Stochastic optimization:
Beyond stochastic gradients and convexity

Part I

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Context

Machine learning for large-scale data

- Large-scale supervised machine learning: large $d$, large $n$
  - $d$: dimension of each observation (input) or number of parameters
  - $n$: number of observations
- Examples: computer vision, advertising, bioinformatics, etc.
Search engines - Advertising - Marketing

Tour de France 2014  Translate this page
www.letour.fr
Tour de France 2014 - Site officiel de la célèbre course cycliste Le Tour de France. Contient les itinéraires, coureurs, équipes et les infos des Tours passés.

Tour de France (cyclisme) — Wikipédia  Translate this page
fr.wikipedia.org/wiki/Tour_de_France_(cyclisme)
Le Tour de France est une compétition cycliste par étapes créée en 1903 par Henri Desgrange et Géo Lefèvre, chef de la rubrique cyclisme du journal L'Auto.
Histoire · Médialisation du ... · Équipes et participation
Visual object recognition
Context

Machine learning for large-scale data

• Large-scale supervised machine learning: large $d$, large $n$
  – $d$: dimension of each observation (input), or number of parameters
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• Examples: computer vision, advertising, bioinformatics, etc.

• Ideal running-time complexity: $O(dn)$
Machine learning for large-scale data

- **Large-scale supervised machine learning**: large $d$, large $n$
  - $d$: dimension of each observation (input), or number of parameters
  - $n$: number of observations

- **Examples**: computer vision, advertising, bioinformatics, etc.

- **Ideal running-time complexity**: $O(dn)$

- **Going back to simple methods**
  - Stochastic gradient methods (Robbins and Monro, 1951)

- **Goal**: Present recent progress
Outline

1. Introduction/motivation: Supervised machine learning
   - Optimization of finite sums
   - Existing optimization methods for finite sums

2. Convex finite-sum problems
   - Linearly-convergent stochastic gradient method
   - SAG, SAGA, SVRG, SDCA, MISO, etc.
   - From lazy gradient evaluations to variance reduction

3. Non-convex problems

4. Parallel and distributed settings

5. Perspectives
References

• Textbooks and tutorials
  – Nesterov (2004): *Introductory lectures on convex optimization*
  – Bertsekas (2016): *Nonlinear programming*
  – Bottou et al. (2016): *Optimization methods for large-scale machine learning*

• Research papers
  – See end of slides
  – Slides available at www.ens.fr/~fbach/
Parametric supervised machine learning

- **Data**: $n$ observations $(x_i, y_i) \in X \times Y$, $i = 1, \ldots, n$

- **Prediction function** $h(x, \theta) \in \mathbb{R}$ parameterized by $\theta \in \mathbb{R}^d$
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- **Motivating examples**
  - Linear predictions: $h(x, \theta) = \theta^\top \Phi(x)$ with features $\Phi(x) \in \mathbb{R}^d$
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• **Motivating examples**
  
  – Linear predictions: \( h(x, \theta) = \theta^\top \Phi(x) \) with features \( \Phi(x) \in \mathbb{R}^d \)
  
  – Neural networks: \( h(x, \theta) = \theta_m^\top \sigma(\theta_{m-1}^\top \sigma(\cdots \theta_2^\top \sigma(\theta_1^\top x))) \)
Parametric supervised machine learning

- **Data:** $n$ observations $(x_i, y_i) \in \mathcal{X} \times \mathcal{Y}$, $i = 1, \ldots, n$

- **Prediction function** $h(x, \theta) \in \mathbb{R}$ parameterized by $\theta \in \mathbb{R}^d$

- **(regularized) empirical risk minimization:** find $\hat{\theta}$ solution of

  $$
  \min_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} \ell(y_i, h(x_i, \theta)) + \lambda \Omega(\theta)
  $$

  data fitting term + regularizer
Usual losses

- **Regression**: $y \in \mathbb{R}$
  - Quadratic loss $\ell(y, h(x, \theta)) = \frac{1}{2}(y - h(x, \theta))^2$
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  - Logistic loss \( \ell(y, h(x, \theta)) = \log(1 + \exp(-yh(x, \theta))) \)
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- **Structured prediction**
  - Complex outputs \( y \) (\( k \) classes/labels, graphs, trees, or \( \{0, 1\}^k \), etc.)
  - Prediction function \( h(x, \theta) \in \mathbb{R}^k \)
  - Conditional random fields (Lafferty et al., 2001)
  - Max-margin (Taskar et al., 2003; Tsochantaridis et al., 2005)
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data fitting term + regularizer
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  data fitting term + regularizer

