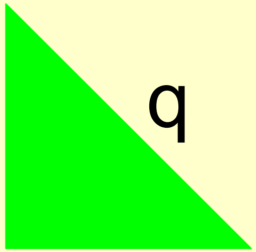
Greek Arguments

Proposition (Pythagoras):

$$\sqrt{2} \notin \mathbb{Q}$$

$$q^2 = 1^2 + 1^2 = 2$$



$$q = \frac{n}{d}, \quad (n,d) = 1$$

$$d^2 = 2n^2; \quad d = 0, n = 1 \pmod 2$$

$$d = 2d', \quad n^2 = 2d'^2; \quad n = 0 \pmod 2 !!!$$

Paradox (Zeno):

$$\sum_{N \in \mathbb{N}} \frac{1}{2^N} = 2 ?$$

Definition (Eudoxus):

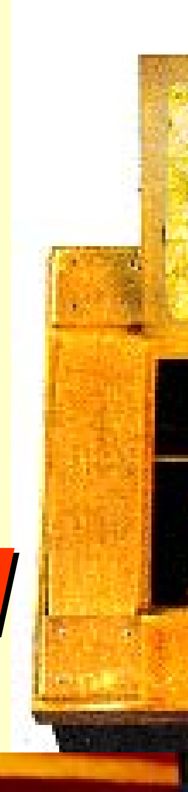
$$a \neq b \quad \text{if} \quad a < c < b, \quad \text{or} \quad b < c < a, \quad \text{for some finite} \quad c.$$

$$a = b, \quad \text{if} \quad a \neq c \quad \text{implies} \quad b \neq c, \quad \text{for all finite} \quad c.$$

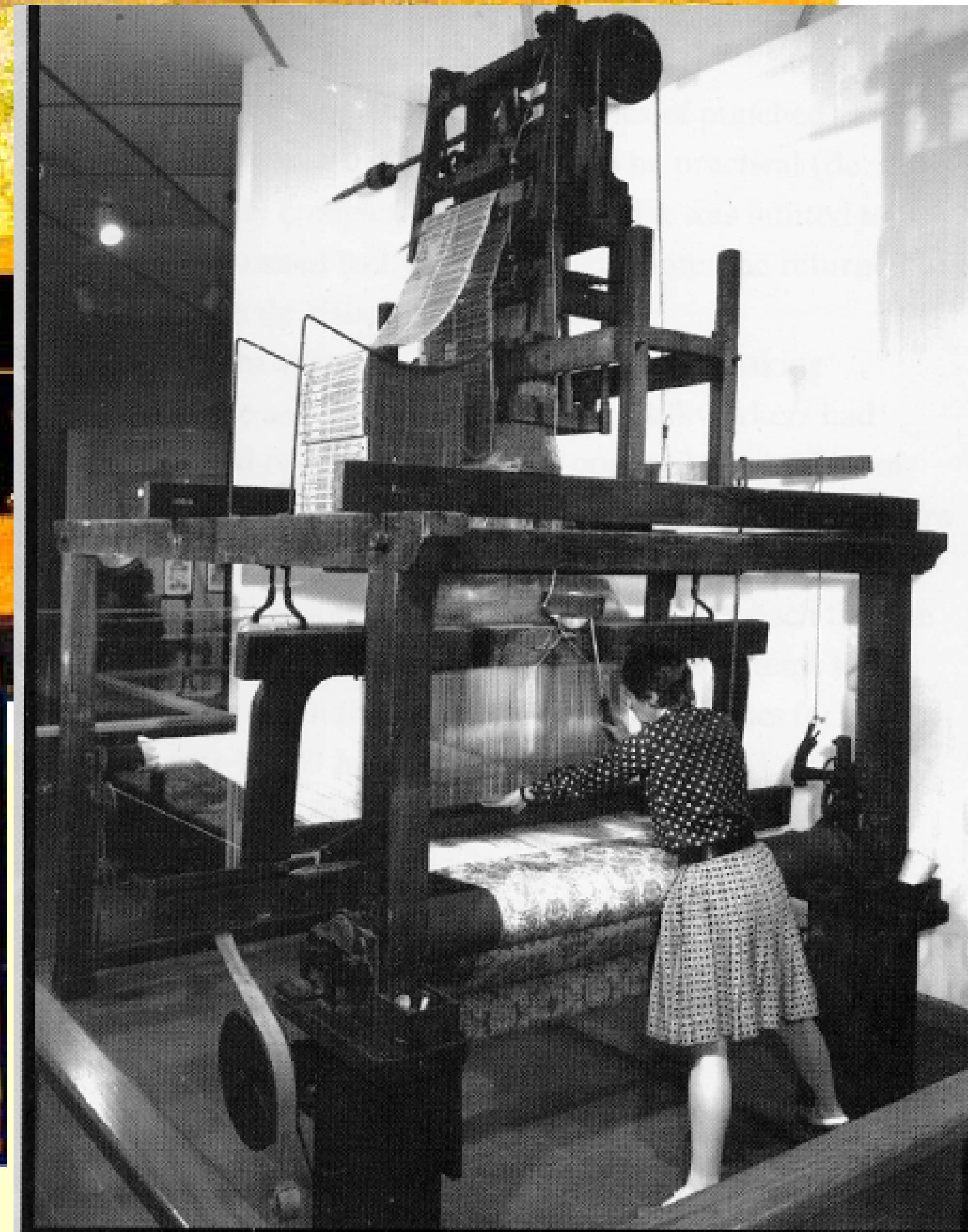
Paradox (liar):

