Equilibria of Atomic Flow Games are not Unique*

Umang Bhaskar† Lisa Fleischer† Darrell Hoy‡ Chien-Chung Huang§

Abstract

In routing games with infinitesimal players, it follows from well-known convexity arguments that equilibria exist and are unique (up to induced delays, and under weak assumptions on delay functions). In routing games with players that control large amounts of flow, uniqueness has been demonstrated only in limited cases: in 2-terminal, nearly-parallel graphs; when all players control exactly the same amount of flow; when latency functions are polynomials of degree at most three.

In this work, we answer an open question posed by Cominetti, Correa, and Stier-Moses (ICALP 2006) and show that there may be multiple equilibria in atomic player routing games. We demonstrate this multiplicity via two specific examples. In addition, we show our examples are topologically minimal by giving a complete characterization of the class of network topologies for which unique equilibria exist. Our proofs and examples are based on a novel characterization of these topologies in terms of sets of circulations.

1 Introduction

In this paper, we study a distributed routing problem that arises in both electronic communication and transportation: network congestion games. In the problem we study, there is a network and a set of players; and each player wants to route flow from a source to a destination along links in the network. The total flow on a link determines the delay of the link. Each player routes flow to minimize the average delay of her flow.

In the Internet, we can view a player as a manager of an overlay network, seeking to route the traffic she controls. In transportation, we can view a player as a shipping company, or a bus company, routing multiple drivers to minimize average delay experienced by her fleet.

As described in the network congestion game above, each player controls a discrete, or atomic, amount of the total flow. These games are called *atomic splittable flow games* [2, 13]. A special case of this setting, described as early as 1952 by Wardrop [18], is where each player controls a negligible, infinitesimal, or nonatomic, amount of flow. In the scenarios above, these players could correspond to individual messages or drivers. In the recent literature, games with these infinitesimal players only are called *nonatomic flow games*.

The atomic setting is captured by the nonatomic setting with the addition of collusion [5]. *Collusion* describes the behavior of a subset of players that choose to cooperate by forming a coalition to reduce the average delay faced by players in the coalition. If the set of coalitions is fixed, then a nonatomic flow game with collusion is an atomic splittable flow game; a coalition in the nonatomic setting corresponds to a player in the atomic setting.

We study equilibria of collusive flow games, equivalently, the equilibria of atomic splittable flow games. An equilibrium in this setting (also called a Nash equilibrium or an equilibrium flow) is a flow where no player can unilaterally change her routing strategy and reduce its total delay.

In the nonatomic game, under weak assumptions on the delay functions, equilibria exist and if there are multiple equilibria, the induced delays on each edge are the same (e.g., [14]). Thus we say that equilibria are "unique up to induced delays."

In atomic splittable flow games equilibria also exist [10], but uniqueness is less well-understood. Previous work shows that there is a unique Nash equilibrium (again, under weak assumptions on the delay functions, and up to induced delays) in a few special cases:

1. if all players are of the same *type*: they control the same amount of flow, and have the same source and destination [10];

2. if the delay functions are all monomials of the same degree; or they are all polynomials of degree ≤ 3 [1];

3. if the network is a *two-terminal nearly-parallel* graph [11].

In contrast to these uniqueness results, it is known that if different players see *different* delay functions on the same edge, multiple equilibrium flows may exist [10, 11].

*This work was supported in part by NSF grants CCF-0515127 and CCF-0728869.

†Dartmouth College, 6211 Sudikoff, Hanover, NH 03755, USA. Email: {umang, lkf}@cs.dartmouth.edu

‡Email: darrell.hoy@gmail.com. This work was done while a student at Dartmouth College.

§Max-Planck-Institut für Informatik, 66123 Saarbrücken, Germany. Email: villars@mpi-inf.mpg.de. This work was done while a graduate student at Dartmouth.

Our Contributions. We settle the question of uniqueness of equilibria for atomic splittable flow games. We give the first examples of multiple equilibria in routing games when all players using an edge experience the same delay on that edge; and we give a complete characterization of graph topologies that have a unique equilibrium (under weak assumptions on the delay functions, and up to induced delays). To summarize the results, we use $\mathcal{L}$ to denote a natural class of delay functions (defined in Section 2).

1. For two players, there is a unique equilibrium for any choice of edge delays in $\mathcal{L}$ if and only if the network is a *generalized series-parallel* graph. For multiple players of only two *types*, there is a unique equilibrium for any choice of delays in $\mathcal{L}$ if and only if the network is an *s-t-series-parallel* graph.

2. For more than two types of players, there is a unique equilibrium for any choice of delays in $\mathcal{L}$ if and only if the network is a *generalized nearly-parallel* graph.

The second result generalizes a result of Richman and Shimkin [11] on uniqueness in nearly-parallel graphs. Our proofs and examples are based on new characterizations of both generalized series-parallel and generalized nearly-parallel graphs in terms of sets of circulations.

Related Work. Atomic splittable flow games are the subject of many recent papers. Harker [4] introduced a model in which both nonatomic and atomic players are present and investigated the existence and uniqueness of Nash equilibrium. Catoni and Pallottino [2] first showed that the social cost (total delay) in atomic splittable flow games may exhibit idiosyncratic behavior. The social cost in this setting has been further studied in [3, 5, 13, 15]. Swamy [16] investigated how to toll the players to achieve optimal flows.

Stackelberg routing games [17] are related to atomic splittable flow games in that there is at least one player that controls a significant amount of flow. In this setting, there is a player whose objective is not to minimize the delay of her own traffic but some other objective, for example, to minimize the total delay of all traffic [12, 16].

2 Model & Definitions

Let $G = (V, E)$ be a directed graph. For ease of notation only, we assume without loss of generality that for each pair of nodes $\{u, w\}$, there is at most one arc in E. That is, there are neither parallel or anti-parallel multi-arcs.*

The vector f, indexed by edges $e \in E$, is defined as a *flow* of *volume* v if the following conditions are satisfied. Here $f_{u,w}$ represents the flow on arc (u, w).

$$\begin{aligned}
\sum_w f_{u,w} - \sum_w f_{w,u} &= 0, \ \forall u \in V - \{s,t\}. \\
\sum_w f_{s,w} - \sum_w f_{w,s} &= v. \\
f_e &\geq 0, \ \forall e \in E.
\end{aligned}$$

If there are several flows $\{f^1, f^2, \cdots, f^k\}$, the total flow f is defined as $f := \sum_{b=1}^{k} f^b$. We define $f^{-b} := \sum_{j \neq b} f^j = f - f^b$.

