Simple Functional Encryption Schemes for Inner Products

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1. Overview of the results
   - What is Functional Encryption?
   - Inner Product functionality
   - What does simple mean? What do we achieve?
2. The Framework
   - Overview of the framework
   - Example
   - Proof of security
   - Generalization
3. Work in progress
   - What is there left to do?
   - Thank you!

Brief history

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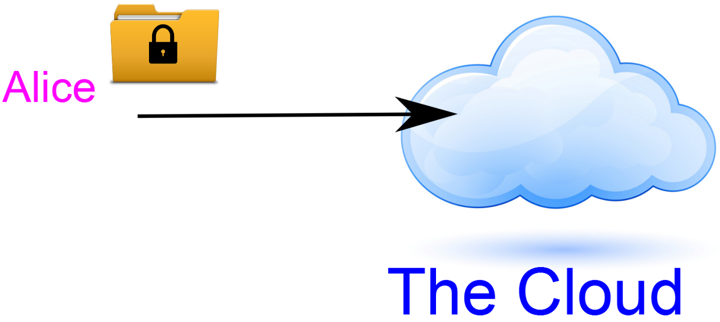
Enables keys that give partial information.

Motivation



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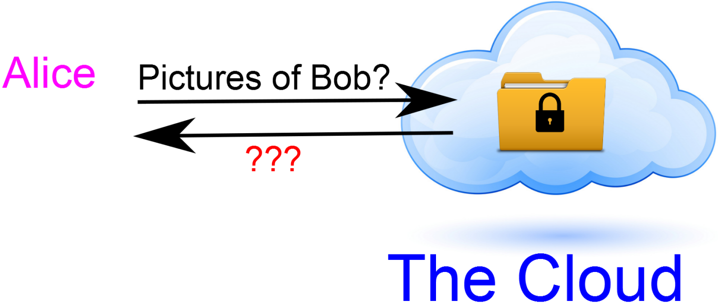

Formal definition

$$\text{Functionality } \mathcal{F} : \mathcal{K} \times \mathcal{X} \to \mathcal{M}$$
$$(k, x) \mapsto \mathcal{F}(k, x)$$

Secret key for k : $\mathbf{sk}_k \leftarrow msk$

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Alice gets Bob's pictures in her data.

Security

Intuitively:
$\mathbf{sk}_k$ doesn't leak any more information than $\mathcal{F}(k, x)$

Even if there are collusions !
$\mathbf{sk}_k$ and $\mathbf{sk}'_k$ don't leak more information than $\mathcal{F}(k, x)$ and $\mathcal{F}(k', x)$

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The server doesn't access Alice's private data other than needed.

Even if there are collusions !

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Pictures of Jean and pictures of Jacques don't make pictures of Jean-Jacques.

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- Designing efficient functional encryption for access control...

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- This work: figuring out what we can do with simple assumption

Inner Product functionality

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$$(\mathbf{y}, \mathbf{x}) \to < \mathbf{x}, \mathbf{y} >$$

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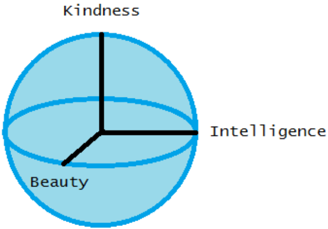
Motivation example: Online dating system



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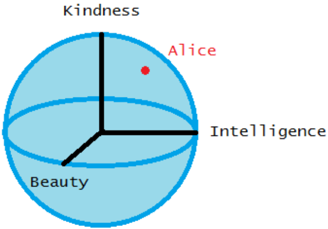

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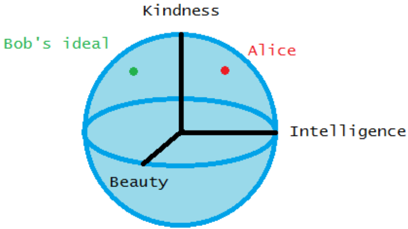

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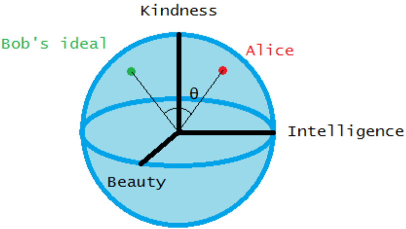

