# Algorithmic Aspects of Embeddability

## Higher-Dimensional Analogues of Graph Planarity


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joint work with

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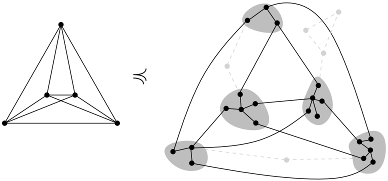
# Starting Point: Graphs & Planarity

- A graph (=1-dimensional complex) $G$ is planar if it can be embedded into the plane $\mathbb{R}^2$ (equivalently, into the sphere $S^2$)
- Classical notion in topology, graph theory, discrete and computational geometry, theoretical computer science
- Combinatorics & Structure
  - Characterization of planar graphs by forbidden minors $K_5$, $K_{3,3}$ (Kuratowski 1930, K. Wagner 1937)



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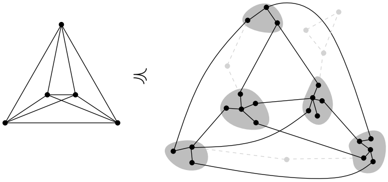

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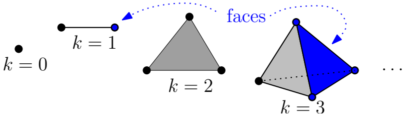

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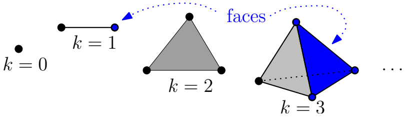


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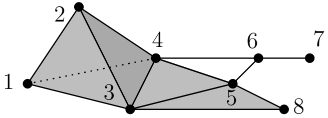

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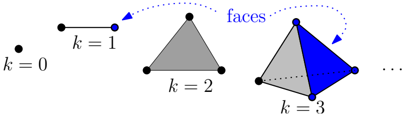


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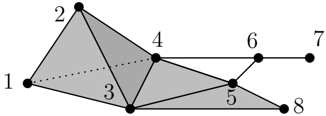


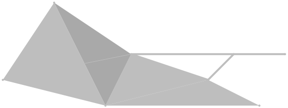

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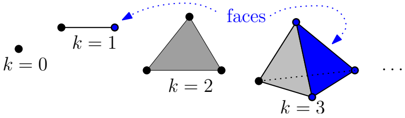


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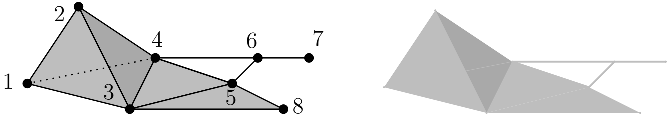


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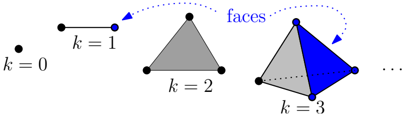


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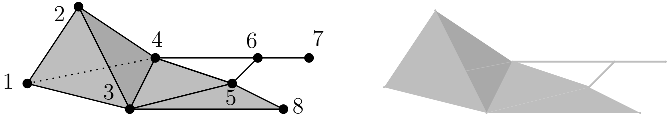


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- Graphs: 1-dimensional special case

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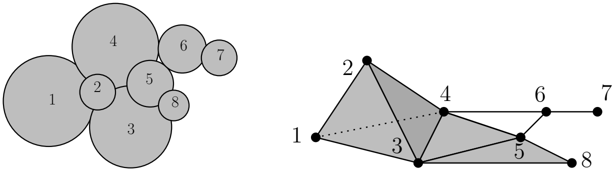

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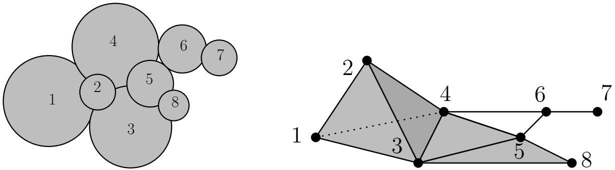