A circulation is typically defined as a flow of volume 0. However, we are interested in circulations that arise as the difference of two flows f and g of the same value, and thus they may take positive and negative values. To distinguish these circulations from flows, we denote a circulation vector as $\vec{f}$, and define it as satisfying the following constraint

$$\sum_w \vec{f}_{u,w} - \sum_w \vec{f}_{w,u} = 0, \ \forall u \in V.$$

If $g_e - f_e \geq 0$, the circulation $g - f$ is in the same direction as the original directed graph G on edge e; on the other hand, if $g_e - f_e < 0$, the circulation $g - f$ is in the opposite direction on edge e. For two circulations $\vec{f}$ and $\vec{g}$, if $\vec{f}_e \cdot \vec{g}_e \geq 0$, the two circulations are in the same direction; on the other hand, if $\vec{f}_e \cdot \vec{g}_e < 0$, they are in the opposite direction.

An *atomic splittable flow game* is a tuple $(G = (V, E), K, l)$ where $K = \{(v_1, s_1, t_1), (v_2, s_2, t_2), \ldots, (v_k, s_k, t_k)\}$ is a set of triples indicating the flow volume, the source, and the destination for each of the k players, and l is a vector of delay functions for edges in G. We denote by $l_e : \mathbb{R}^+ \to \mathbb{R}^+$ the delay function for edge e.

For a flow f player b incurs a delay $C_e(f^b, f^{-b}) := f_e^b l_e(f_e)$ on edge e. His total delay is $C(f^b, f^{-b}) := \sum_{e \in E} C_e(f^b, f^{-b})$. Rational players want to minimize their delays. A flow is at equilibrium if no player can unilaterally alter his flow and reduce his total delay. Formally,

Definition 2.1. (Nash Equilibrium) *In an atomic splittable game $(G, \{(v_1, s_1, t_1), (v_2, s_2, t_2), \cdots, (v_k, s_k, t_k)\}, l)$, flow f is a Nash equilibrium flow if and only if, for every player b and every flow g of volume v_b,*

$$C(f^b, f^{-b}) \leq C(g, f^{-b}). \tag{2.1}$$

*For each multi-arc, we can introduce a new node z, and replace one arc (u, w) with arcs (u, z) and (z, w); the first with delay function of the original arc and the second with delay 0.

In this paper, we use the terms equilibrium, equilibrium flow, and Nash equilibrium interchangeably.

We consider a natural class of delay functions $\mathcal{L}$. A delay function $l_e(\cdot)$ belongs to the class $\mathcal{L}$ if it is: (1) non-negative, (2) non-decreasing, and (3) strictly semi-convex. A function $l_e(\cdot)$ is (strictly) semi-convex if $xl_e(x)$ is (strictly) convex. For convenience, we restrict to strict semi-convexity to eliminate multiplicities in equilibrium flows caused by the existence of constant functions. For example, on a 2-vertex graph with 2 parallel links and delay functions on both links constant and equal to 1, there are an infinite number of equilibrium flows, since each player can split his flow arbitrarily in any equilibrium. Our results handle semi-convex functions by relaxing uniqueness of equilibria to "uniqueness up to induced latencies."

The *marginal delay* for player b of adding flow to path p is given by

$$L_p^b(f) = \sum_{e \in p} l_e(f_e) + f_e^b l_e'(f_e) \tag{2.2}$$

The following lemma follows easily from Karush-Kuhn-Tucker conditions (see [9], for example) applied to a player b's minimization problem.

LEMMA 2.2. *Flow f is a Nash equilibrium flow if and only if for any player b and any two directed paths p and q between the same pair of vertices such that $f_p^b > 0$,*

$$L_p^b(f) \le L_q^b(f). \tag{2.3}$$

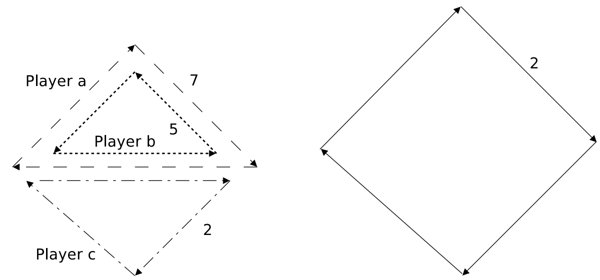
This lemma implies that in an equilibrium flow, a player never sends flow along a directed cycle. It also implies that for any player b and Nash equilibrium flow f, the marginal delay $L_p^b(f)$ on all paths p with $f_e^b > 0$ $\forall e \in p$ will be identical. Thus for a player b and equilibrium flow f we can simply write $L^b(f)$ for this value.

3 Agreeing Cycles and Uniqueness of Equilibria

In this section, we introduce the notion of *agreeing cycles* and connect them to the uniqueness of equilibria.

DEFINITION 3.1. *Let $G = (V, E)$ be an undirected graph and $\vec{f}^1, \vec{f}^2, \cdots, \vec{f}^k$ be k circulations on G. A cycle C in G is a j-*agreeing cycle *if it is a directed cycle in $\vec{f}^j$ and $\forall e \in C$, $\vec{f}_e \cdot \vec{f}_e^j \ge 0$.*

Figure 1 shows an example of agreeing cycles for players a and c. Definition 3.1 is based on *undirected* graphs. When we discuss directed graphs and consider the circulation $g - f$ derived from the two flows f and g of the same volume, if we have a cycle C agreeing with the circulation $g - f$, this means that $g - f$ has directed cycle flow along C, but this does not imply that C is also a directed cycle in the original *directed* graph.



(i) Circulations for 3 players

(ii) Total circulation

Figure 1: Agreeing cycles

LEMMA 3.2. *Let $(G, \{(v_1, s_1, t_1), (v_2, s_2, t_2), \cdots, (v_k, s_k, t_k)\}, l)$ be an atomic splittable flow game, where G is a directed multi-graph and $l_e \in \mathcal{L}$ for any edge e. If there are two equilibrium flows f and g, then none of the k circulations $g^j - f^j, 1 \le j \le k$, on graph G, has an agreeing cycle with $g - f$.*

Proof. Suppose that C is an r-agreeing cycle. We will show that this leads to a contradiction. See Figure 2 for an illustration of the paths used in the proof. We first consider how edges along cycle C are oriented in the original directed graph. By Lemma 2.2, no player sends flow along a directed cycle. Therefore, there are some vertices on cycle C for which both incident edges in C are outgoing edges; similarly, an equal number of vertices on C must have two incoming edges on cycle C. We designate the former set as $v_0^O, v_1^O, \cdots, v_{k-1}^O$ and the latter set as $v_0^I, v_1^I, \cdots, v_{k-1}^I$. Without loss of generality, assume that the agreeing cycle C runs in the direction of $v_0^O, v_0^I, v_1^O, v_1^I, \cdots$ in the *undirected* graph. By this assumption, the player r sends more flow on the path starting from v_j^O to v_{j-1}^I, modulo k, in flow f than in flow g; similarly, he sends more flow on the path from v_j^O to v_j^I, modulo k, in g than in f. In the following, for ease of exposition, the indices will always be taken modulo k.