- **Optimization:** optimization of regularized risk training cost
Parametric supervised machine learning

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\( \ell \) data fitting term + regularizer

- **Optimization**: optimization of regularized risk

- **Statistics**: guarantees on \( \mathbb{E}_{p(x,y)} \ell(y, h(x, \theta)) \)
Smoothness and (strong) convexity

A function \( g : \mathbb{R}^d \rightarrow \mathbb{R} \) is \( L \)-smooth if and only if it is twice differentiable and

\[
\forall \theta \in \mathbb{R}^d, \left| \text{eigenvalues}[g''(\theta)] \right| \leq L
\]
Smoothness and (strong) convexity

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- Machine learning
  - with $g(\theta) = \frac{1}{n} \sum_{i=1}^{n} \ell(y_i, h(x_i, \theta))$
  - Smooth prediction function $\theta \mapsto h(x_i, \theta) + \text{smooth loss}$
**Smoothness and (strong) convexity**

- A twice differentiable function $g : \mathbb{R}^d \to \mathbb{R}$ is convex if and only if
  \[
  \forall \theta \in \mathbb{R}^d, \text{ eigenvalues} \left[ g''(\theta) \right] \geq 0
  \]
Smoothness and (strong) convexity

• A twice differentiable function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is $\mu$-strongly convex if and only if

$$\forall \theta \in \mathbb{R}^d, \text{ eigenvalues } [g''(\theta)] \geq \mu$$
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- A twice differentiable function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is $\mu$-strongly convex if and only if
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- Condition number $\kappa = L/\mu \geq 1$

(small $\kappa = L/\mu$)  (large $\kappa = L/\mu$)
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- Convexity in machine learning
  - With $g(\theta) = \frac{1}{n} \sum_{i=1}^{n} \ell(y_i, h(x_i, \theta))$
  - Convex loss and linear predictions $h(x, \theta) = \theta^\top \Phi(x)$
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- Relevance of convex optimization
  - Easier design and analysis of algorithms
  - Global minimum vs. local minimum vs. stationary points
  - Gradient-based algorithms only need convexity for their analysis
Smoothness and (strong) convexity

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  \[ \forall \theta \in \mathbb{R}^d, \text{eigenvalues} [g''(\theta)] \geq \mu \]

- **Strong convexity in machine learning**
  - With \( g(\theta) = \frac{1}{n} \sum_{i=1}^{n} \ell(y_i, h(x_i, \theta)) \)
  - Strongly convex loss and linear predictions \( h(x, \theta) = \theta^\top \Phi(x) \)
Smoothness and (strong) convexity

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• **Strong convexity in machine learning**

  – With $g(\theta) = \frac{1}{n} \sum_{i=1}^{n} \ell(y_i, h(x_i, \theta))$
  
  – Strongly convex loss and linear predictions $h(x, \theta) = \theta^\top \Phi(x)$
  
  – Invertible covariance matrix $\frac{1}{n} \sum_{i=1}^{n} \Phi(x_i)\Phi(x_i)^\top \Rightarrow n \geq d$
  
  – Even when $\mu > 0$, $\mu$ may be arbitrarily small!
Smoothness and (strong) convexity

- A twice differentiable function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is $\mu$-strongly convex if and only if
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- **Strong** convexity in machine learning
  - With $g(\theta) = \frac{1}{n} \sum_{i=1}^{n} \ell(y_i, h(x_i, \theta))$
  - Strongly convex loss and linear predictions $h(x, \theta) = \theta^\top \Phi(x)$
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- Adding regularization by $\frac{\mu}{2} \|\theta\|^2$
  - creates additional bias unless $\mu$ is small, but reduces variance
  - Typically $L/\sqrt{n} \geq \mu \geq L/n$
Iterative methods for minimizing smooth functions