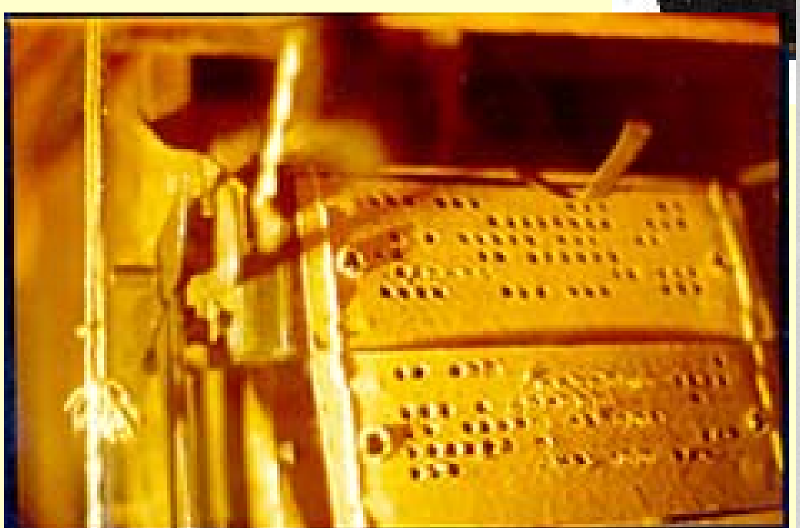
- *All Cretes are liars.*
- *Epimenides, the Crete, says: I lie!*

