced and studied SIR percolation, extending the original model of Gilbert.

2010: Andrews Baccelli, and Ganti adapted the SIR model for mobile or cellular phone networks.

Research field explodes: more and more engineering papers using stochastic geometry, motivated by denser phone networks and new technologies being deployed to handle YouTube etc traffic
Coverage models

- Classic models of wireless communication networks involve transmitters (and receivers) scattered over some region.
- Transmitter locations often form a homogeneous Poisson process $\Phi = \{X_i\}$ with density $\lambda$ on the plane $\mathbb{R}^2$.
- **Boolean model**: each transmitter $X_i \in \Phi$ has a circular transmission range (i.e., forms a disc).
- Each transmission radius can be a fixed constant or a radium variable.
- Very geometrical or intuitive model.
Figure: Green discs representing transmission ranges of transmitters in a wireless network. Picture by B. Błaszczyszyn.
1961: Birth of continuum percolation with Gilbert’s paper, which featured a model with fixed transmission discs.

Create a undirected graph: two points $X_1$ and $X_2$ are connected if their discs overlap the centres of each other.

Gilbert showed a critical threshold (for density or disc radius) existed for the infinite or “giant” component.
Another contender: Voronoi tessellation

\[ C_i = \{ y \in \mathbb{R}^2 : |y - x| \leq |y - X_i| : \forall X_i \in \Phi \} \]
Voronoi tessellation and Boolean Model are special cases of the general germ-grain coverage model. Specifically:

- **General germ-grain coverage model** \{ (X_i, C_i) \}
- \{ X_i \} are germs forming a point process \( \Phi \) on \( \mathbb{R}^2 \)
- \( C_i = C_i(X_i, \Phi) \) are grains consisting of (possibly dependent) random closed subsets of \( \mathbb{R}^2 \).
- Often Poisson point process forms the “germs”
- Discs or Voronoi cells form the “grains” for the Boolean model or Voronoi tessellation respectively

Baccelli and Błaszczyszyn (2001) introduced a new coverage model with dependent grains:

- **SIR coverage model** \{ (X_i, C_i) \}, where each \( C_i \) is a SIR cell formed by the SIR to be greater than some threshold \( \tau \)
- SIR coverage model is also called shot-noise coverage model in Chiu, Stoyan, Kendall, Mecke (2013).
Wireless network of transmitters \( \{X_i\} \) in \( \mathbb{R}^2 \) and a single (or “typical”) user located at the origin \( o \).

- \( R_i \) is the power received \( o \) of a signal originating from a transmitter at \( X_i \in \mathbb{R}^2 \).

- Performance metric is signal-to-interference-ratio (SIR) in relation to a base station at \( X_i \)

\[
\text{SIR}(X_i, o) = \frac{R_i}{\sum_{j \neq i} R_j}, \quad \text{SINR}(X_i, o) = \frac{R_i}{\sum_{j \neq i} R_j + N}
\]

(Constant \( N \geq 0 \) is negligible in high density networks, so consider SIR).

- One important quantity: coverage probability of a single is defined as

\[
\mathcal{P}_c(\tau) := \mathbb{P}(\max_i \text{SIR}(X_i, o) > \tau)
\]

where \( \tau \) is the (technology-dependent) SIR threshold.
Standard models

- Standard network model: base stations \( \{X_i\} \) positioned as a homogeneous Poisson process \( \Phi \) with density \( \lambda \) on \( \mathbb{R}^2 \)

- Standard propagation model:

\[
R_i = F_i \ell(|X_i|) = \frac{F_i}{|X_i|^\beta},
\]

- \( \ell(x) = (|x|)^{-\beta} \) is the path-loss/attenuation function for constant \( \beta > 2 \)

- Independent and identically-distributed (iid) random variables \( F_i \) represent propagation effects such as multi-path fading ie signals colliding with obstacles.