- **Assumption:** \( g \) convex and \( L \)-smooth on \( \mathbb{R}^d \)

- **Gradient descent:**
  \[
  \theta_t = \theta_{t-1} - \gamma_t g'(\theta_{t-1})
  \]

\[
 g(\theta_t) - g(\theta^*) \leq O\left(\frac{1}{t}\right)
\]

\[
 g(\theta_t) - g(\theta^*) \leq O\left(e^{-t\frac{\mu}{L}}\right) = O\left(e^{-t/\kappa}\right)
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(small \( \kappa = \frac{L}{\mu} \))

(large \( \kappa = \frac{L}{\mu} \))
Iterative methods for minimizing smooth functions

• **Assumption:** $g$ convex and $L$-smooth on $\mathbb{R}^d$

• **Gradient descent:** $\theta_t = \theta_{t-1} - \gamma_t g'(\theta_{t-1})$

$$g(\theta_t) - g(\theta_*) \leq O(1/t)$$
$$g(\theta_t) - g(\theta_*) \leq O((1-\mu/L)^t) = O(e^{-t(\mu/L)})$$ if $\mu$-strongly convex

(small $\kappa = L/\mu$) (large $\kappa = L/\mu$)
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- **Gradient descent**: $\theta_t = \theta_{t-1} - \gamma_t g'(\theta_{t-1})$
  - $O(1/t)$ convergence rate for convex functions
  - $O(e^{-t/\kappa})$ *linear* if strongly-convex

- **Newton method**: $\theta_t = \theta_{t-1} - g''(\theta_{t-1})^{-1}g'(\theta_{t-1})$
  - $O(e^{-\rho^2 t})$ *quadratic rate*
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- **Key insights for machine learning** (Bottou and Bousquet, 2008)
  1. No need to optimize below statistical error
  2. Cost functions are averages
  3. Testing error is more important than training error
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Stochastic gradient descent (SGD) for finite sums

$$\min_{\theta \in \mathbb{R}^d} g(\theta) = \frac{1}{n} \sum_{i=1}^{n} f_i(\theta)$$

- **Iteration:** $\theta_t = \theta_{t-1} - \gamma_t f'_{i(t)}(\theta_{t-1})$
  - Sampling with replacement: $i(t)$ random element of $\{1, \ldots, n\}$
  - Polyak-Ruppert averaging: $\bar{\theta}_t = \frac{1}{t+1} \sum_{u=0}^{t} \theta_u$
Stochastic gradient descent (SGD) for finite sums

\[ \min_{\theta \in \mathbb{R}^d} g(\theta) = \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) \]

- **Iteration:** \( \theta_t = \theta_{t-1} - \gamma_t f'_{i(t)}(\theta_{t-1}) \)
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  - Polyak-Ruppert averaging: \( \bar{\theta}_t = \frac{1}{t+1} \sum_{u=0}^{t} \theta_u \)

- **Convergence rate** if each \( f_i \) is convex \( L \)-smooth and \( g \) \( \mu \)-strongly-convex:
  \[ \mathbb{E} g(\bar{\theta}_t) - g(\theta^*) \leq \begin{cases} O\left(\frac{1}{\sqrt{t}}\right) & \text{if } \gamma_t = \frac{1}{(L \sqrt{t})} \\ O\left(\frac{L}{(\mu t)}\right) = O\left(\frac{\kappa}{t}\right) & \text{if } \gamma_t = \frac{1}{(\mu t)} \end{cases} \]

- No adaptivity to strong-convexity in general
- Adaptivity with self-concordance assumption (Bach, 2014)
- Running-time complexity: \( O(d \cdot \kappa/\varepsilon) \)
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   - Linearly-convergent stochastic gradient method
   - SAG, SAGA, SVRG, SDCA, etc.
   - From lazy gradient evaluations to variance reduction

3. **Non-convex problems**

4. **Parallel and distributed settings**

5. **Perspectives**
Stochastic vs. deterministic methods

• Minimizing \( g(\theta) = \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) \) with \( f_i(\theta) = \ell(y_i, h(x_i, \theta)) + \lambda \Omega(\theta) \)
Stochastic vs. deterministic methods