Properties

Inner product is very interesting:

- lots of applications
- easy to compute - only need additions if one vector is known
- still non-trivial: $|\mathcal{K}|$ is exponential in ℓ
- theoretically interesting problem - enables any computation in NC^0

Inherent security limitation

$< \mathbf{x}, \mathbf{y} >$ gives a lot of information about $\mathbf{x}$
ℓ well chosen secret keys reveals everything

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Additive ElGamal, based on Decisional Diffie-Hellman (DDH) assumption Lattice based Public Key Encryption scheme, based on the Learning With Errors (LWE) assumption

Efficient

Ciphertext size is $\ell + 1$ elements
Key size is 1 element

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This is really close to information theoretical optimal for correctness

Selective IND-CPA security

The resulting scheme is secure under selective chosen plaintext attacks

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- $\mathcal{A}$ receives $\{\mathbf{sk}_\mathbf{y}\}$
- $\mathcal{A}$ guesses b'

How to apply our framework?

Our framework is easy to instantiate:

Pick a good Public Key Encryption scheme

requires structural properties stated later

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And it's done ! (and safe !)

The additively homomorphic ElGamal public key encryption scheme

$$\begin{aligned} \text{Public parameters} &: p, \mathcal{G}, g \\ \text{Secret key} &: s \\ \text{Public key} &: g^s \\ \text{Ciphertext for } m &: (g^r, g^{rs} g^m) \end{aligned}$$

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Correctness

$$\frac{g^{rs} g^{m}}{(g^{r})^{s}} = g^{m}$$

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Now onto correctness...

Using homomorphism to decrypt the inner product

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$$\prod_i (g^{rs_i} g^{x_i})^{y_i} = (g^r)^{\sum_i y_i s_i} g^{\sum_i x_i y_i}$$

First trick

You can change easily the basis used in the whole scheme

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Given a matrix $\mathbf{P}$, a ciphertext $\mathbf{ct}_{\vec{x}}$, and the master secret key $\vec{s}$

You can generate a new ciphertext $\mathbf{ct}_{\mathbf{P}\vec{x}}$ using the homomorphism, and a new master secret key $\mathbf{P}\vec{s}$

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Indeed, $\mathcal{A}$ can only ask secret keys for $\vec{y}$ such that $< \vec{y}, \vec{x_1} - \vec{x_0} >= 0$

So a basis having $\vec{x_1} - \vec{x_0}$ as first vector verifies this

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$\square$

What properties do we need?

2 properties:

Randomness Reuse $g^r, g^{r\vec{s}} g^{\vec{x}}$ is safe

In this case, it is an instance of ElGamal with secret keys r and randomnesses s_i

Homomorphism of message and key

$$g^{rs_1+x_1} g^{rs_2+x_2} = g^{r(s_1+s_2)+(x_1+x_2)}$$

How to generalize?

To generalize, replace:

- $s \rightarrow sk$
- $g^s \rightarrow pk$
- $g^r \rightarrow C(r)$
- $g^{rs+x} \rightarrow Enc(pk, x; r)$

the LWE assumption

$$\begin{aligned} \text{Public parameters} &: q, n, m, \mathbf{A} \in \mathbb{Z}_q^{m\times n} \\ \text{Secret key} &: \vec{s} \in \mathbb{Z}_q^m \\ \text{Public key} &: \mathbf{A}\vec{s} + \vec{e} \in \mathbb{Z}_q^m & \vec{e} \leftarrow \chi^m \\ \text{Ciphertext for } x &: (\vec{r}\mathbf{A}, \vec{r}(\mathbf{A}\vec{s} + \vec{e}) + \lfloor \frac{q}{2} \rceil x) & \vec{r} \leftarrow \{0,1\}^{1\times m} \end{aligned}$$

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Advantages

- Avoid small space restriction of additive ElGamal
- Post-quantum

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Inconvenients

Noisy setup - proof is more subtle

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- Adaptive security
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- Function privacy
  In private setting - $\mathcal{A}$ doesn't know what his key compute
- Find other interesting fitting PKE
  Paillier-like cryptosystem would solve the small space restrictions
- etc.

Thank you!

Thank you for your attention!