- \( F_i \) is often assumed to be exponentially or log-normally distributed

- Under this model, define the SINR cell as

\[
C_i = \left\{ y \in \mathbb{R}^2 : \frac{F_i \ell(|y - X_i|)}{\gamma \sum_{j \neq i} F_j \ell(|y - X_j|) + N} \geq \tau \right\}
\]

where \( 0 \leq \gamma \leq 1 \) is an “interference cancellation factor” – depends on the technology.
SIR cells for SIR threshold $\tau = 0.5$

Figure: $\lambda = 10$, $\ell(r) = (Kr)^{-\beta}$, $K = 8000$, $\beta = 3$. Picture by B. Błaszczyszyn
SIR cells for SIR threshold $\tau = 0.4$

Figure: $\lambda = 10$, $\ell(r) = (Kr)^{-\beta}$, $K = 8000$, $\beta = 3$. Picture by B. Błaszczyszyn
SIR cells for SIR threshold $\tau = 0.3$

Figure: $\lambda = 10$, $\ell(r) = (Kr)^{-\beta}$, $K = 8000$, $\beta = 3$. Picture by B. Błaszczyszyn
SIR cells for SIR threshold $\tau = 0.2$

Figure: $\lambda = 10$, $\ell(r) = (Kr)^{-\beta}$, $K = 8000$, $\beta = 3$. Picture by B. Błaszczyszyn
SIR cells for SIR threshold $\tau = 0.1$

Figure: $\lambda = 10$, $\ell(r) = (Kr)^{-\beta}$, $K = 8000$, $\beta = 3$. Picture by B. Błaszczyszyn
SINR coverage model links some classic stochastic geometry models

Lemma

A single user can be covered by (at maximum) $k$ transmitters in the entire network if $\frac{1}{(k+1)} \leq \tau < \frac{1}{k}$

- **SIR cell:**

  \[ C_i = \left\{ y \in \mathbb{R}^2 : \frac{F_i \ell(|y - X_i|)}{\gamma \sum_{j \neq i} F_j \ell(|y - X_j|) + N} \geq \tau \right\} \]

- When $\gamma = 0$ (no interference), the SINR cells are independent $\Rightarrow$ Boolean model approximations. Constant $F_i$ gives Gilbert’s disc model.

- When $N = 0$ (no noise) and $\beta \to \infty$, then SINR cells converge to Voronoi cells.

- Playing with $N \to 0$ and $\beta \to \infty$, the SINR model becomes the Johnson-Mehl model (for a simple Poisson birth process on $\mathbb{R}^2 \times [0, t]$).
Create a undirected graph: two points $X_1$ and $X_2$ are connected if at both points their respective SIRs exceed some threshold $\tau$.


Figure: Path-loss model: $\ell(r) = \min(1, r^{-3})$. Plot by B. Błaszczyszyn

Increasing density $\lambda$ may destroy the giant infinite component(s).
Tractable results for a single user: fading invariance

- Under standard Poisson model with singular path-loss function \( \ell(r) = r^{-\beta} \), Poisson mapping theorem says that the signal power values

\[
\Theta := \{ R_i \} = \left\{ \frac{F_i}{|X_i|^{\beta}} : X_i \in \Phi \right\}
\]

form an inhomogeneous Poisson point process on the positive real line with intensity measure

\[
Q([t, \infty)) := \lambda \pi \mathbb{E}(F^{2/\beta}) t^{-2/\beta}
\]

- Use exponential \( F \) and Laplace transforms, remove assumption

- To a single user, a deterministic (or even random non-Poisson) transmitter configuration \( \phi = \{ x_i \} \subset \mathbb{R}^2 \) can still appear Poisson

- Keeler, Ross, and Xia (2014) showed that as iid \( F_i \to 0 \) (in distribution), the point process of power values converges to an inhomogeneous Poisson point process

- For a single user, define the *SIR point process* on the positive half-line \( \mathbb{R}^+ \) as

\[
\{ Z_i \} := \left\{ \frac{R_i}{I - R_i} : R_i \in \Theta \right\}, \quad (1)
\]

where \( I = \sum R_i^{-1} \) is total interference in the network.
Tractable results for a single user: $k$-coverage probability