We now define the path on C from v_j^O to v_{j-1}^I as p_j^f and the path from v_j^O to v_j^I as p_j^g. Furthermore, there must exist directed simple paths starting from s_r and ending at $v_j^O, 0 \le j \le k-1$; let these paths be p_j^O. Similarly, there must exist directed simple paths starting from $v_j^I, 0 \le j \le k-1$ and ending at t_r; let these paths be p_j^I.

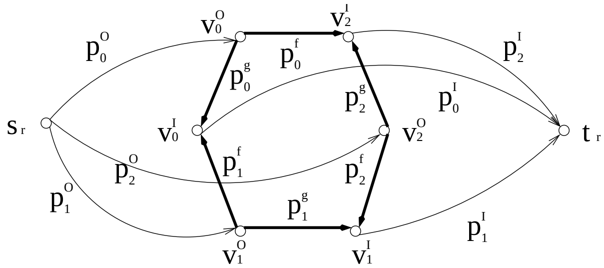


Figure 2: An illustration of all paths along, coming to, and leaving cycle C.

Consider equilibrium flow f. Since the path composed of p_j^O and p_j^f and the path composed of p_{j-1}^O and p_{j-1}^g both start with source s_r and end in the vertex v_{j-1}^I, Lemma 2.2 implies that

$$L^r_{p_j^O}(f) + L^r_{p_j^f}(f) \le L^r_{p_{j-1}^O}(f) + L^r_{p_{j-1}^g}(f), \tag{3.4}$$

for $0 \le j \le k-1$. If we sum up inequality (3.4) over all j, all $L^r_{p_j^O}$ terms will cancel, and we have

$$\sum_{j=0}^{k-1} L^r_{p_j^f}(f) \le \sum_{j=0}^{k-1} L^r_{p_j^g}(f). \tag{3.5}$$

A symmetric argument yields the following relation for g.

$$\sum_{j=0}^{k-1} L^r_{p_j^g}(g) \le \sum_{j=0}^{k-1} L^r_{p_j^f}(g). \tag{3.6}$$

Player r sends more flow on p_j^f in flow f than in flow g, so $f^r_{p_j^f} > g^r_{p_j^f}$. Since the r-agreeing cycle C runs in the direction of $v_0^O, v_o^I, v_1^O, v_1^I, \cdots$, so by Definition 3.1, we have that $g_{p_j^f} \le f_{p_j^f}$. These two facts imply

$$\begin{array}{rcl} L^r_{p_j^f}(f) & = & \sum_{e\in p_j^f} L^r_e(f) \\ & = & \sum_{e\in p_j^f} l_e(f) + f^r_e l_e(f) \\ & > & \sum_{e\in p_j^f} l_e(g) + g^r_e l_e(g) \\ & = & \sum_{e\in p_j^f} L^r_e(g) = L^r_{p_j^f}(g) \end{array} \tag{3.7}$$

In a similar manner, for path p_j^g, we can show that

$$L^r_{p_j^g}(f) < L^r_{p_j^g}(g). \tag{3.8}$$

Combining inequality (3.5), (3.7), and (3.8), we derive

$$\sum_{j=0}^{k-1} L^r_{p_j^f}(g) < \sum_{j=0}^{k-1} L^r_{p_j^f}(f) \le \sum_{j=0}^{k-1} L^r_{p_j^g}(f) < \sum_{j=0}^{k-1} L^r_{p_j^g}(g),$$

contradicting inequality (3.6). This gives the required proof. ∎

The lemma has two implications. The first is that if we can identify a class of graphs in which any set of circulations always has an agreeing cycle, then we can prove that this class of graphs has a unique equilibrium for any atomic splittable flow game. We use this below to demonstrate the uniqueness of equilibria in generalized nearly-parallel graphs with an arbitrary number of players, in generalized series-parallel graphs with 2 players, and in series-parallel graphs with multiple players where each player is one of two types.

The second implication is that in order to find an example with multiple equilibria, it is necessary to construct a set of circulations that do not have an agreeing cycle. In Section 4 we use this as a basis to construct examples showing that our characterizations of uniqueness are complete.

3.1 Two-Player Games and Series-Parallel Graphs We start by defining s-t-series-parallel and generalized series-parallel graphs. Their definitions use the following operations: (i) *Copy*: for any edge $e = (u, v)$, add a parallel edge $e' = (u, v)$; (ii) *Split*: for any edge $e = (u, v)$, replace it by a new vertex w and edges (u, w) and (w, v), and (iii) *Add*: for any vertex u, add a new vertex w and a new edge (u, w). A graph is *s-t-series-parallel* if it is a single edge $e = (s, t)$, or is constructed by any sequence of copy and split operations from (s, t). A graph is *generalized series-parallel* if it is a single edge, or if it is created by any sequence of copy, split and add operations from a generalized series-parallel graph. Equivalently, a graph is *generalized series-parallel* if and only if it does not contain a K_4 minor. (K_4 is a clique on four vertices. A graph H is a minor of graph G if H is isomorphic to a graph that can be obtained by zero or more edge contractions on a subgraph of G.) The equivalence of these two definitions for generalized series-parallel graphs was shown in [6, 8].

Lemma 3.3 gives a novel characterization for generalized series-parallel graphs, based on sets of circulations. For the rest of this section, for two-player games we use r(ed) and b(lue) to denote the two players, so that f^r is the red flow and f^b is the blue flow.

LEMMA 3.3. *An undirected graph G is a generalized series-parallel graph if and only if, given any two cir-*

culations $\vec{f^b}$ and $\vec{f^r}$ on G, there is either a red or a blue agreeing cycle.

Proof. ($\Rightarrow$) We prove by induction on the number of copy / split / add operations performed to generate G. The base case is a single edge connecting two vertices, where the theorem is vacuously true.

For the induction step, suppose that the last operation is a split operation, breaking the edge $e'' = (u, v)$ into two edges $e = (u, w)$ and $e' = (w, v)$. Remove w and replace the two edges e and e' with $e'' = (u, v)$ and update the circulations as follows: $\vec{f^b_{e''}} = \vec{f^b_e}, \vec{f^r_{e''}} = \vec{f^r_e}$. Applying the induction hypothesis, we have an agreeing cycle C. If C does not involve edge e'', we are done. If $e'' \in C$, we have another agreeing cycle $C' = (C - \{e''\}) \cup \{e, e'\}$ in the original graph, by expanding flow f on e'' to e and e' in the obvious way.