Pascal

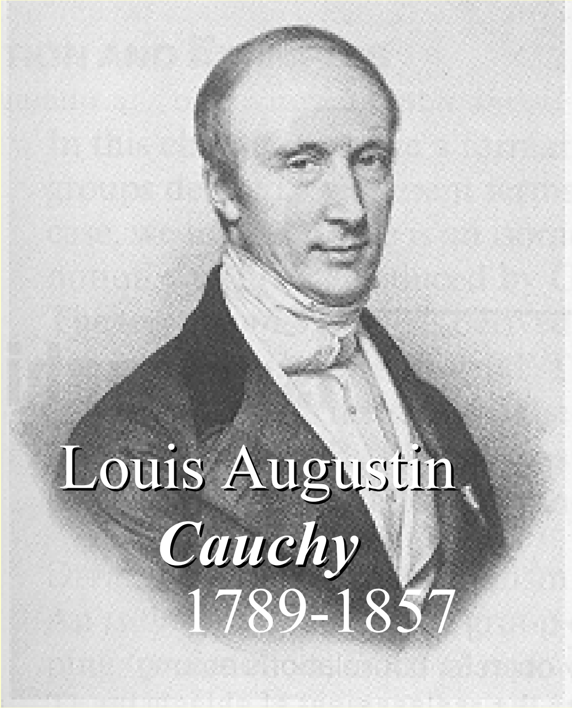


Jacquard



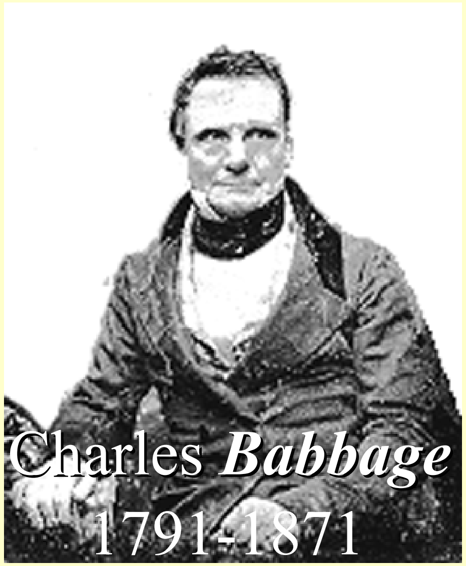


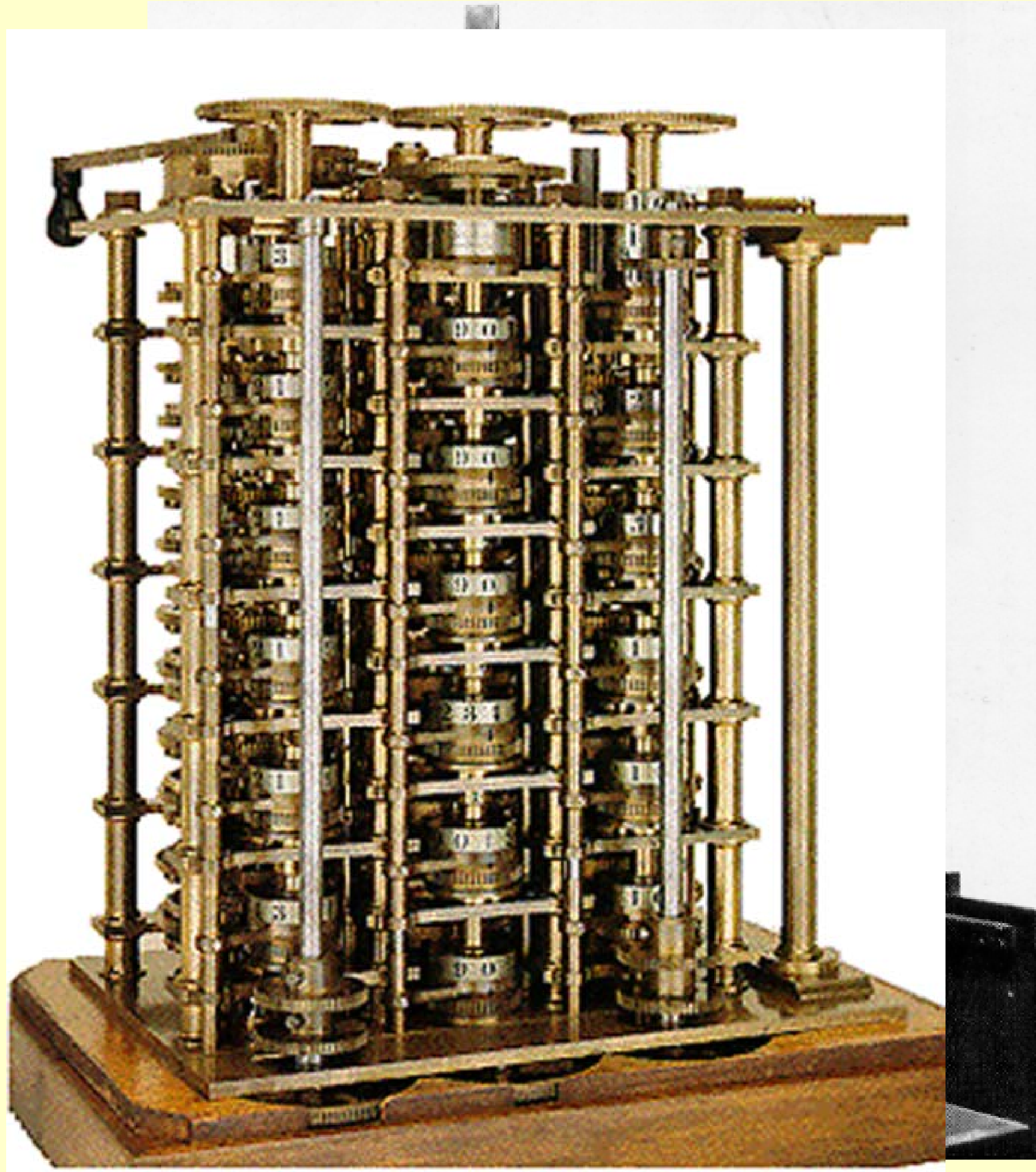
Real Constructions



1. Choose a representation: sequences, intervals, binary, continued fractions, …
2. Define the four operations +, -, x, / within the representation.
3. Show isomorphism between the representations: field **R** is the equivalence class.

Babbage





Transcendental

Theorem (d'Alembert, Gauss):

A degree n polynomial $P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$ has n complex roots: $P(z) = \prod_{0<k\le n} (z - r_k)$.

Liouville - 1851

$$\sum_{n\in\mathbb{N}} \frac{1}{2^{n!}} \notin \mathbb{A}$$

Hermitte - 1873

$$e \notin \mathbb{A}$$

Lindemann - 1882

$$\pi \notin \mathbb{A}$$

Set Theory

Theorem (Cantor):

1. $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = |\mathbb{A}|$

2. $|\mathbb{N}| < |\mathbb{R}| = |\mathbb{C}| = |\mathbb{N} \to \mathbb{N}|$

Cantor - 1874

$$N_0 = {}_2 b_0^0 b_1^0 b_2^0 b_3^0 \cdots$$