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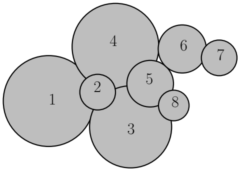


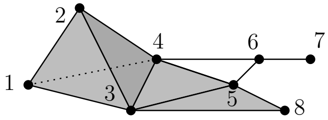


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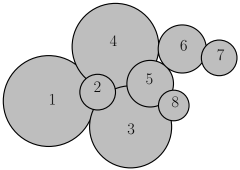


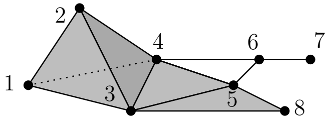


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        - *monotone graph properties* and evasiveness

# Embeddings of simplicial complexes

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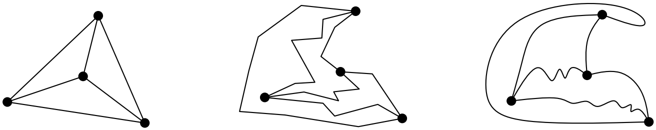

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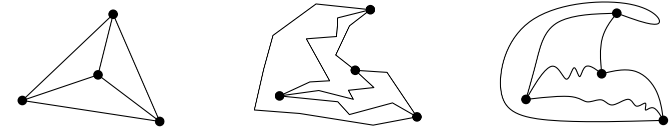

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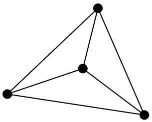

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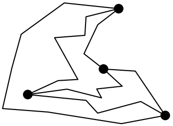

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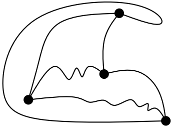

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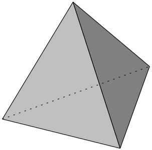
- For graphs in the plane, TOP/PL/LINEAR embeddability are equivalent (only *one* notion of planarity).
  - TOP $\Rightarrow$ PL: easy compactness argument,
  - PL $\Rightarrow$ LINEAR: nontrivial [Steinitz,Fáry].

# Different Types of Embeddings

Embeddings $X \hookrightarrow \mathbb{R}^d$ of a simplicial complex, $\dim X = k$

- Subtle differences in higher dimensions ($d \geq 3$)



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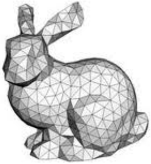

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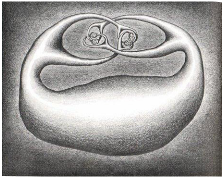

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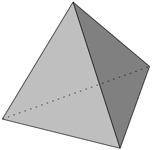

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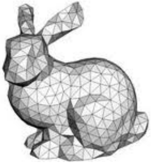

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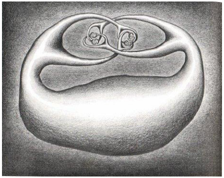

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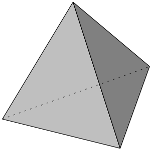

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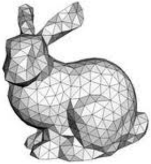

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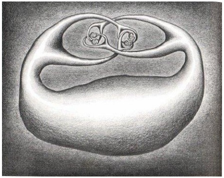

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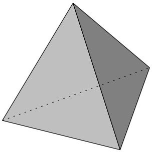

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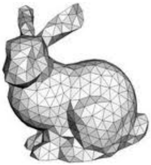

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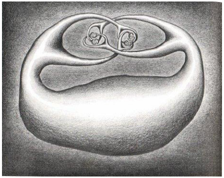

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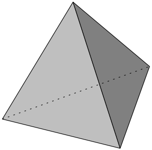

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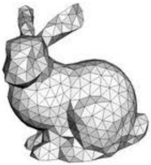

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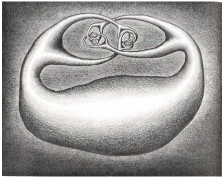

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- Linear embeddability always in PSPACE (solvability of polynomial inequalities in real variables).
- For algorithmic questions we consider PL embeddability

# Algorithmic Embeddability Testing

$k \leq d$ fixed positive integers
$\text{EMBED}_{k \to d}$ is the following algorithmic problem:

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| Input: | A simplicial complex $K$ of dimension (at most) $k$. |
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- $d \geq 2k+1$ trivial: embeds always (general position).
- For $d = 2k$, there exist $k$-dimensional complexes not embeddable into $\mathbb{R}^{2k}$:
  - complete $k$-complex $K^k_{2k+3} = \text{skel}_k(\Delta^{2k+2})$ (all simplices of dimension $\leq k$ on $2k+3$ vertices)
  - complete multipartite $k$-complex $K^k_{3,\dots,3}$
  - for $k \geq 2$, infinitely other minimally non-embeddable complexes (no straightforward analogue of Kuratowski)

# Algorithmic Embeddability: Classical Results

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- but no prior systematic study from a computational viewpoint (unlike its cousin, knot theory, isotopy of embeddings of the circle $S^1$ into $\mathbb{R}^3$).

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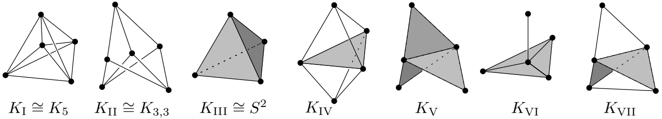
- $\mathrm{EMBED}_{2\to 2}$: characterization by forbidden subcomplexes (Halin, Jung 1964) yields $O(n)$ algorithm.



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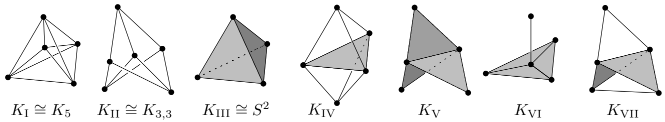


- van Kampen obstruction (van Kampen 1932; Shapiro, Wu), yields polynomial-time algorithm for $\text{EMBED}_{k\to 2k}$, $k \geq 3$.

# Current State of Knowledge: Complexity of EMBED$_{k \to d}$

| $k$ \ $d$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | P | | | | | | | | | | | | |
| 2 | P | D | NPh | | | | | | | | | | |
| 3 | | D | NPh | NPh | P | | | | | | | | |
| 4 | | | NPh | und | NPh | NPh | P | | | | | | |
| 5 | | | | und | und | NPh | NPh | P | P | | | | |
| 6 | | | | | und | und | NPh | NPh | NPh | P | P | | |
| 7 | | | | | | und | und | NPh | NPh | NPh | P | P | P |

und = algorithmically undecidable [Matoušek, Tancer, W.]
NPh = NP-hard [Matoušek, Tancer, W.]
D = algorithmically decidable [Matoušek, Sedgwick, Tancer, W.]
P = polynomial-time solvable; new results based on algorithmic homotopy classification of (equivariant) maps [Čadek, Krčál, Matoušek, Sergeraert, Vokřínek, W.]

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und = algorithmically undecidable [Matoušek, Tancer, W.]
NPh = NP-hard [Matoušek, Tancer, W.]
D = algorithmically decidable [Matoušek, Sedgwick, Tancer, W.]
P = polynomial-time solvable; new results based on algorithmic homotopy classification of (equivariant) maps [Čadek, Krčál, Matoušek, Sergeraert, Vokřínek, W.]

**Dividing line**: metastable range $d \geq 3(k+1)/2$ [Haefliger–Weber]
(*small dimensions* $d = 2, 3$ somewhat exceptional)

# The deleted product obstruction and Haefliger–Weber

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**Theorem (Haefliger–Weber)**

*If $K$ is a $k$-dimensional simplicial complex and $d \geq \frac{3(k+1)}{2}$ (metastable range) then $K$ embeds in $\mathbb{R}^d$ iff there is an equivariant map $K^2_\Delta \to_{\mathbb{Z}_2} S^{d-1}$.*

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### Remark

For all $(d,k)$ outside the metastable range, $d \geq 3$, the deleted product obstruction is known to be incomplete (Segal, Spież, Freedman, Krushkal, Teichner, A. Skopenkov).

