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A CONSTRUCTIVE CHARACTERIZATION OF THE LATTICES OF ALL RETRACTIONS, PRECLOSURE, QUASI-CLOSURE AND CLOSURE OPERATORS ON A COMPLETE LATTICE

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1. Introduction

We give a constructive characterization of the complete lattices of all retractions, preclosure, quasi-closure and closure operators on a complete lattice. Our general approach is the following: in order to study the structure of the set $\Gamma \subseteq (L \to L)$ of operators ρ on a complete lattice L satisfying a given axiom A, we show that ρ has property A if and only if it is a fixed point of some monotone operator F on the complete lattice $(L \to L)$ proving that Γ is the set of fixed points of F. Then using Cousot & Cousot's constructive version of Tarski's lattice theoretical fixed point theorem, we constructively characterize the infimum, supremum, union and intersection of the complete lattice Γ which are defined by means of limits of stationary transfinite iteration sequences for F. Variants of this argument are used when F is a clousre operator (in which case the constructive version of Tarski's theorem amounts to Ward's theorem) or when the operators with property A are the postfixed points of F or the common fixed points of two functionals. The reasoning is repeated when Γ is characterized by means of more than one axiom.

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2. Preliminaries

1.1. Let $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ be a non-empty *complete lattice* with *partial ordering* $\sqsubseteq$, *least upper bound* $\sqcup$, *greatest lower bound* $\sqcap$. The *infimum* $\perp$ of L is $\sqcap L$, the *supremum* $\top$ of L is $\sqcup L$. (Birkhoff's standard reference book [2] provides the necessary background material).

1.2. Let $\theta \in (L \to M)$ be a total function from the complete lattice $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ to the complete lattice $M(\sqsubseteq', \perp', \top', \sqcup', \sqcap')$. θ is a *join-morphism* when $(\forall x, y \in L, \theta(x \sqcup y) = \theta(x) \sqcup' \theta(y))$. θ is a *complete join-morphism* when $(\forall S \subseteq L, \theta(\sqcup S) = \sqcup' \theta(S))$. The dual notions are the ones of *meet-morphism* and *complete meet-morphism*.

1.3. Using Church[3]'s lambda notation (so that $f \in (L \to M)$ is $\lambda x.f(x)$) let us recall that the set $(L \to L)$ of all operators on the complete lattice $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ is a complete lattice $(\sqsubseteq', \perp', \top', \sqcup', \sqcap')$ where $((f \sqsubseteq' g) \Leftrightarrow (\forall x \in L,\ f(x) \sqsubseteq g(x)))$, $\perp' = \lambda x. \perp$, $\top' = \lambda x. \top$, $\sqcup' = \lambda S.(\lambda x. \sqcup \{f(x) : f \in S\})$, $\sqcap' = \lambda S.(\lambda x. \sqcap \{f(x) : f \in S\})$. In the following we will omit the primes so that the distinction between $\sqsubseteq$, $\perp$, $\top$, $\sqcup$, $\sqcap$ and $\sqsubseteq'$, $\perp'$, $\top'$, $\sqcup'$, $\sqcap'$ will be contextual.

1.4. A *retraction* ρ on L is an operator on L (i. e. $\rho \in (L \to L)$) which is *monotone* (i. e. $\forall x, y \in L, (x \sqsubseteq y) \Rightarrow (\rho(x) \sqsubseteq \rho(y))$) and *idempotent* (i. e. $\rho = \rho \circ \rho$ that is $\forall x \in L, \rho(x) = \rho(\rho(x))$).

1.5. An *upper preclosure operator* $\bar{\rho}$ on L is monotone, idempotent and satisfies the *upper connectivity axiom* $(\forall x \in L,\ \bar{\rho}(x \sqcup \bar{\rho}(x)) = \bar{\rho}(x))$. The dual notion is the one of *lower preclosure operator* (Ladegaillerie, [6, Def. 1]).

1.6. A *quasi-closure operator* ρ on L is monotone, *comparing* (i. e. $\forall x \in L$, either $x \sqsubseteq \rho(x)$ or $\rho(x) \sqsubseteq x$) and satisfies the *connectivity axiom* $(\forall x \in L,\ \rho(x \sqcap \rho(x)) = \rho(x \sqcup \rho(x))$, Bernard[1, p. 6]). Notice that monotony and connectivity axiom imply $(\forall x \in L,\ \rho(x) = \rho(x \sqcap \rho(x)) = \rho(x \sqcup \rho(x)))$ and using the comparing hypothesis this in turn implies idempotence.