- Minimizing $g(\theta) = \frac{1}{n} \sum_{i=1}^{n} f_i(\theta)$ with $f_i(\theta) = \ell(y_i, h(x_i, \theta)) + \lambda \Omega(\theta)$

- Batch gradient descent: $\theta_t = \theta_{t-1} - \gamma_t g'(\theta_{t-1}) = \theta_{t-1} - \frac{\gamma_t}{n} \sum_{i=1}^{n} f'_i(\theta_{t-1})$
  
  - Linear (e.g., exponential) convergence rate in $O(e^{-t/\kappa})$
  - Iteration complexity is linear in $n$
**Stochastic vs. deterministic methods**

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• Stochastic gradient descent: $\theta_t = \theta_{t-1} - \gamma_t f_{i(t)}'(\theta_{t-1})$
  
  – Sampling with replacement: $i(t)$ random element of $\{1, \ldots, n\}$
  – Convergence rate in $O(\kappa/t)$
  – Iteration complexity is independent of $n$
Stochastic vs. deterministic methods

- Minimizing $g(\theta) = \frac{1}{n} \sum_{i=1}^{n} f_i(\theta)$ with $f_i(\theta) = \ell(y_i, h(x_i, \theta)) + \lambda \Omega(\theta)$

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- Stochastic gradient descent: $\theta_t = \theta_{t-1} - \gamma_t f'_{i(t)}(\theta_{t-1})$
Stochastic vs. deterministic methods

- **Goal** = best of both worlds: Linear rate with $O(d)$ iteration cost
  Simple choice of step size
Stochastic vs. deterministic methods

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  Simple choice of step size
Accelerating gradient methods - Related work

- **Generic acceleration** (Nesterov, 1983, 2004)

\[
\theta_t = \eta_{t-1} - \gamma_t g'(\eta_{t-1}) \quad \text{and} \quad \eta_t = \theta_t + \delta_t(\theta_t - \theta_{t-1})
\]
Accelerating gradient methods - Related work

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  \[ \theta_t = \eta_{t-1} - \gamma_t g'(\eta_{t-1}) \quad \text{and} \quad \eta_t = \theta_t + \delta_t (\theta_t - \theta_{t-1}) \]

  - Good choice of momentum term \( \delta_t \in [0, 1) \)

  \[ g(\theta_t) - g(\theta^*) \leq O(1/t^2) \]

  \[ g(\theta_t) - g(\theta^*) \leq O(e^{-t\sqrt{\mu/L}}) = O(e^{-t/\sqrt{\kappa}}) \] if \( \mu \)-strongly convex

  - **Optimal rates** after \( t = O(d) \) iterations (Nesterov, 2004)
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- **Optimal rates** after \( t = O(d) \) iterations (Nesterov, 2004)

- Still \( O(nd) \) iteration cost: complexity \( = O(nd \cdot \sqrt{\kappa} \log \frac{1}{\varepsilon}) \)
Accelerating gradient methods - Related work

• Constant step-size stochastic gradient
  – Solodov (1998); Nedic and Bertsekas (2000)
  – Linear convergence, but only up to a fixed tolerance
Accelerating gradient methods - Related work

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• Stochastic methods in the dual (SDCA)
  – Shalev-Shwartz and Zhang (2013)
  – Similar linear rate but limited choice for the $f_i$’s
  – Extensions without duality: see Shalev-Shwartz (2016)
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- **Constant step-size stochastic gradient**
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  - Extensions without duality: see Shalev-Shwartz (2016)

- **Stochastic version of accelerated batch gradient methods**
  - Tseng (1998); Ghadimi and Lan (2010); Xiao (2010)
  - Can improve constants, but still have sublinear $O(1/t)$ rate
Stochastic average gradient
(Le Roux, Schmidt, and Bach, 2012)