# Hardness of $\text{EMBED}_{2\to4}$: A Sketch

## Theorem

*It is NP-hard to decide whether a given 2-complex embeds into $\mathbb{R}^4$.*

- Reduction from 3-SAT: for every 3-CNF formula $\varphi$, e.g.,

$$\varphi = (x_1 \vee \bar{x}_2 \vee x_4) \wedge (x_1 \vee \bar{x}_4 \vee x_5) \wedge \ldots,$$

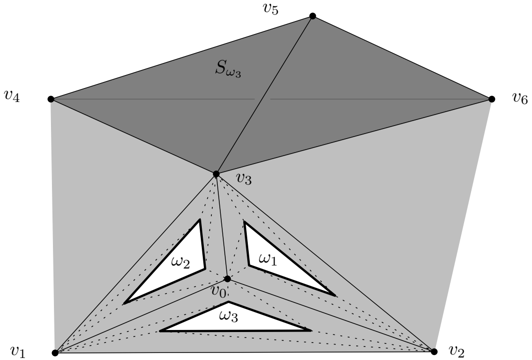
  construct a 2-dimensional simplicial complex $K_\varphi$ such that

$$\varphi \text{ is satisfiable} \Leftrightarrow K_\varphi \hookrightarrow \mathbb{R}^4$$

- $K_\varphi$ is built from clause gadgets and conflict gadgets
- Gadgets based on examples of Freedman, Krushkal and Teichner showing that the van Kampen obstruction is incomplete for embeddings into $\mathbb{R}^4$.

# Clause Gadget

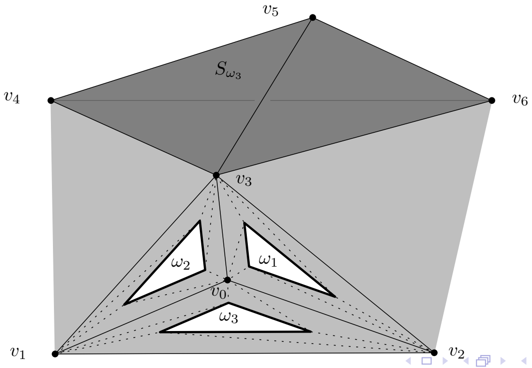
- start with $K_7^2$ (all triangles on 7 vertices)
- make small holes (openings) in the interiors of three triangles sharing a vertex
- for each opening, there is a complementary 2-sphere



# Linking Lemma

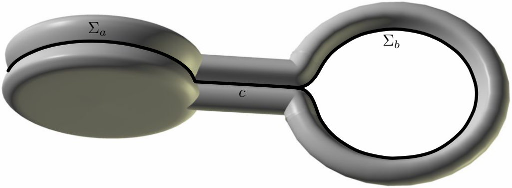
## Lemma

1. *For every PL embedding $f\colon G \hookrightarrow \mathbb{R}^4$, there is an opening $\omega_i$ such that the images $f(\partial\omega_i)$ and $f(S_{\omega_i})$ have odd linking number.*
2. *For every $i$, there exists and embedding such that only $f(\partial\omega_i)$ and $f(S_{\omega_i})$ are linked.*



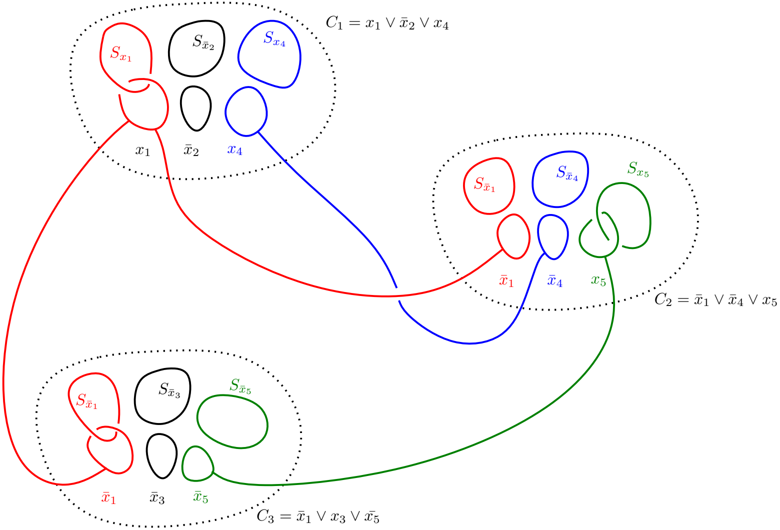
# Conflict Gadget

- Squeezed torus, obtained by glueing an octagon to "two circles with a stick".