$$N_1 = {}_2 b_0^1 b_1^1 b_2^1 b_3^1 \cdots$$
$$N_2 = {}_2 b_0^2 b_1^2 b_2^2 b_3^2 \cdots$$
$$N_3 = {}_2 b_0^3 b_1^3 b_2^3 b_3^3 \cdots$$
$$\vdots$$

$$d_n = 1 - b_n^n$$
$$D = {}_2 d_0 \cdots d_n \cdots$$
$$D \notin \{N_0 \cdots N_n \cdots\}$$
$$|\mathbb{N}| < |{}_2\mathbb{Z}|$$

Paradox (Russel):

$$r = \{e : \ e \notin e\}$$
$$r \in r ?$$

Halting Problem

Hilbert - 1901

A(0,r,a)	=	a
A(1,0,a)	=	a+1
A(1,r+1,a)	=	A(a,r,a)
A(i+1,r,a)	=	A(i,r,A(1,r,a))

for $z > 2$ {

for $2 < n < z$ { for $x < z$ { for $y < x$ {

assert $x^n + y^n \neq z^n$

}}}}

$$27 \Rightarrow 41 \Rightarrow^{8} 91 \Rightarrow^{9} 593 \Rightarrow^{32} 2308 \Rightarrow^{5} 5 \Rightarrow 8 \Rightarrow^{4} 0$$

F(1)	=	0
F(2n)	=	F(n)
F(2n+1)	=	F(3n+2)

$$21 \Rightarrow 32 \Rightarrow 16 \Rightarrow 8 \Rightarrow 4 \Rightarrow 2 \Rightarrow 1 \Rightarrow 0$$

Computable Function

Thesis (Church,Turing):

A function is computable if it may be represented by a λ-expression or, equivalently, by a Turing machine.