- **Stochastic average gradient (SAG) iteration**
  - Keep in memory the gradients of all functions \( f_i, i = 1, \ldots, n \)
  - Random selection \( i(t) \in \{1, \ldots, n\} \) with replacement
  - Iteration: \( \theta_t = \theta_{t-1} - \frac{\gamma_t}{n} \sum_{i=1}^{n} y_i^t \) with \( y_i^t = \begin{cases} f_i'(\theta_{t-1}) & \text{if } i = i(t) \\ y_{i}^{t-1} & \text{otherwise} \end{cases} \)
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functions $g = \frac{1}{n} \sum_{i=1}^{n} f_i$ \hspace{1cm} $f_1 \hspace{0.2cm} f_2 \hspace{0.2cm} f_3 \hspace{0.2cm} f_4 \hspace{0.2cm} \cdots \hspace{0.2cm} f_{n-1} \hspace{0.2cm} f_n$

gradients $\in \mathbb{R}^d$ $\frac{1}{n} \sum_{i=1}^{n} y^t_i$ \hspace{1cm} $y^t_1 \hspace{0.2cm} y^t_2 \hspace{0.2cm} y^t_3 \hspace{0.2cm} y^t_4 \hspace{0.2cm} \cdots \hspace{0.2cm} y^t_{n-1} \hspace{0.2cm} y^t_n$
Stochastic average gradient 
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    \end{cases}$$

functions 

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$\begin{array}{cccccccc} f_1 & f_2 & f_3 & f_4 & \cdots & f_{n-1} & f_n \end{array}$

gradients $\in \mathbb{R}^d$  
$\begin{array}{cccccccc} \frac{1}{n} \sum_{i=1}^{n} y_i^t & y_1^t & y_2^t & y_3^t & y_4^t & \cdots & y_{n-1}^t & y_n^t \end{array}$
Stochastic average gradient
(Le Roux, Schmidt, and Bach, 2012)

• Stochastic average gradient (SAG) iteration
  
  – Keep in memory the gradients of all functions \( f_i, i = 1, \ldots, n \)
  – Random selection \( i(t) \in \{1, \ldots, n\} \) with replacement
  – Iteration:
    \[
    \theta_t = \theta_{t-1} - \frac{\gamma_t}{n} \sum_{i=1}^{n} \ y_i^t \text{ with } y_i^t = \begin{cases} 
    f'_i(\theta_{t-1}) & \text{if } i = i(t) \\
    y_i^{t-1} & \text{otherwise}
  \end{cases}
    \]

• Stochastic version of incremental average gradient (Blatt et al., 2008)
Stochastic average gradient
(Le Roux, Schmidt, and Bach, 2012)

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• Stochastic version of incremental average gradient (Blatt et al., 2008)

• **Extra memory requirement**: $n$ gradients in $\mathbb{R}^d$ in general

• **Linear supervised machine learning**: only $n$ real numbers
  
  – If $f_i(\theta) = \ell(y_i, \Phi(x_i)^\top \theta)$, then $f_i'(\theta) = \ell'(y_i, \Phi(x_i)^\top \theta) \Phi(x_i)$
Stochastic average gradient - Convergence analysis

• Assumptions
  
  – Each $f_i$ is $L$-smooth, $i = 1, \ldots, n$
  – $g = \frac{1}{n} \sum_{i=1}^{n} f_i$ is $\mu$-strongly convex
  – constant step size $\gamma_t = 1/(16L)$ - no need to know $\mu$
**Stochastic average gradient - Convergence analysis**

- **Assumptions**
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  - constant step size $\gamma_t = 1/(16L)$ - no need to know $\mu$

- **Strongly convex case** (Le Roux et al., 2012; Schmidt et al., 2016)
  \[
  \mathbb{E}[g(\theta_t) - g(\theta_*)] \lesssim \text{cst} \times \left(1 - \min\left\{ \frac{1}{8n}, \frac{\mu}{16L} \right\} \right)^t
  \]
  - Linear (exponential) convergence rate with $O(d)$ iteration cost
  - After one pass, reduction of cost by $\exp\left(- \min\left\{ \frac{1}{8}, \frac{n\mu}{16L} \right\} \right)$
  - NB: in machine learning, may often restrict to $\mu \geq L/n$
    \[\Rightarrow\] constant error reduction after each effective pass
Running-time comparisons (strongly-convex)