- Can be embedded into $\mathbb{R}^3$ if one of the circles is "free" (not linked with any obstacles); asymmetry in the embedding.
- Cannot be embedded into $\mathbb{R}^4$ if both circles are blocked (linked with 2-spheres).

# Reduction Sketch



# Algorithmic Embeddability in $\mathbb{R}^3$

- EMBED$_{2\to3}$ and EMBED$_{3\to3}$ can be reduced, possibly with exponential-time overhead, to the following question: Given a compact 3-manifold $X$ with boundary, does it embed in $S^3$?
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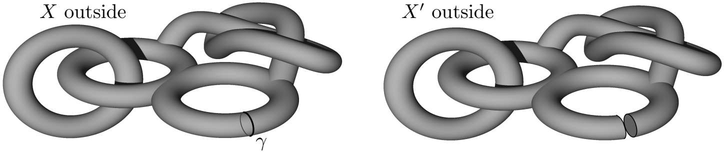
- **Strategy**: "Guess" a meridian $\gamma$, glue a thickened disk to $X$ along $\gamma$. Preserves embeddability, simplifies $\partial X$. Recurse.




- Base of the recursion: $S^3$-recognition [Rubinstein–Thompson]

# Algorithmic Embeddability in $\mathbb{R}^3$, cont'd

Key technical result, proved using normal surface theory:

**Theorem (Short Meridians; Matoušek, Sedgwick, Tancer, W.)**

*Suppose that $X$ is a 3-manifold with boundary*[1] *that embeds in $S^3$. Then there exists (a possibly different) embedding of $X$ for which there is a short meridian $\gamma$, i.e., an essential*[2] *normal curve $\gamma \subset \partial X$ bounding a disk in $S^3 \setminus X$ such that the length of $\gamma$, measured as the number of intersections of $\gamma$ with the edges of the triangulation, is bounded by a computable function of the number of tetrahedra.*

---

[1] Caveat: We first need to do some preprocessing to ensure that $X$ has certain helpful technical properties:

- $X$ is *irreducible*, neither a ball nor an $S^3$,
- $X$ has *incompressible boundary*,
- $X$ is equipped with a 0-efficient triangulation.

[2] Meaning that $\gamma$ does not bound a disk in $\partial X$.

# New Results on Homotopy Classification and Extensions

## Theorem (ČKMSVW)

*Assume we are given the following input: simplicial complexes* $A \subseteq X$ *and* $f \colon A \to S^r$.

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## Remarks

- Generalizes a classical algorithm (Brown, 1957) to compute $[X, Y]$ for $Y$ with all *homotopy groups* $\pi_i(Y)$ *finite*, $i \leq \dim X$
- Generalization to equivariant maps [Čadek, Krčál, Vokřínek]
- Extension problem undecidable for input $f\colon A \to S^r$, $\dim X = 2r$, $r$ even.

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    (Currently, we compute very implicit representations of homotopy classes)
  - Algorithmic Haefliger–Weber?
  - Recent result [Freedman–Krushkal]: In the case $d = 2k$, $k \geq 3$, an exponential number of subdivisions is sufficient and sometimes necessary.

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- **Embeddability in other ambient manifolds**?
  - Given a 3-manifold $M$ and a 2-complex $K$, it is NP-hard to decide whether $K \hookrightarrow M$. True even under the additional assumption that $K$ is a (non-orientable) surface! [Burton, de Mesmay, W.]
  - Is the problem in NP? Yes for odd Euler genus nonorientable surfaces. Even Euler genus?

Thank you for your attention!