1.7. An *upper closure operator* $\bar{\rho}$ on L is monotone, idempotent and *extensive* (i. e. $\lambda x.x \sqsubseteq \bar{\rho}$), (Moore[9]). Dually a *lower closure operator* $\underline{\rho} \in (L \to L)$ is monotone, idempotent and *reductive* (i e. $\underline{\rho} \sqsubseteq \lambda x.x$).

1.8. Let $\bar{\rho}$ be an upper closure operator on $L(\sqsubseteq,\bot,\top,\sqcup,\sqcap)$. For all $x \in L$, the set $\{y \in \bar{\rho}(L) : x \sqsubseteq y\}$ is not empty and $\bar{\rho}(x)$ is its least element (Monteiro & Ribeiro[8, Th. 5.2]).

1.9. Let $R \subseteq L$ and $\bar{\rho} \in (L \to R)$ be such that for any $x \in L$, $\bar{\rho}(x)$ is the least element of the set $\{y \in R : x \sqsubseteq y\}$ then $\bar{\rho}$ is an upper closure operator and $R = \bar{\rho}(L)$, (Monteiro & Ribeiro[8, Th. 5.3]).

1.10. Let $\bar{\rho}$ be an upper closure operator on L, then the image $\bar{\rho}(L)$ of L by $\bar{\rho}$ is a complete lattice $\bar{\rho}(L)(\sqsubseteq, \bar{\rho}(\bot), \top, \lambda S.\bar{\rho}(\sqcup S), \sqcap)$ which is a complete sublattice of L if and only if $\bar{\rho}$ is a complete join-morphism, (Ward[14, Th. 4.1]).

1.11. Applying the duality principle it follows that if $R \subseteq L$ and $\bar{\rho}$, $\underline{\rho} \in (L \to L)$ are such that for any $x \in L$, $\bar{\rho}(x)$ is the least element of the set $\{y \in R : x \sqsubseteq y\}$ and $\underline{\rho}(x)$ is the greatest element of the set $\{y \in R : y \sqsubseteq x\}$ then $R = \bar{\rho}(L) = \underline{\rho}(L)$ is a complete sublattice $R(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ of L whereas $\bar{\rho}$ is a complete join-morphism and $\underline{\rho}$ is a complete meet-morphism.

1.12. Let $\bar{\rho}_1$ and $\bar{\rho}_2$ be upper closure operators on L. Then according to Ore[12, p. 525], $\bar{\rho}_1 \circ \bar{\rho}_2$ and $\bar{\rho}_2 \circ \bar{\rho}_1$ are upper closure operators on L if and only if $\bar{\rho}_1$ and $\bar{\rho}_2$ are *commuting* (i. e. $\bar{\rho}_1 \circ \bar{\rho}_2 = \bar{\rho}_2 \circ \bar{\rho}_1$) in which case $\bar{\rho}_1 \circ \bar{\rho}_2(L) = \bar{\rho}_2 \circ \bar{\rho}_1(L) = \bar{\rho}_1(L) \cap \bar{\rho}_2(L)$.

1.13. Let $L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ be a complete lattice, μ the smallest ordinal such that the class $\{\delta : \delta \in \mu\}$ has a cardinality greater than the cardinality of L and $f \in (L \to L)$. The *upper iteration sequence for f starting with* $d \in L$ is the μ-termed sequence $< X^\delta, \delta \in \mu >$ of elements of L defined by transfinite recursion in the following way:

— $X^0 = d$

— $X^\delta = f(X^{\delta-1})$ for every successor ordinal $\delta \in \mu$

— $X^\delta = \bigsqcup_{\alpha<\delta} X^\alpha$ for every limit ordinal $\delta \in \mu$

(The dual *lower iteration sequence* is defined by:

— $X^\delta = \bigsqcap_{\alpha<\delta} X^\alpha$ for every limit ordinal $\delta \in \mu$)

1.14. We say that the sequence $< X^\delta, \delta \in \mu >$ is *stationary* if and only if $(\exists \varepsilon \in \mu : (\forall \beta \in \mu, (\varepsilon \leqslant \beta) \Rightarrow (X^\varepsilon = X^\beta)))$ in which case the