- **Assumptions:** \( g(\theta) = \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) \)
  - Each \( f_i \) convex \( L \)-smooth and \( g \) \( \mu \)-strongly convex

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<tr>
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<td>( d \times (n + \frac{L}{\mu}) \times \log \frac{1}{\varepsilon} )</td>
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- NB-1: for (accelerated) gradient descent, \( L \) = smoothness constant of \( g \)
- NB-2: with non-uniform sampling, \( L \) = average smoothness constants of all \( f_i \)'s
Running-time comparisons (strongly-convex)

- **Assumptions:**
  \[ g(\theta) = \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) \]
  
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- **Beating two lower bounds** (Nemirovski and Yudin, 1983; Nesterov, 2004): with additional assumptions

  1. stochastic gradient: exponential rate for finite sums
  2. full gradient: better exponential rate using the sum structure
Running-time comparisons (non-strongly-convex)

- **Assumptions:** \( g(\theta) = \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) \)
  - Each \( f_i \) convex \( L \)-smooth
  - **Ill conditioned problems:** \( g \) may not be strongly-convex \((\mu = 0)\)

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- Adaptivity to potentially hidden strong convexity
- No need to know the local/global strong-convexity constant
Stochastic average gradient
Implementation details and extensions

• **Sparsity in the features**
  – Just-in-time updates ⇒ replace $O(d)$ by number of non zeros
  – See also Leblond, Pedregosa, and Lacoste-Julien (2016)

• **Mini-batches**
  – Reduces the memory requirement + block access to data

• **Line-search**
  – Avoids knowing $L$ in advance

• **Non-uniform sampling**
  – Favors functions with large variations

• See [www.cs.ubc.ca/~schmidtm/Software/SAG.html](http://www.cs.ubc.ca/~schmidtm/Software/SAG.html)
Experimental results (logistic regression)

quantum dataset
\( (n = 50\,000, \, d = 78) \)

rcv1 dataset
\( (n = 697\,641, \, d = 47\,236) \)
Experimental results (logistic regression)

quantum dataset  
\( (n = 50\,000, \, d = 78) \)

rcv1 dataset  
\( (n = 697\,641, \, d = 47\,236) \)
Before non-uniform sampling

protein dataset
\( (n = 145\,751, \, d = 74) \)

sido dataset
\( (n = 12\,678, \, d = 4\,932) \)
After non-uniform sampling

**protein dataset**

\((n = 145\,751, d = 74)\)

**sido dataset**

\((n = 12\,678, d = 4\,932)\)
Linearly convergent stochastic gradient algorithms

- Many related algorithms
  - SAG (Le Roux, Schmidt, and Bach, 2012)
  - SDCA (Shalev-Shwartz and Zhang, 2013)
  - SVRG (Johnson and Zhang, 2013; Zhang et al., 2013)
  - MISO (Mairal, 2015)
  - Finito (Defazio et al., 2014b)
  - SAGA (Defazio, Bach, and Lacoste-Julien, 2014a)
  - ...

- Similar rates of convergence and iterations
Linearly convergent stochastic gradient algorithms

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  – ...

• Similar rates of convergence and iterations

• Different interpretations and proofs / proof lengths
  – Lazy gradient evaluations
  – Variance reduction
Variance reduction

- **Principle**: reducing variance of sample of $X$ by using a sample from another random variable $Y$ with known expectation

$$Z_\alpha = \alpha(X - Y) + \mathbb{E}Y$$

- $\mathbb{E}Z_\alpha = \alpha\mathbb{E}X + (1 - \alpha)\mathbb{E}Y$
- $\text{var}(Z_\alpha) = \alpha^2\left[\text{var}(X) + \text{var}(Y) - 2\text{cov}(X, Y)\right]$
- $\alpha = 1$: no bias, $\alpha < 1$: potential bias (but reduced variance)
- Useful if $Y$ positively correlated with $X$
Variance reduction

- **Principle**: reducing variance of sample of $X$ by using a sample from another random variable $Y$ with known expectation

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  - $EZ_\alpha = \alpha EX + (1 - \alpha) EY$
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  - Useful if $Y$ positively correlated with $X$