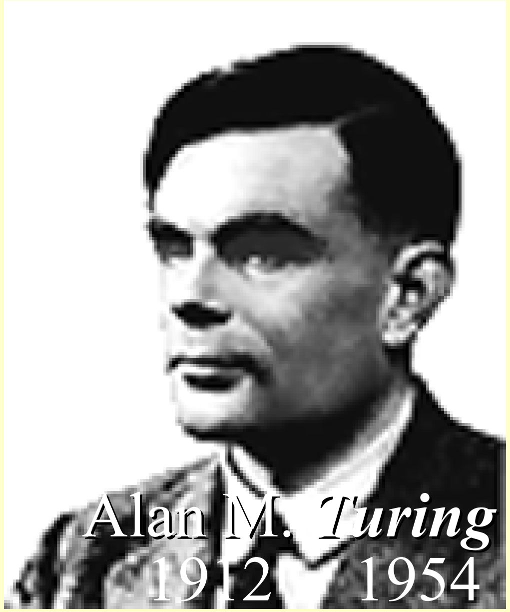
Equivalent:

- **Lisp, C, Java, CAML …**
- **I86, Alpha, DSC …**

Theorem (Cantor):

- The set of computable functions $\mathbf{N} \rightarrow \mathbf{N}$ is ***enumerable*.**
- The set of functions $\mathbf{N} \rightarrow \mathbf{N}$ is ***not enumerable*.**

Halting Problem



Theorem (Turing 1936):

The Halting problem is undecidable.

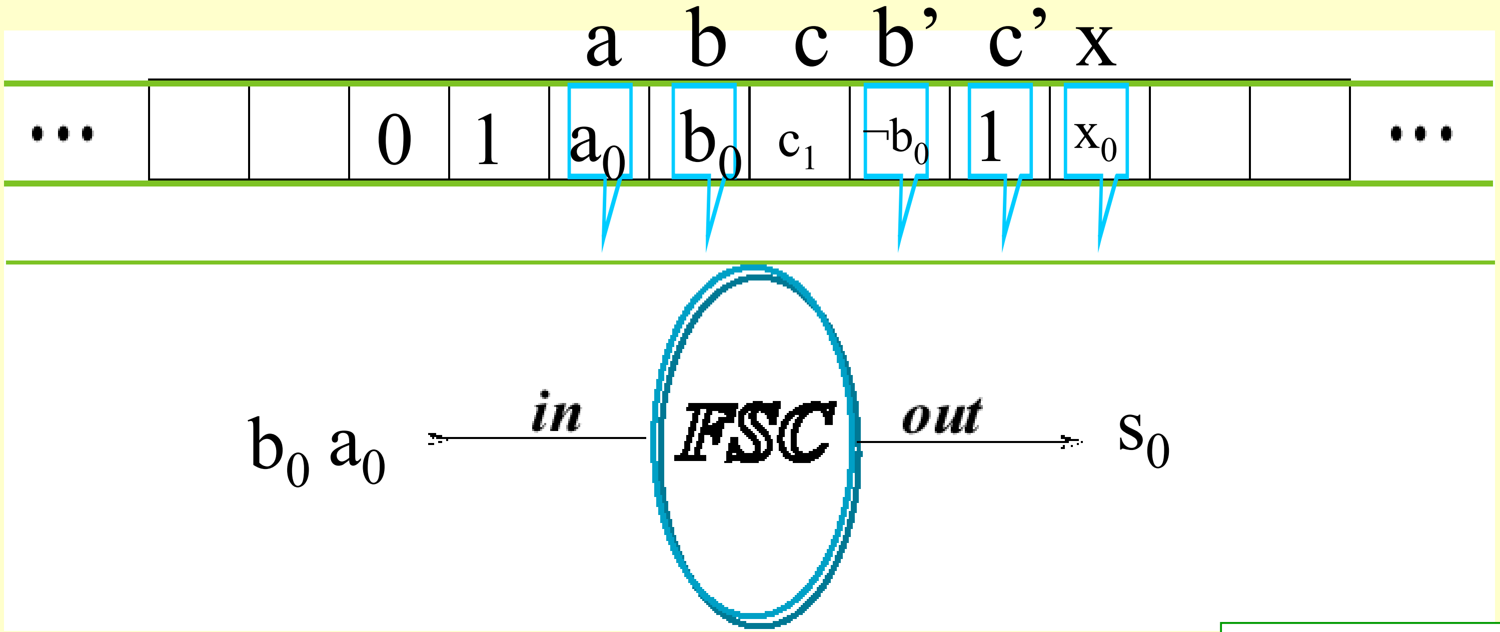
(def *Halts*(f x) ???)