Suppose that the last operation is an add operation (adding a dangling vertex), then the circulations will not go to the newly-added vertex, so the induction step is trivial.

Now we consider the final case that the last operation is a copy operation, adding $e' = (u, v)$ to the graph as a parallel link of $e = (u, v)$. To avoid trivialities, we assume that the cycle composed of e and e' is not an agreeing cycle. Now we merge e and e' into a new link e'' and update the circulations accordingly as follows: $\vec{f^b_{e''}} = \vec{f^b_e} + \vec{f^b_{e'}}, \vec{f^r_{e''}} = \vec{f^r_e} + \vec{f^r_{e'}}$. Induction hypothesis implies that in the reduced graph, we have an agreeing cycle C. If C does not involve e'', we are done trivially. If not, without loss of generality, we assume that cycle C is a red agreeing cycle. If $\vec{f^r_{e''}} \cdot \vec{f^b_{e''}} \geq 0$, then at least one of e or e' can replace e'' in the original graph for the agreeing cycle. On the other hand, if $\vec{f^r_{e''}} \cdot \vec{f^b_{e''}} < 0$, then since C is a red agreeing cycle, $|\vec{f^r_{e''}}| \geq |\vec{f^b_{e''}}|$, the red flow either has more flow volume than the blue or is in the same direction as the blue on either e or e'. In either case, this edge e or e' can replace e'' in the original graph for the construction of the agreeing cycle.

($\Leftarrow$) In Section 4 we give an example pictured in Figure 5(iii) of two circulations in a K_4 graph that do not have an agreeing cycle. By the second definition for generalized series-parallel graphs, any graph that is not generalized series-parallel contains K_4 as a minor. ∎

Applying Lemma 3.2 to Lemma 3.3 implies the next theorem.

THEOREM 3.4. *There is a unique Nash equilibrium flow for the two-player game $(G, \{(v_b, s_b, t_b), (v_r, s_r, t_r)\}, l)$, where $G = (V, E)$ is a directed generalized series-parallel graph and $l_e \in \mathcal{L}$ for each edge $e \in E$.*

This holds even when both players have different sources and destinations.

Our next theorem extends Theorem 3.4 to more than two players, as long as each player is standard and is one of two types. For a routing game on an s-t-series-parallel graph, a player is *standard* if his source is s and his sink is t.

THEOREM 3.5. *There is a unique Nash equilibrium flow for the game $(G, \{(v_1, s, t), (v_2, s, t), \cdots, (v_k, s, t)\}, l)$ where $G = (V, E)$ is a directed s-t-series-parallel graph, $l_e \in \mathcal{L}$ $\forall e \in E$ and $v_i \in \{v_b, v_r\}$ $\forall i$.*

This theorem is tight in the following sense: in Section 4.3 we give an example with *three* types of standard players in an s-t-series-parallel graph that has multiple equilibria. The proof of this theorem makes use of the following lemma about standard players in series-parallel graphs. The lemma provides bounds on the amount of flow a player can put on an edge in an equilibrium flow, relative to other players. It extends a result of [10] restricted to parallel link graphs.

LEMMA 3.6. *Let f be an equilibrium flow for the atomic splittable flow game $(G, \{(v_1, s, t), (v_2, s, t), \cdots, (v_k, s, t)\}, l)$ where $G = (V, E)$ is an s-t-series-parallel graph, and $l_e \in \mathcal{L}$, $\forall e \in E$. Then for any pair of players b and r and any edge e, $f^b_e > f^r_e$ if and only if $v_b > v_r$.*

LEMMA 3.7. *Let G be an s-t-series-parallel graph. If f^b and f^r are two $s-t$ flows such that $v(f^b|_G) > v(f^r|_G)$, then there exists an $s-t$ path p such that $\forall e \in p$, $f^b_e > f^r_e$.*

Proof. In the SP decomposition of G, every time the graph is split into two parallel graphs G_1 and G_2, $v(f^b|_G) = v(f^b|_{G_1}) + v(f^b|_{G_2})$. Thus if $v(f^b|_G) > v(f^r|_G)$, at least one of the parallel graphs must have more flow on it from b than from r. ∎

Proof of Lemma 3.6. Suppose $v_b \leq v_r$ and $f^b_e > f^r_e$ for some edge e. We will derive a contradiction. Consider the decomposition tree of graph G. Edge e is a leaf node. We climb up the tree until we reach the last node G_1 fulfilling the property $v(f^b|_{G_1}) > v(f^r|_{G_1})$. Furthermore, let G_2 be the parent of G_1 in the tree, and by our choice of G_1, $v(f^b|_{G_2}) \leq v(f^r|_{G_2})$. Such a graph G_2 must consist of two series-parallel graphs in parallel, one of which is G_1. If G_2 consists of two series-parallel graphs in series, then $v(f^b|_{G_2}) > v(f^r|_{G_2})$, a contradiction. Furthermore, by the volumes of flows on G_1 and G_2, we have $v(f^b|_{G_2 \setminus G_1}) < v(f^r|_{G_2 \setminus G_1})$.

Let the source and sink of G_1 and G_2 be s' and t'. By Lemma 3.7, there is a path p in G_1 such that $\forall e \in p$ $f^b_e > f^r_e$, moreover, since all players share the same delay function on each edge, we have

$$\begin{array}{rcl} L_p^b(f) & = & \sum_{e\in p} L_e^b(f) \\ & = & \sum_{e\in p} l_e(f) + f_e^b l_e'(f) \\ & > & \sum_{e\in p} l_e(f) + f_e^r l_e'(f) = L_p^r(f). \end{array}$$

Similarly, there exists a path p' starting from s' and ending at t' in $G_2 \backslash G_1$ such that $\forall e \in p'$, $f_e^b < f_e^r$ and as all players share the same delay function on each edge, we have

$$\begin{array}{rcl} L_{p'}^b(f) & = & \sum_{e\in p'} L_e^b(f) \\ & = & \sum_{e\in p'} l_e(f) + f_e^b l_e'(f) \\ & < & \sum_{e\in p'} l_e(f) + f_e^r l_e'(f) \\ & = & \sum_{e\in p'} L_e^r(f) = L_{p'}^r(f) \end{array}$$

Moreover, by Lemma 2.2, we have $L_p^b(f) \leq L_{p'}^b(f)$ and $L_p^r(f) \geq L_{p'}^r(f)$. Putting all these together, we derive $L_{p'}^r(f) \leq L_p^r(f) < L_p^b(f) \leq L_{p'}^b(f) < L_{p'}^r(f)$, a contradiction. ∎