- **Application to gradient estimation** (Johnson and Zhang, 2013; Zhang, Mahdavi, and Jin, 2013)
  - SVRG: $X = f'_{i(t)}(\theta_{t-1}), Y = f'_{i(t)}(\tilde{\theta}), \alpha = 1$, with $\tilde{\theta}$ stored
  - $EY = \frac{1}{n} \sum_{i=1}^{n} f'_i(\tilde{\theta})$ full gradient at $\tilde{\theta}$, $X - Y = f'_{i(t)}(\theta_{t-1}) - f'_{i(t)}(\tilde{\theta})$
Stochastic variance reduced gradient (SVRG)  
(Johnson and Zhang, 2013; Zhang et al., 2013)

• Initialize $\tilde{\theta} \in \mathbb{R}^d$

• For $i_{\text{epoch}} = 1$ to $\#$ of epochs
  
  – Compute all gradients $f'_i(\tilde{\theta})$; store $g'(\tilde{\theta}) = \frac{1}{n} \sum_{i=1}^n f'_i(\tilde{\theta})$
  – Initialize $\theta_0 = \tilde{\theta}$
  – For $t = 1$ to length of epochs
    
    $$\theta_t = \theta_{t-1} - \gamma \left[ g'(\tilde{\theta}) + (f'_{i(t)}(\theta_{t-1}) - f'_{i(t)}(\tilde{\theta})) \right]$$
  – Update $\tilde{\theta} = \theta_t$

• Output: $\tilde{\theta}$

– No need to store gradients - two gradient evaluations per inner step
– Two parameters: length of epochs + step-size
– Same linear convergence rate as SAG, simpler proof
**Stochastic variance reduced gradient (SVRG)**
*(Johnson and Zhang, 2013; Zhang et al., 2013)*

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Interpretation of SAG as variance reduction

- **SAG update**: \( \theta_t = \theta_{t-1} - \frac{\gamma}{n} \sum_{i=1}^{n} y^t_i \) with 
  \[y^t_i = \begin{cases} 
  f'_i(\theta_{t-1}) & \text{if } i = i(t) \\
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  – Biased update (expectation w.r.t. to $i(t)$ not equal to full gradient)
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- **SVRG update**: \( \theta_t = \theta_{t-1} - \gamma \left[ \frac{1}{n} \sum_{i=1}^{n} f_i'(\tilde{\theta}) + (f_i'(t)(\theta_{t-1}) - f_i'(\tilde{\theta})) \right] \)
  
  – Unbiased update
Interpretation of SAG as variance reduction

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  - Unbiased update

- **SAGA update:** \( \theta_t = \theta_{t-1} - \gamma \left[ \frac{1}{n} \sum_{i=1}^{n} y_i^{t-1} + (f_i'(	heta_{t-1}) - y_{i(t)}^{t-1}) \right] \)
  
  - Defazio, Bach, and Lacoste-Julien (2014a)
  
  - Unbiased update without epochs
SVRG vs. SAGA

- **SAGA update**: \( \theta_t = \theta_{t-1} - \gamma \left[ \frac{1}{n} \sum_{i=1}^{n} y_{i}^{t-1} + (f'_{i(t)}(\theta_{t-1}) - y_{i(t)}) \right] \)

- **SVRG update**: \( \theta_t = \theta_{t-1} - \gamma \left[ \frac{1}{n} \sum_{i=1}^{n} f'_{i}(\tilde{\theta}) + (f'_{i(t)}(\theta_{t-1}) - f'_{i(t)}(\tilde{\theta})) \right] \)

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<th>SVRG</th>
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<td><strong>Storage of gradients</strong></td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Epoch-based</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td><strong>Parameters</strong></td>
<td>step-size</td>
<td>step-size &amp; epoch lengths at least 2</td>
</tr>
<tr>
<td>Gradient evaluations per step</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Adaptness to strong-convexity</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Robustness to ill-conditioning</td>
<td>yes</td>
<td>no</td>
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– See Babanezhad et al. (2015)
Proximal extensions

- **Composite optimization problems**: \( \min_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) + h(\theta) \)

  - \( f_i \) smooth and convex
  - \( h \) convex, potentially non-smooth
Proximal extensions