(def *Loop*() (*Loop*))

(def *Contradicts*(f)
  (if (*Halts* f f)
    (*Loop*)
    0))

(*Halts* *Contradicts* *Contradicts*)

Turing Machine



in ⇨ a

in ⇨ b

mux(b,0,1) ⇨ b'

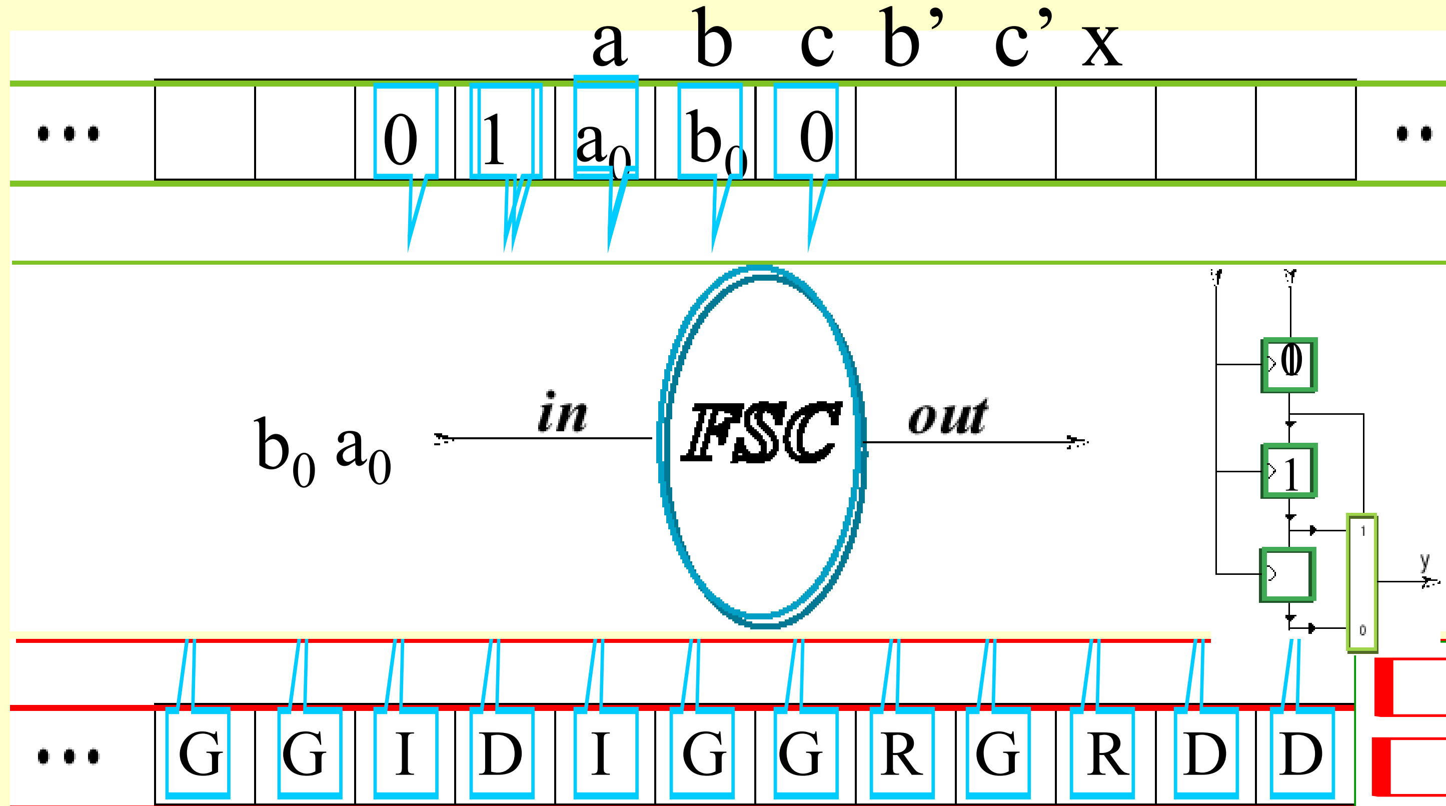
mux(c,0,1) ⇨ c'