Proof of Theorem 3.5. Suppose that there are two Nash equilibrium flows f and g. We will derive a contradiction. Let the players of the first type be $b_1, b_2, \cdots, b_k$ and those of the second type be $r_1, r_2, \cdots, r_h$. By Lemma 3.6, on every edge e, $f_e^{b_i} = f_e^b$, $g_e^{b_i} = g_e^b$, $\forall i \in \{1, 2, \cdots, k\}$; similarly, $f_e^{r_i} = f_e^r$, $g_e^{r_i} = g_e^r$, $\forall i \in \{1, 2, \cdots, h\}$. As a result, all circulations $\vec{B}^i = g^{b_i} - f^{b_i}, 1 \leq i \leq k$, are identical (i.e., on every edge, their flow values are the same and they are in the same direction); so are all circulations $\vec{R}^j = g^{r_j} - f^{r_j}, 1 \leq j \leq h$. Now we "bundle" all blue and red circulations into two aggregated circulations $\vec{B} = \sum_{i=1}^k \vec{B}^i$ and $\vec{R} = \sum_{j=1}^h \vec{R}^j$.

As an s-t-series-parallel graph is a special case of generalized series-parallel graphs, and as there are only two aggregated circulations $\vec{B}$ and $\vec{R}$, we can apply Lemma 3.3 to graph G and assert that there is an agreeing cycle C. Without loss of generality, let C be a r(ed)-agreeing cycle and the sum of the two circulation $\vec{B}$ and $\vec{R}$ be $\vec{D}$. Given any edge $e \in C$, by Definition 3.1, we know that $\vec{R}_e \cdot \vec{D}_e \geq 0$, furthermore, since $\vec{R} = h\vec{R}^i$, we have $\vec{R}_e^i \cdot \vec{D}_e \geq 0$. Finally, observe that $\vec{D} = \vec{B} + \vec{R} = \sum_{i=1}^k \vec{B}^i + \sum_{j=1}^h \vec{R}^j$. So C is also an agreeing cycle among the $k+h$ (small) circulations. Lemma 3.2 implies that f and g cannot be both equilibrium flows. This is a contradiction. ∎

3.2 Multiplayer Games and Nearly-Parallel Graphs In this section, we prove that a generalized nearly-parallel graph has a unique Nash equilibria with any number of players of any number of types.

We begin with some definitions. A *merge* operation takes a graph $G = (V, E)$ and any two vertices $v_1, v_2 \in V$. It replaces v_1 and v_2 with a single vertex v and each edge $e = (u, w) \in E$ with $u \in \{v_1, v_2\}$ by an edge $e' = (v, w)$. A *two-terminal network* is a triple (G, s, t) where s and t are nodes in the graph G and each vertex and each edge belong to at least one path from s to t. Vertices s and t are called the terminals of G.

Definition 3.8. *A* two-terminal nearly-parallel *graph can be constructed by starting with any of the five two-terminal networks in Figure 3 and applying the following operations any number of times in any order.*

1. series-join*: given (G_1, s_1, t_1), (G_2, s_2, t_2), both nearly-parallel, merge t_1 and s_2 into a single node to produce (G, s_1, t_2).*
2. edge-split*: for an edge $(u, v) \in G$, introduce a vertex w and replace (u, v) with two new edges (u, w) and (w, v).*

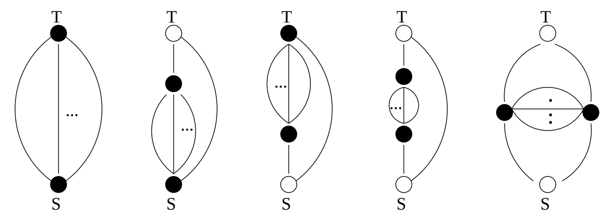


Figure 3: The five basic units of nearly-parallel graphs. The vertices labeled as S and T are the terminals.

Milchtaich [7] characterized nearly-parallel networks as follows: a two-terminal network G is nearly-parallel if and only if it does not contain the five-arc graph shown in Figure 6(i) as a minor.

Equivalently, we define a *component* to be a graph consisting of a set of vertex-disjoint paths connecting two nodes called *hubs*. The solid circles shown in Figure 3 are the hubs by this definition. We will call such a component a *hub-component*. Any two vertices in the hub-component may be the terminals. A two-terminal nearly-parallel graph is any graph composed of hub-components using series-joins.

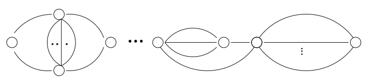
For graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *component-join* operation consists of merging any two vertices $v_1 \in V_1$ and $v_2 \in V_2$ into a single vertex v.

Definition 3.9. *A* generalized nearly-parallel graph *is any graph that can be constructed from hub-components using component-join operations.*

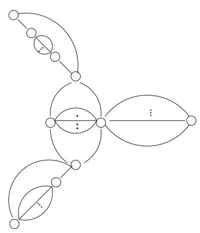
The key difference between two-terminal nearly-parallel graphs and generalized nearly-parallel graphs is that the latter allows hub-components to be connected in a tree structure, while the former only allows them to be connected in a line as demonstrated in Figure 4.

Note that the graph in Figure 6(i) does not contain a cut-node. Since the nodes formed by component-join operations are all cut-nodes, if the graph in Figure 6(i) is a minor of G, it must be a minor of a bi-connected component of G. This, it must be a minor of a nearly-parallel graph. Using Milchtaich's characterization of nearly-parallel two-terminal graphs mentioned above, it follows that a graph is generalized nearly-parallel if and only if it does not contain this graph as a minor.

The next theorem characterizes generalized nearly-parallel graphs in terms of agreeing cycles.



(i) Two-terminal nearly-parallel graph.



(ii) Generalized nearly-parallel graph.

Figure 4: A comparison of two-terminal nearly-parallel graphs and generalized nearly-parallel graphs.

THEOREM 3.10. *An undirected graph G is generalized nearly-parallel if and only if given any set of circulations $\{\vec{f}^1, \vec{f}^2, \cdots, \vec{f}^k\}$, there exists a j-agreeing cycle, for some $1 \leq j \leq k$.*

Proof. ($\Rightarrow$) We first consider the case that G is a single hub-component with hubs v and w. Let $\vec{f}$ be the sum of all the k circulations. Since every circulation $\vec{f^j}$ sends the same amount of flow on every edge along a given path p from v to w, we can reduce p to a single edge e_p. If $\vec{f_e} = 0\,\forall e \in E$, then every $\vec{f^j}$ agrees with $\vec{f}$. So we can assume there is an edge e with $|\vec{f_e}| > 0$. Re-orient all edges from w to v so that $\vec{f_e} > 0$ means that $\vec{f}$ goes from w to v while $\vec{f_e} < 0$ means $\vec{f}$ is from v to w on edge e. Choose an edge $\hat{e}$ satisfying $\vec{f_{\hat{e}}} < 0$ and let $\mathbb{F}_{wv}$ be the set of circulations that agree with $\vec{f}$ on $\hat{e}$: $\mathbb{F}_{wv} = \{\vec{f^j} | \vec{f^j_{\hat{e}}} \cdot \vec{f_{\hat{e}}} > 0\}$. A circulation in $\mathbb{F}_{wv}$ restricted to $G\backslash\{\hat{e}\}$ is a flow from w to v. Moreover, the total circulation $\vec{f}$ restricted to $G\backslash\{\hat{e}\}$ is also a flow from w to v. So there exists a non-empty set of edges $E^+ = \{e | \vec{f_e} > 0\}$.

