Popularity, Mixed Matchings, and Self-duality

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Abstract

Our input instance is a bipartite graph $G = (A \cup B, E)$ where A is a set of applicants, B is a set of jobs, and each vertex $u \in A \cup B$ has a preference list ranking its neighbors in a strict order of preference. For any two matchings M and T in G, let $\phi(M,T)$ be the number of vertices that prefer M to T. A matching M is *popular* if $\phi(M,T) \geq \phi(T,M)$ for all matchings T in G. There is a utility function $w : E \to \mathbb{Q}$ and we consider the problem of matching applicants to jobs in a popular and utility-optimal manner. A popular *mixed matching* could have a much higher utility than all popular matchings, where a mixed matching is a probability distribution over matchings, i.e., a mixed matching $\Pi = \{(M_0, p_0), \ldots, (M_k, p_k)\}$ for some matchings $M_0, \ldots, M_k$ and $\sum_{i=0}^{k} p_i = 1$, $p_i \geq 0$ for all i. The function $\phi(\cdot,\cdot)$ easily extends to mixed matchings; a mixed matching Π is popular if $\phi(\Pi, \Lambda) \geq \phi(\Lambda, \Pi)$ for all mixed matchings Λ in G.

Motivated by the fact that a popular mixed matching could have a much higher utility than all popular matchings, we study the popular fractional matching polytope $\mathcal{P}_G$. Our main result is that this polytope is half-integral and in the special case where a stable matching in G is a perfect matching, this polytope is integral. This implies that there is always a max-utility popular mixed matching Π such that $\Pi = \{(M_0, \frac{1}{2}), (M_1, \frac{1}{2})\}$ where M_0 and M_1 are matchings in G. As Π can be computed in polynomial time, an immediate consequence of our result is that in order to implement a max-utility popular mixed matching in G, we need just *a single* random bit.

We analyze $\mathcal{P}_G$ whose description may have exponentially many constraints via an extended formulation with a linear number of constraints. The linear program that gives rise to this formulation has an unusual property: *self-duality*. In other words, this linear program is identical to its dual program. This is a rare case where an LP of a natural problem has such a property. The self-duality of this LP plays a crucial role in our proof of half-integrality of $\mathcal{P}_G$.

We also show that our result carries over to the *roommates* problem, where the graph G need not be bipartite. The polytope of popular fractional matchings is still half-integral here and so we can compute a max-utility popular half-integral matching in G in polynomial time. To complement this result, we also show that the problem of computing a max-utility popular (integral) matching in a roommates instance is NP-hard.

1 Introduction

Let $G = (A \cup B, E)$ be a bipartite graph on n vertices and m edges where A is called the set of applicants, B is called the set of jobs, and every vertex $u \in A \cup B$ has a preference list ranking its neighbors in a strict order of preference. Such a graph G is also referred to as an instance of the *stable marriage* problem with strict and possibly incomplete preference lists. Moreover, a utility function $w : E \to \mathbb{Q}$ is given, where $w(a,b)$ is the utility of matching applicant a with job b. Our goal is to match applicants to jobs such that this matching is *popular* and has the maximum utility among all popular matchings, where the utility $w(M)$ of a matching M is the sum of utilities of all edges in M.

The notion of popularity was introduced by Gärdenfors [18] in 1975. For any two matchings M and M' in G, let $\phi(M, M')$ be the number of vertices that *prefer* M to M', where we say a vertex $u \in A \cup B$ prefers matching M to matching M' if either u is matched in M and unmatched in M' or u is matched in both the matchings and $M(u)$ ranks better than $M'(u)$ in u's preference list. We say M is *more popular than* M' if $\phi(M, M') > \phi(M', M)$.

DEFINITION 1. *A matching M is* popular *if $\phi(M, M') \geq \phi(M', M)$ for every matching M' in G, i.e., $\Delta(M, M') \geq 0$ where $\Delta(M, M') = \phi(M, M') - \phi(M', M)$.*

Thus a popular matching never loses an election (where vertices cast votes) and so a popular matching can be considered to be "globally stable" since an election cannot force a migration from a popular matching to any other matching. The notion of popularity is naturally appealing and is less demanding than the notion of *stability*. A matching is stable if it has no blocking edges: an edge (a,b) blocks matching M if both a and b prefer each other to their respective assignments in M. The existence of stable matchings and the Gale-Shapley algorithm [16] to find one are classical results in algorithms. It is easy to show that every stable matching is popular [18].