**Theorem (Błaszcyszyn, Karray, Keeler 2015)**

Under a homogeneous Poisson model with density $\lambda$ and singular path-loss function $\ell(r) = r^{-\beta}$, the $k$-coverage probability of a single user is

$$
P_{c}^{(k)}(\tau) = \sum_{n=k}^{\lfloor 1/\tau \rfloor} (-1)^{n-k} \binom{n-1}{k-1} \frac{(2/\beta)^{n-1}}{\tau^{n-(2n)/\beta}} \frac{C(\beta)^n}{[C(\beta)]^n} \mathcal{J}_{n,\beta}(\tau_n),$$

where $\tau_n = \frac{\tau}{1-(n-1)^2}$, $C(\beta) = \frac{2\pi}{\beta \sin(2\pi/\beta)}$, and for $x \geq 0$,

$$
\mathcal{J}_{n,\beta}(x) = \int_{[0,1]^{n-1}} \prod_{i=1}^{n-1} v_i^{i(2/\beta+1)-1} (1 - v_i)^{2/\beta} \prod_{i=1}^{n-1} [x + \eta_i(\{v_i\})] dv_1 \ldots dv_{n-1} \tag{2}
$$

where $\eta_i(\{v_i\}) := (1 - v_i) \prod_{k=i+1}^{n-1} v_k$.

$\mathcal{J}_{1,\beta}(x) = 1$ so for $\tau > 1$ gives $P_{c}^{(1)}(\tau) = 1/[\tau^{2/\beta} C(\beta)]$.

$\mathcal{J}_{2,\beta}(x)$ is a sum of two hypergeometric functions $\mathcal{F}_{1}^{2}$. 
Define the signal-to-total-interference ratio or *STIR process* on \((0, 1]\) as

\[
\{Z'_i\} := \left\{ \frac{Y_i^{-1}}{I} : Y_i \in \Theta \right\}, \quad Z_i = \frac{Z'_i}{1-Z'_i}, \quad Z'_i = \frac{Z_i}{1+Z_i}
\]

For parameters \(0 \leq \alpha < 1\) and \(\theta > -\alpha\), introduce a sequence of random variables \(\tilde{V}_1 = U_1, \quad \tilde{V}_i = (1-U_1)\ldots(1-U_{i-1})U_i, \quad i \geq 2\), where \(U_1, U_2\) are independent beta variables such that each \(U_i\) has \(B(1-\alpha, \theta + i\alpha)\) distribution. \(\sum_{i=1}^{\infty} \tilde{V}_i = 1\) with probability one.

Denote the decreasing order statistics of \(\{\tilde{V}_i\}\) by \(\{V_i\}\) such that \(V_1 \geq V_2 \geq \ldots\).

Define the two-parameter Poisson-Dirichlet distribution with parameters \(\alpha\) and \(\theta\), abbreviated as \(PD(\alpha, \theta)\), to be the distribution of \(\{V_i\}\).

See Pitman and Yor (1997) for interesting and useful results.

**Proposition (Błaszczyszyn and Keeler (2014))**

*The sequence \(\{Z'_i\}\) is equal in distribution to \(\{V_i\}\) for \(\alpha = 2/\beta\) and \(\theta = 0\). In other words, the STIR process \(\{Z'_i\}\) is a \(PD(2/\beta, 0)\) point process.*
Summary:

- For information theoretic reasons, Baccelli and Błaszczyszyn (2001) introduced the SIR coverage model that bridges some well-known models.
- Results exist on SIR (continuum) percolation by Dousse and friends.
- To a single user under *strong* and *independent* fading, networks appear Poisson or can be approximated with Poisson networks Keeler, Ross and Xia (2014).
- For single user and simple path-loss function, interesting SIR results exist, many observed independently in physics (e.g., Ruelle’s cascade model) and mathematics (e.g., work of Pitman and Yor).

Research directions:

- Conditions for SINR model or purely geometric models.
- Study multiple users/receivers and multi-hop scenarios.
- Dynamic situation with movement of transmitters and users.
- Introduce dependence into fading variables e.g., Gaussian fields.
- Use techniques from large-deviation theory to tackle the problem in the high density setting.
References:
Pitman and Yor, *The two-parameter Poisson-Dirichlet distribution derived from a stable subordinator*, 1997
Baccelli and Błaszczyszyn, *On a coverage process ranging from the Boolean model to the Poisson Voronoi tessellation, with applications to wireless communications*, 2001
Dousse, Baccelli and Thiran, *Impact of Interferences on Connectivity in Ad Hoc Networks*, 2005
Dousse, Franceschetti, Macris, Meester and Thiran, *Percolation in the Signal to Interference Ratio Graph*, 2006
Błaszczyszyn and Keeler, *Studying the SINR process of the typical user in Poisson networks by using its factorial moment measures*, 2015
Błaszczyszyn and Keeler *SINR in wireless networks and the two-parameter Poisson-Dirichlet process*, 2014