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  \[
  \min_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) + h(\theta)
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  - $h$ convex, potentially non-smooth
  - Constrained optimization: $h(\theta) = 0$ if $\theta \in K$, and $+\infty$ otherwise
  - Sparsity-inducing norms, e.g., $h(\theta) = \|\theta\|_1$
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**Proximal methods (a.k.a. splitting methods)**

- Extra projection / soft thresholding step after gradient update
- See, e.g., Combettes and Pesquet (2011); Bach, Jenatton, Mairal, and Obozinski (2012); Parikh and Boyd (2014)
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  - See, e.g., Combettes and Pesquet (2011); Bach, Jenatton, Mairal, and Obozinski (2012); Parikh and Boyd (2014)

- **Directly extends to variance-reduced gradient techniques**
  
  - Same rates of convergence
Acceleration

- **Similar guarantees for finite sums**: SAG, SDCA, SVRG (Xiao and Zhang, 2014), SAGA, MISO (Mairal, 2015)

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- **Acceleration for special algorithms** (e.g., Shalev-Shwartz and Zhang, 2014; Nitanda, 2014; Lan, 2015)

- **Catalyst** (Lin, Mairal, and Harchaoui, 2015)
  - Widely applicable generic acceleration scheme
From training to testing errors

- **rcv1 dataset** \((n = 697\,641, d = 47\,236)\)

  - NB: IAG, SG-C, ASG with optimal step-sizes in hindsight

### Training cost

<table>
<thead>
<tr>
<th>Effective Passes</th>
<th>Objective minus Optimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10^10</td>
</tr>
<tr>
<td>10</td>
<td>10^-20</td>
</tr>
<tr>
<td>20</td>
<td>10^-10</td>
</tr>
<tr>
<td>30</td>
<td>10^-10</td>
</tr>
<tr>
<td>40</td>
<td>10^-10</td>
</tr>
<tr>
<td>50</td>
<td>10^-10</td>
</tr>
</tbody>
</table>

[Graph showing training cost with labels for different algorithms: IAG, L-BFGS, SG-C, ASG, SAG]
From training to testing errors

- **rcv1** dataset ($n = 697\ 641$, $d = 47\ 236$)
  - NB: IAG, SG-C, ASG with optimal step-sizes in hindsight

![Graphs showing training and testing cost](attachment:graphs.png)
SGD minimizes the testing cost!

- **Goal**: minimize $f(\theta) = \mathbb{E}_{p(x,y)} \ell(y, \theta^\top \Phi(x))$
  
  - Given $n$ independent samples $(x_i, y_i), i = 1, \ldots, n$ from $p(x, y)$
  - Given a **single pass** of stochastic gradient descent
  - Bounds on the excess testing cost $\mathbb{E} f(\bar{\theta}_n) - \inf_{\theta \in \mathbb{R}^d} f(\theta)$
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  – Optimal for non-smooth losses (Nemirovski and Yudin, 1983)
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- **Constant-step-size SGD**
  
  - Linear convergence up to the noise level for strongly-convex problems (Solodov, 1998; Nedic and Bertsekas, 2000)
  - Full convergence and robustness to ill-conditioning?
Robust averaged stochastic gradient
(Bach and Moulines, 2013)

- Constant-step-size SGD is convergent for least-squares
  - Convergence rate in $O(1/n)$ without any dependence on $\mu$
  - Simple choice of step-size (equal to $1/L$)
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- Convergence in $O(1/n)$ for smooth losses with $O(d)$ online Newton step
Conclusions - Convex optimization

- Linearly-convergent stochastic gradient methods
  - Provable and precise rates
  - Improves on two known lower-bounds (by using structure)
  - Several extensions / interpretations / accelerations
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  – Extension to saddle-point problems (Balamurugan and Bach, 2016)
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• **What’s next:** non-convexity, parallelization, extensions/perspectives
Postdoc opportunities in downtown Paris

- Machine learning group at INRIA - Ecole Normale Supérieure
  - Two postdoc positions (2 years)
  - One junior researcher position (4 years)
References


L. Xiao. Dual averaging methods for regularized stochastic learning and online optimization. *Journal