mux(a,b,b') ⇨ x

mux(x,c,c') ⇨ out

mux(x,b,c) ⇨ c

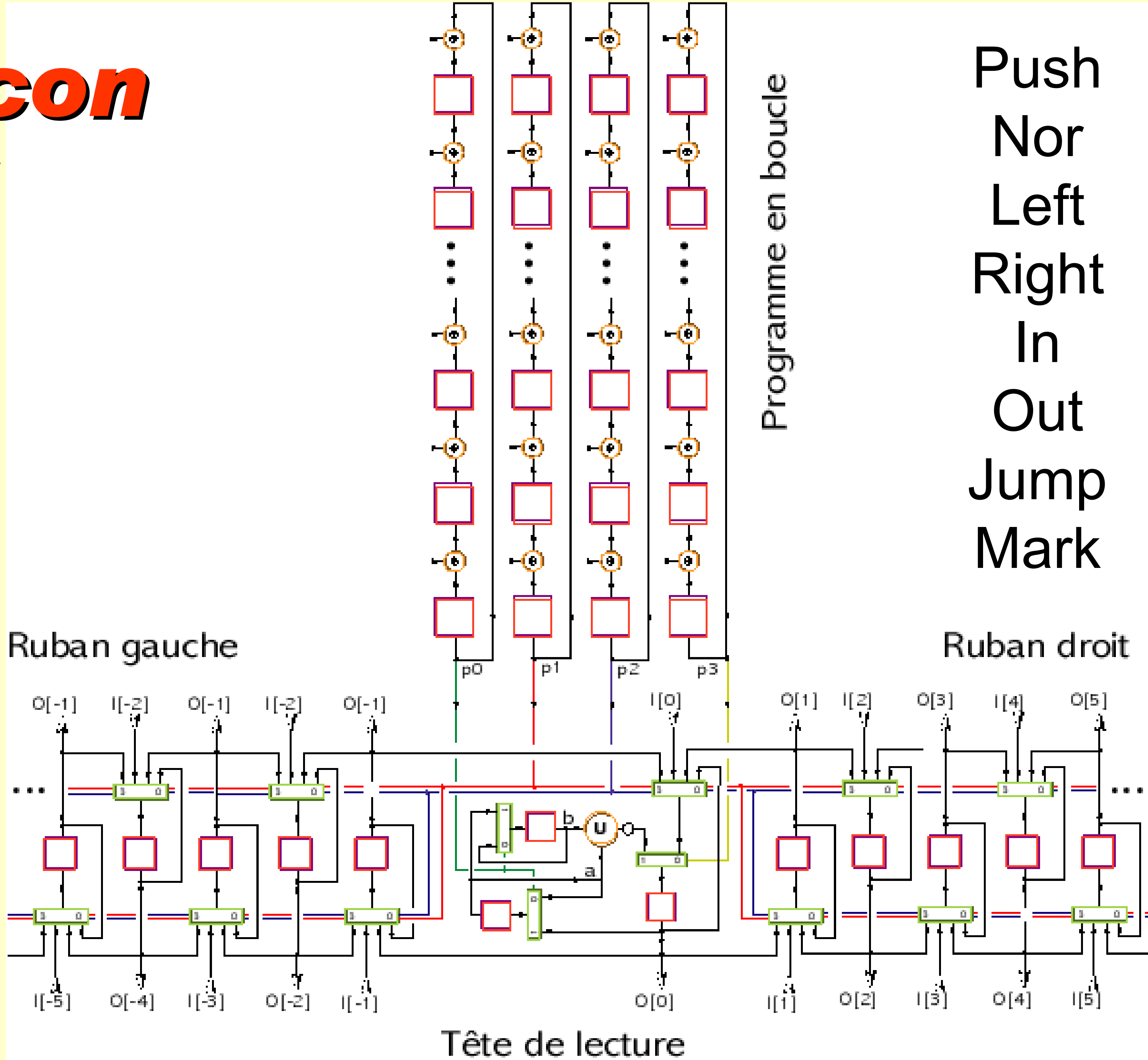
Universal Turing Machine



in⇨a
in⇨b
mux(b,0,1)⇨b’
mux(c,0,1)⇨c’
mux(a,b,b’)⇨x
mux(x,c,c’)⇨out
mux(x,b,c)⇨c

GGI
DI
GGRGRDD DRDDW
GGGGRGRDDDDRDDW
GRGGRGRDDDW
DRGGRDRO
GRGRDDRGW

Silicon TM


Push
Nor
Left
Right
In
Out
Jump
Mark
Programme en boucle
Ruban gauche
Ruban droit
p0
p1
p2
p3
O[-1]
I[-2]
O[-1]
I[-2]
O[-1]
I[0]
O[1]
I[2]
O[3]
I[4]
O[5]
b
a
U
I[-5]
O[-4]
I[-3]
O[-2]
I[-1]
O[0]
I[1]
O[2]
I[3]
O[4]
I[5]
Tête de lecture

Computable Real

Let $Fe_z=1$ if

$\exists n>2, \exists x<y<z: x^n + y^n = z^n$,

else $Fe_z=0$. Let

$i(N)=\sum_{z\leq N} \frac{Fe_z}{2^z}, s(N) = i(N) + \frac{1}{2^N}$,

$Fe(N)=[i(N),s(N)]$, and $Fe=Fe(\infty)$.

$$\varphi = \frac{1+\sqrt{5}}{2}$$

$$\varphi = 1 + \frac{1}{\varphi}$$

$$\varphi = /1\ 1\ 1\ \cdots/$$

$$\varphi = \bigcap_{0\leq N} \phi(N).$$

$$\phi(N) = [\frac{F_{2N+2}}{F_{2N+1}}, \frac{F_{2N+1}}{F_{2N}}]$$

$$\frac{1}{1} < \frac{3}{2} < \frac{8}{5} < \frac{21}{13} < \cdots < \varphi < \cdots \quad 8 \quad 3 \quad 1 \quad 0$$

$$[1,\infty] \supset [\frac{3}{2}, 2] \supset [\frac{8}{5}, \frac{5}{3}] \supset [\frac{21}{13}, \frac{13}{8}] \supset \cdots \supset [\varphi, \varphi]$$

A computable real $r \in \mathbb{R}$ is the limit

$$r = r(\infty) = \bigcap_{0\leq N} r(N)$$

of a sequence $r(0)\cdots r(n)\cdots$ of intervals

$r(n) = [i(n), s(n)]$ such that:

(i) $i(n), s(n) \in \mathbb{Q}$