We assume that there is no agreeing cycle and will show that this assumption leads to a contradiction. Let $\mathbb{F} = \{\vec{f^j} | \vec{f^j_e} \cdot \vec{f_e} > 0,$ for some $e \in E^+.\}$. If there is a circulation $\vec{f^j} \in \mathbb{F} \cap \mathbb{F}_{wv}$, then there is a j-agreeing cycle composed of $\{\hat{e}, e\}$ for some $e \in E^+$. Thus, we can assume $\mathbb{F} \cap \mathbb{F}_{wv} = \emptyset$. Furthermore, if there is a circulation $\vec{f^j} \in \mathbb{F}$ and an edge $e \in E - (E^+ \cup \{\hat{e}\})$ with $|\vec{f^j_e}| > 0$ and $\vec{f^j_e} \cdot \vec{f_e} \geq 0$, then there is a j-agreeing cycle composed of $\{e, e'\}$ for some $e' \in E^+$. Thus, if $\vec{f^j} \in \mathbb{F}$ and $\vec{f^j_e} < 0$ for some $e \in G\backslash\{\hat{e}\}$, then $e \in E^+$. Note that the total flow on edges in E^+ is equal to the sum of the circulations in $\mathbb{F}$ restricted to E^+ and the circulations not in $\mathbb{F}$ restricted to E^+. Formally:

$$\left|\sum_{e \in E^+} \vec{f_e}\right| = \left(\left| \sum_{e \in E^+} \sum_{j: \vec{f^j_e} \cdot \vec{f_e} > 0, \vec{f^j} \in \mathbb{F}} \vec{f^j_e} \right| - \left| \sum_{e \in E^+} \sum_{j: \vec{f^j_e} \cdot \vec{f_e} < 0, \vec{f^j} \in \mathbb{F}} \vec{f^j_e} \right| \right) - \left| \sum_{e \in E^+} \sum_{j: \vec{f^j_e} \cdot \vec{f_e} < 0, \vec{f^j} \notin \mathbb{F}} \vec{f^j_{e'}} \right|$$

The first term must be *strictly* greater than 0, since $\vec{f}$ restricted to $G\backslash\{e\}$ is a flow from w to v. The sum of the two terms inside the brackets is ≤ 0, since $\mathbb{F} \cap \mathbb{F}_{wv} = \emptyset$ and if $\vec{f^j} \in \mathbb{F}$ and $\vec{f^j_e} < 0$, then $e \in E^+$. Combining these two facts, we arrive at a contradiction when G is a single hub-component.

Now suppose G is composed of a set of hub-components G_1, G_2, $\cdots$. Observe that a circulation $\vec{f^j}$ can be decomposed into a set of cycle flows, each of which must reside entirely within a hub-component G_i. So $\vec{f^j}$ can be decomposed into a set of disjoint circulations, each of which is entirely contained in a hub-component G_i. The above argument implies that any G_i that has non-zero circulation must contain an agreeing cycle for some circulation $\vec{f^j}$.

($\Leftarrow$) Figure 6(i) shows an example containing a five-arc graph with three circulations that do not have an agreeing cycle. Any graph which is not nearly-parallel must contain such a five-arc graph as a minor. ∎

It follows from this characterization and from Lemma 3.2 that for atomic splittable flow games in generalized nearly-parallel graphs, there is a unique equilibrium.

THEOREM 3.11. *Let $(G,\ \{(v_1, s_1, t_1),\ (v_2, s_2, t_2),\ \cdots,\ (v_k, s_k, t_k)\},\ l)$ be an atomic splittable flow game, where $l_e \in \mathcal{L}\ \forall e \in E$. If graph G is a generalized nearly-parallel graph, then there is a unique equilibrium flow.*

4 Multiple Equilibria

In this section, we exhibit the first examples of Nash equilibria flows in atomic splittable flow games where all delay functions are shared.

As mentioned in Section 1, if all delay functions are polynomials of degree ≤ 3, then there is a unique Nash equilibrium flow up to induced latencies. Therefore, to construct an example with multiple equilibria, we need highly-nonlinear functions. However, such functions can be hard to reason with.

To help with this, we use "elbow" functions, as shown in Figure 5(i), to simulate a convex function with higher second derivatives. Given two Nash equilibrium flows f and g, suppose l_e is an unknown function in $\mathcal{L}$. It is noteworthy that the following six values $f_e, g_e, l_e(f_e), l_e(g_e), l'_e(f_e), l'_e(g_e)$ are critical, because they determine the marginal delays in the two equilibrium flows f and g on edge e. How the function l_e behaves in other places except f_e and g_e does not really matter. As a consequence, we can use two simple linear functions $\tilde{l}_e^1, \tilde{l}_e^2$ to simulate such a function l_e. The key is to make sure that the intersection of $\tilde{l}_e^1$ and $\tilde{l}_e^2$ is "in between" f_e and g_e.

The second important idea in our construction is how we design flow differences. If there are really two Nash equilibrium flows f and g, there cannot be an agreeing cycle among the set of circulations $\{g^i - f^i\}_{1 \leq i \leq k}$, as shown in Lemma 3.2.