There are many polynomial time algorithms to compute a max-utility stable matching in G [12, 13, 15, 22, 33, 36, 37] – several of these use linear programming on the stable matching polytope $\mathcal{S}_G$, which is the convex hull of the 0-1 edge incidence vectors of stable matchings in G. The utility of a max-utility popular matching could be much more than that of a max-utility stable matching; for instance, when all utilities are 1, a max-utility popular matching is the same as a max-size popular matching and a stable matching is a *min-size* popular matching [20].

Mixed matchings. A *mixed matching* is a probability distribution over matchings, i.e., a mixed matching $\Pi = \{(M_0, p_0), \ldots, (M_k, p_k)\}$, where $M_0, \ldots, M_k$ are matchings in G and $\sum_{i=0}^{k} p_i = 1$, $p_i \geq 0$ for all i, and the utility of Π is $\sum_{i=0}^{k} p_i \cdot w(M_i)$. Our objective is to find a max-utility

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matching under the constraints of popularity — so as to achieve highest utility, what we seek should be a popular mixed matching rather than a popular (pure) matching.

In economics, mixed matchings are called random assignments and they are used in the design of mechanisms to guarantee various desirable properties, such as efficiency, fairness, and strategy-proofness (see [7, 23] and the references therein). The function $\phi(M,M')$ defined earlier easily extends to $\phi(\Pi,M')$ as follows: $\phi(\Pi,M') = \sum_{i=0}^{k} p_i \cdot \phi(M_i,M')$, where $\Pi = \{(M_0,p_0),\ldots,(M_k,p_k)\}$. The definition of $\phi(M',\Pi))$ is analogous.

DEFINITION 2. *A mixed matching Π is popular if $\phi(\Pi,M') \geq \phi(M',\Pi)$ for all matchings M' in G, i.e., $\Delta(\Pi,M') \geq 0$, where $\Delta(\Pi,M') = \phi(\Pi,M') - \phi(M',\Pi)$.*

Suppose $\Lambda = \{(N_0,q_0),\cdots,(N_h,q_h)\}$ is another mixed matching. Let $\Delta(\Pi,\Lambda) = \sum_{j=0}^{h} q_j \cdot \Delta(\Pi,N_j)$. Then it follows easily from Definition 2 that if Π is popular then $\Delta(\Pi,\Lambda) \geq 0$ for all mixed matchings Λ in G. Thus a popular mixed matching never loses to any integral or mixed matching. As an allocation mechanism, a popular mixed matching has several nice properties, such as population-consistency and composition-consistency. We refer the reader to [5] for details.

A popular mixed matching need not be a probability distribution over popular matchings in G. Such an example was shown in [25] and we show a much simpler example here in Fig. 1. Let $A = \{a_0,a_1,a_2\}$, $B = \{b_1,b_2\}$, and $E = A \times B$. The preference list of every applicant is the same: $b_1 \succ b_2$, i.e., b_1 is the top choice and then b_2; the preference list of every job is the same: $a_1 \succ a_2 \succ a_0$. This instance admits exactly one popular matching: this is the stable matching $S = \{(a_1,b_1),(a_2,b_2)\}$ (the blue matching in Fig. 1). Consider the mixed matching $\Pi = \{(S,\frac{1}{2}),(M,\frac{1}{2})\}$ where $M = \{(a_1,b_2),(a_2,b_1)\}$ (the red matching in Fig. 1). Note that Π is outside the convex hull of popular matchings – we will show in Section 2 that Π is popular. Whenever $w(a_1,b_2)+w(a_2,b_1)$ is larger than $w(a_1,b_1)+w(a_2,b_2)$, the utility of Π is higher than that of S. Thus the utility of a max-utility popular mixed matching can be much higher than that of a popular matching.

Mixed matchings are closely related to *fractional* matchings. A fractional matching $\vec{x} = (x_e)_{e \in E}$ in G is a point in $\mathbb{R}^m_{\geq 0}$ that satisfies $\sum_{e \in E(v)} x_e \leq 1$ for every vertex v, where $E(v)$ is the set of edges incident on vertex v. In a bipartite graph, a fractional matching is equivalent to a mixed matching (Birkhoff-von Neumann theorem).