(ii) $r(0) \supseteq \cdots \supseteq r(n) \supseteq r(n+1) \supseteq \cdots$

(iii) $0 = \lim_{n\to\infty} | s(n) - r(n) |$

(iv) Function $r \in \mathbb{N} \to [\mathbb{Q}, \mathbb{Q}]$ is computable.

(i) The number Fe is a computable real $Fe \in \mathbb{R}$.

(ii) $Fe = 0$ if and only if Wiles is right!

Halting Reduction

Proposition:

Testing if an arbitrary computable real is equal to zero, is ***not*** a computable operation.

Halting reduction:

1. Let $w = {}_2w_0 \ldots w_t \ldots$ be a (binary) write command, effectively derived from the Turing machine.
2. Attach a Zeno counter to the machine, so that: $c(0)=0$, $c(t+1)=c(t)$ when $w_t=0$, and $c(t+1)=1+c(t)/2$ when $w_t=1$.
3. Number $c=c(1/0)$ is a computable real, such that:
   1. $c=0$ if and only if $w=0$;
   2. $c<2$ if and only if w is an integer;
   3. $c=2$ if and only if the computation does not terminate.

Not Computable

Theorem (Rice):

All computable functions over the real numbers are continuous.

Not computable:

1. The sign of r.
2. The integer part of r.
3. The fractional part of r.
4. The decimal representation of r.
5. The binary representation of r.
6. Any non-redundant representation of r.

Computable Real

Proposition:

The computable real numbers **R** form a field.

Restrictions:

$$0 = (-a) + a, \quad a \neq \infty, a \neq \frac{0}{0}$$

$$1 = \frac{1}{a} \times a, \quad a \neq 0, a \neq \infty, a \neq \frac{0}{0}$$

Proof:

Define the 4 operations: interval arithmetic.