4.1 Constructing the Examples

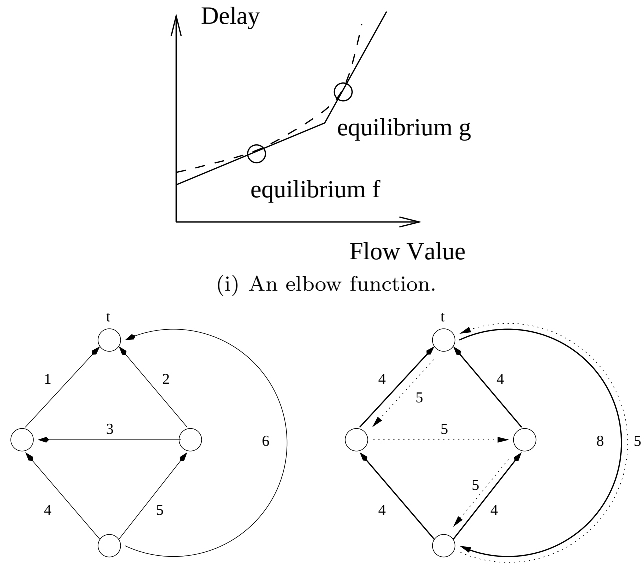
As stated above, the first step in constructing the examples is finding a set of circulations to be the pattern for $g-f$ so that there is no agreeing cycle. The circulations for the examples are given in Table 3 and Table 7. The next step is to choose flows and delay functions so that the marginal delays across parallel paths satisfy Equation 2.3 for both f and g. Additionally, the delay functions must satisfy the properties for elbow functions described above. Writing down these equalities and including nonnegativity and flow conservation constraints gives us a system which, though underconstrained, is quadratic in the unknowns, and an example is simply a feasible solution to this system of equalities and inequalities.

To find such a solution, we proceed by picking values for the unknowns. For each edge, we try to pick flows and delay functions so that the flows constructed thus far satisfy equilibrium conditions. In both examples, the highest indexed edge is what we call the "long edge". The values for the unknowns on this long edge are picked last. Hence, in choosing these values to satisfy the system of equalities and inequalities we have very little freedom, and these values are quite unappealing.

For the second example, picking a large number for the number of players of type a allows us greater freedom in choosing values for the set of circulations, which makes it easier to construct the example.

4.2 Multiple Equilibria with Two Players

In our first example, there are only two players. The graph is shown in Figure 5(ii). Our example is tight in the following sense: it is K_4, the smallest graph that is not a generalized series-parallel graph. This provides the counterpoint to Lemma 3.3.



(i) An elbow function.

(ii) The counter example, K_4.

(iii) The two circulations $g^b - f^b$ (solid line) and $g^r - f^r$ (dotted line).

Figure 5: The delay functions and the graph we use in constructing the examples.

The flow difference shown in Figure 5(iii) does not have an agreeing cycle. The blue player b has flow volume 1425.42125 and the red player r has flow volume 285.42125. The delay functions are the same for edges e_2 and e_4, and for edges e_1 and e_5. The detailed delay functions are in Table 1, edge flows for equilibrium flow f are given in Table 2 and the change in flow $g - f$ are given in Table 3. The marginal delays for each path are in Table 4.[†]

[†]The example can be verified by computing for each path the marginal delay as in Equation 2.2, and confirming that Lemma 2.2 indeed holds for equilibrium flows f and g. For example, for blue player b for equilibrium flow f the marginal delay in using the path formed by edges e_4 and e_1 is $0.02 \times (500 + 500) + 670 + 0.55 \times (500 + 600) + 65 = 690 + 670 = 1360$, and this is less than or equal to the marginal delay on any other $s - t$ path.

Edge	Delay function
e_1, e_5 *	$\begin{cases} 0.55x + 65 & \text{if } x \geq 599.34 \\ 0.1694x + 293.1103 & \text{otherwise} \end{cases}$
e_2,e_4	$0.02x + 670$
e_3	$0.06x + 208$
e_6 *	$\begin{cases} x + 323.7362 & \text{if } x \geq 609.5 \\ 0.569915x + 585.8053 & \text{otherwise} \end{cases}$

Table 1: Delay functions

Edge	Player b	Player r	Total
e_1	500	100	600
e_2	500	0	500
e_3	0	100	100
e_4	500	0	500
e_5	500	100	600
e_6	425.42125	185.42125	610.8425

Table 2: Edge flows for equilibrium flow f

Edge	Player b	Player r	Total
e_1	4	-5	-1
e_2	4	0	4
e_3	0	-5	-5
e_4	4	0	4
e_5	4	-5	-1
e_6	-8	5	-3

Table 3: Change in flow $g - f$

Path	$L_p^b(f)$	$L_p^r(f)$	$L_p^b(g)$ *	$L_p^r(g)$ *
e_4,e_1	1360	1130	1170.1185	1090.7539
e_5,e_3,e_1	1554	1120	1191.617	1040.7478
e_5,e_2	1360	1130	1170.1185	1090.7539
e_6	1360	1120	1170.1185	1040.7478

Table 4: Marginal delays for equilibrium flows f and g on all possible paths

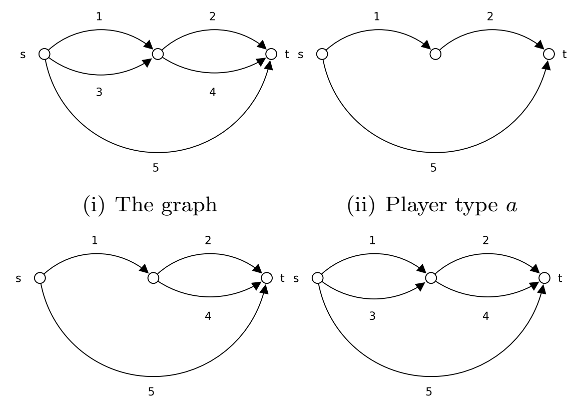
*Values are accurate to 4 decimal places

4.3 Multiple Equilibria with Three Types of Players Our next example demonstrates existence of multiple equilibria on the series-parallel graph in Figure 6(i). This is the smallest graph on which there can be two equilibria, since removing an edge creates a nearly-parallel graph. As with the previous example, we use piece-wise linear delay functions to represent the delay functions on the edges.

The example uses 200 players but only 3 types of players: a, b and c. There are 198 players of type a, and one each of types b and c. Table 5 gives the delay functions for each edge. Table 6 gives the flow on each edge for the first equilibrium flow and Table 7 gives the difference in equilibrium flows, $g - f$. Tables 8 and 9 list the marginal delays on each edge for the players for flows f and g. Since the graph is series-parallel, by Lemma 3.6 all players of type a have the same flow pattern. Thus the tables give the flow and marginal delay of a single type a player.

5 Extensions

In this work, we have made use of the idea of agreeing cycles to obtain several equilibrium uniqueness results.



(i) The graph (ii) Player type a

(iii) Player type b (iv) Player type c

Figure 6: Graph for the counter example and edges used by the players

In presenting our results, we made two simplifying assumptions:

1. All players use shared delay functions on each edge.
2. For a player b, his delay is measured by the product of his own flow f_e^b and the delay $l_e(f_e)$ on the edge e. Thus his marginal delay is equivalent to $l_e(f_e) + f_e^b l_e'(f_e)$.

Neither assumption is necessary for the proofs of Theorem 3.4 and Theorem 3.11. Even if each player has his own delay function l_e^b, these results still hold. The reason is that both these results are proved relying on the fact that agreeing cycles exist. From the proof of Lemma 3.2, it is not hard to observe that the delay functions used throughout the proof need not be shared delay functions. They can be player-specific functions l_e^b.