The polytope $\mathcal{P}_G \subseteq \mathbb{R}^m$ of all popular fractional matchings in $G = (A \cup B,E)$ involves possibly exponentially many constraints (one for each matching in G). However a compact extended formulation of this polytope was given in [26]. Thus a max-utility popular mixed matching can be computed in polynomial time via linear programming on this polytope.

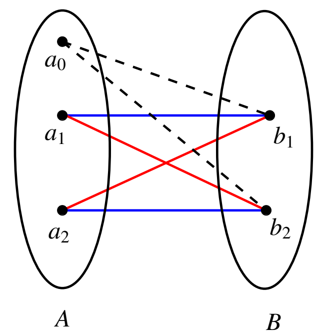

Figure 1: The blue matching $S = \{(a_1,b_1),(a_2,b_2)\}$ is the only popular matching here. The red matching $M = \{(a_1,b_2),(a_2,b_1)\}$ is not popular as $\{(a_0,b_2),(a_1,b_1)\}$ is more popular than M.

However a potential drawback of generalizing from matchings to mixed matchings is that the optimal solution has become more complex to describe and more difficult to implement. A mixed matching can be interpreted as either a lottery over matchings or a time-sharing arrangement (when the mixed matching is viewed as a fractional matching) [30]: in the former case, we need access to several random bits to implement a lottery and the latter case involves sub-dividing jobs and assigning several fractional jobs to an applicant. Thus we may need to deal with an unstructured optimal solution. Our first result is the following.

THEOREM 1.1. *Given an instance $G = (A \cup B,E)$ with strict preference lists and a utility function $w : E \to \mathbb{Q}$, G always has a max-utility popular mixed matching $\Pi = \{(M_0,\frac{1}{2}),(M_1,\frac{1}{2})\}$ where M_0 and M_1 are matchings in G. Moreover, if G admits a perfect stable matching, i.e., a stable matching where no vertex is unmatched, then $\Pi = \{(M,1)\}$ for some matching M in G, i.e., Π is pure.*

Thus our result implies that to achieve max-utility, we just need a *single random bit* to implement the lottery or we can find a time-sharing arrangement that is simple and organized—every vertex has at most two partners and spends the same amount of time with each. Hence we can find a max-utility popular mixed matching that is highly structured. Note that it was already known how to compute a max-utility popular half-integral matching in G in polynomial time [25]. However it was not known whether this was a max-utility popular fractional matching or not.

Our main contribution is to show that this is so by proving that the polytope $\mathcal{P}_G$ is half-integral. Moreover, $\mathcal{P}_G$ is integral when G admits a perfect stable matching (i.e., $|A| = |B|$ and say, preference lists are complete). Also the linear program that we solve in order to find a max-utility popular half-integral matching is simpler than the linear program used in [25] to find a max-utility popular half-integral matching.

Note that it is necessary that preference lists of vertices in G are *strict*, otherwise the half-integrality of $\mathcal{P}_G$ need not hold. Consider the instance in Fig. 1 and let the preference list of every applicant be the same as discussed earlier, i.e., b_1 is preferred to b_2 by each of a_0, a_1, a_2. In the preference lists of both b_1 and b_2, let us now say that the 3 applicants a_0, a_1, a_2 are *tied together*. It is easy to check that that there is no popular matching here. In fact, there is *no* popular mixed matching of the form $\{(M_0, \frac{1}{2}), (M_1, \frac{1}{2})\}$ for any matchings M_0, M_1 in G. However there is a popular mixed matching $\Pi = \{(M_0, \frac{1}{3}), (M_1, \frac{1}{3}), (M_2, \frac{1}{3})\}$ where $M_0 = \{(a_0,b_1),(a_1,b_2)\}$, $M_1 = \{(a_1,b_1),(a_2,b_2)\}$, $M_2 = \{(a_2,b_1),(a_0,b_2)\}$. Thus the polytope of popular fractional matchings is not half-integral here.

The complexity of finding a max-utility popular matching in $G = (A \cup B, E)$ with strict preference lists is currently unknown. In this paper we also consider the max-utility popular matching problem in a *roommates* instance G with strict preference lists, i.e., G is a general graph (not necessarily bipartite), and show the following result.