Moreover, Theorem 3.4 and Theorem 3.11 hold even if the delay of player b takes a more complicated form (instead of simply the product $f_e^b l_e^b(f_e)$.) In particular, we can express the delay for player b on edge e as a function $J_e^b(f_e^b, f_e)$. His marginal delay on edge e can be thus expressed as a function $K_e^b(f_e^b, f_e) = \frac{\partial}{\partial f_e^b} J_e^b(f_e^b, f_e) + \frac{\partial}{\partial f_e} J_e^b(f_e^b, f_e)$. As long as the function $K_e^b(f_e^b, f_e)$ is strictly increasing in both parameters f_e^b and f_e, Lemma 3.2 still holds. This can be inferred by carefully observing Inequalities 3.7 and 3.8. Both (strict) inequalities hold because of the monotonicity of f_e^b and f_e.

Acknowledgment

We thank Shahar Dobzinski and Zoya Svitkina for helpful conversations.

References

[1] Eitan Altman, Tamer Basar, Tania Jimenez, and Nahum Shimkin. Competitive routing in networks with polynomial costs. *IEEE Transactions on Automatic Control*, 47(1):92–96, January 2002.

[2] Stefano Catoni and Stefano Pallottino. Traffic equilibrium paradoxes. *Transportation Science*, 25(3):240–244, August 1991.

[3] Roberto Cominetti, Jose R. Correa, and N. E. Stier-Moses. The impact of oligopolistic competition in networks. *Operation Research*, 2008. to appear, extended abstract entitled "Network games with atomic players" appeared in ICALP 06.

[4] P. Harker. Multiple equilibrium behaviors on networks. *Transportation Science*, 22(1):39–46, 1988.

[5] Ara Hayrapetyan, Éva Tardos, and Tom Wexler. The effect of collusion in congestion games. In *STOC '06: Proceedings of the Thirty-Eighth Annual ACM Symposium on Theory of Computing*, pages 89–98, New York, NY, USA, 2006. ACM Press.

[6] N.M. Korneyenko. Combinatorial algorithms on a class of graphs. *Discrete Applied Mathematics*, 54(2-3):215–217, 1994.

[7] Igal Milchtaich. Topological conditions for uniqueness of equilibrium in networks. *Math. Oper. Res.*, 30(1):225–244, 2005.

[8] T. Nishizeki and N. Chiba. *Planar Graphs: Theory and Algorithms*. North-Holland, 1988.

[9] J. Nocedal and S.T. Wright. *Numerical Optimization*. Springer, 2006.

[10] Ariel Orda, Raphael Rom, and Nahum Shimkin. Competitive routing in multiuser communication networks. *IEEE/ACM Transactions on Networking*, 1(5):510–521, 1993.

[11] Oran Richman and Nahum Shimkin. Topological uniqueness of the Nash equilibrium for selfish routing with atomic users. *Math. Oper. Res.*, 32(1):215–232, 2007.

[12] T. Roughgarden. Stackelberg scheduling strategies. *SIAM Journal on Computing*, 33(2):332–350, 2004.

[13] T. Roughgarden. Selfish routing with atomic players. In *SODA 2005: ACM-SIAM Symposium on Discrete Algorithms*, pages 1184–1185, 2005.

[14] Tim Roughgarden. *Selfish Routing and the Price of Anarchy*. The MIT Press, 2005.

[15] Tim Roughgarden and Eva Tardos. How bad is selfish routing? *Journal of the ACM*, 49(2):236–259, March 2002.

[16] Chaitanya Swamy. The effectiveness of stackelberg strategies and tolls for network congestion games. In *SODA 2007: ACM-SIAM Symposium on Discrete Algorithms*, pages 1133–1142, 2007.

[17] H.R. von Stackelberg. *Marktform und Gleichgewicht*. Springer-Verlag, 1934. English translation entitled as Market Structure and Equilibrium, published in 1952 by Oxford University Press.

[18] J. G. Wardrop. Some theoretical aspects of road traffic research. In *Proc. Institute of Civil Engineers, Pt. II*, volume 1, pages 325–378, 1952.

Edge	Delay function
e_1	$\begin{cases} 0.8x + 265.46961725 & \text{if } x \leq 760.985 \\ 1.14885x & \text{otherwise} \end{cases}$
e_2	$\begin{cases} x + 180.349 & \text{if } x \leq 3606.98 \\ 1.05x & \text{otherwise} \end{cases}$
e_3 *	$\begin{cases} \dfrac{3.06725}{16.02} \times 10^{-3}x + 1195.24990651 \end{cases}$
e_4	$\dfrac{10.75}{15}x + 214.329$
e_5 *	$\begin{cases} 9.14822205205 \times 10^{-7}x & \text{if } x \leq 413000000 \\ 1.23592884775 \times 10^{-3}x - 510060.792550 & \text{if } 413000000 \leq x \leq 414410220 \\ 1.23799886951 \times 10^{-3}x - 510918.630724 & \text{otherwise} \end{cases}$

Table 5: Delay functions

Edge	Player a	Player b	Player c
e_1	0.01	354	405
e_2	0.01	154	3450
e_3 *	0	0	7844.06976744
e_4 *	0	200	4799.06976744
e_5 *	2055014.64831	2408742.13092	5108573.92631

Edge	Total
e_1	760.98
e_2	3605.98
e_3 *	7844.06976744
e_4 *	4999.06976744
e_5 *	414410216.423

Table 6: Edge flows for equilibrium flow f

Edge	Player a	Player b	Player c	Total
e_1	1	-75	-122.99	0.01
e_2	1	-40	-156	2
e_3	0	0	-8.01	-8.01
e_4	0	-35	25	-10
e_5	-1	75	131	8

Table 7: Change in flow $g - f$

Edge	Player a	Player b	Player c
e_1 *	874.26161725	1157.45361725	1198.25361725
e_2	3786.339	3940.329	7236.329
e_3 *	1196.75176188	1196.75176188	1198.25361
e_4	3796.995666	3940.329	7236.329
e_5 *	4660.60061725	5097.78261725	8434.58261725

Table 8: Marginal delays for flow f

Edge	Player a	Player b	Player c
e_1 *	875.4237	1194.79251151	1198.25055
e_2	3789.4395	3908.079	7247.079
e_3 *	1196.75022826	1196.75022826	1198.25055
e_4	3789.829	3908.079	7247.079
e_5 *	4664.8632	5102.87151151	8445.32955

Table 9: Marginal delays for flow g

*Values are approximate and are shown correct to 12 significant digits