THEOREM 1.2. *Given $G = (V,E)$ where each vertex has a strict preference list over its neighbors and a utility function $w : E \to \mathbb{Q}$, the problem of computing a max-utility popular matching in G is NP-hard.*

It is known that the max-utility stable matching problem in a roommates instance is also NP-hard, as shown by Feder [12]. In this paper we also show that it is UGC-hard to design a polynomial time $O(1)$-approximation algorithm for the max-utility popular matching problem in a roommates instance with non-negative edge utilities.

We consider the problem of computing a max-utility popular half-integral matching in a roommates instance. As before, a fractional matching $\vec{x} = (x_e)_{e\in E}$ in G is a point in $\mathbb{R}^m_{\geq 0}$ that satisfies $\sum_{e\in E(u)} x_e \leq 1$ for every vertex u. A *half-integral* matching is a fractional matching $\vec{q}$ where $q_e \in \{0, \frac{1}{2}, 1\}$ for every $e \in E$. Unlike the equivalence between fractional matchings and mixed matchings in bipartite graphs, a fractional matching need not be a mixed matching in a non-bipartite graph G. For instance, let G be the triangle on 3 vertices a, b, c and let $\vec{x}$ be the fractional matching that sets $x_{(a,b)} = x_{(b,c)} = x_{(c,a)} = \frac{1}{2}$; $\vec{x}$ is not a convex combination of matchings in G. However a fractional matching is always a convex combination of half-integral matchings in any graph.

The function $\Delta(\Pi, \Lambda)$ defined earlier for mixed matchings easily generalizes to fractional matchings by summing up the (fractional) votes cast by all the vertices for one fractional matching versus another (Section 5 has these details). Thus two fractional matchings can be compared with respect to popularity and a fractional matching $\vec{x}$ is popular if $\Delta(\vec{x},\vec{y}) \geq 0$ for all fractional matchings $\vec{y}$ in G. Since a fractional matching is a convex combination of half-integral matchings, a fractional matching $\vec{x}$ is popular if $\Delta(\vec{x},\vec{q}) \geq 0$ for all half-integral matchings $\vec{q}$ in G.

We show that when there is a utility function $w : E \to \mathbb{Q}$, there is a polynomial time algorithm to compute a max-utility popular half-integral matching $\vec{q}$ in G and in fact, $\vec{q}$ is also a max-utility popular fractional matching in G. There was no polynomial time algorithm known so far for computing a max-utility popular half-integral matching in a roommates instance.

1.1 Our Techniques. Let $G = (A \cup B, E)$ be an instance of the stable marriage problem with strict preference lists where $|A \cup B| = n$ and $|E| = m$. The polytope $\mathcal{P}_G$ of popular fractional matchings in G has a compact extended formulation $\mathcal{P}'_G \subseteq \mathbb{R}^{m+2n}$ and in this paper, we analyze the linear program that gives rise to $\mathcal{P}'_G$ and discover an unusual property of this linear program. This LP is *self-dual*, i.e., it is exactly identical to its dual program.[1] To the best of our knowledge, this seems to be the first time a natural problem has an LP with this property and our proof on the structure of $\mathcal{P}_G$ uses this self-duality crucially. Prior to our result, the only linear program that had a somewhat similar property is the one used by Roth, Rothblum, and Vande Vate [30] to describe the stable matching polytope $\mathcal{S}_G$. Every stable fractional matching is an optimal solution to their original LP there but also gives rise to an optimal solution in its dual. In the original words of Roth et al. [30]: *"We know of no similarly rich class of linear programs whose primal and dual solutions are related in this way."*

We add n new variables for the extended formulation and by setting these to 0, the description of $\mathcal{P}'_G$ reduces to the description of $\mathcal{S}_G$. This formulation of $\mathcal{S}_G$ is not independent of other descriptions of this polytope [30, 36], however it has a novel and intuitive interpretation where stability is interpreted as the "sum of votes" of any adjacent pair of vertices for each other being at most 0 (see constraint (2.4) in Section 2).

Our technique to prove the integrality of $\mathcal{P}_G$ in the special case when $G = (A \cup B, E)$ admits a perfect stable matching is inspired by the one used by Teo and Sethuraman [36] to show the integrality of $\mathcal{S}_G$. However as the description of the extended popular fractional matching polytope $\mathcal{P}'_G$ is more general than that of $\mathcal{S}_G$, our task is more involved here. We use the fact that G admits a perfect stable matching to first conclude that for any integral matching M in G, there is a *witness* $(\alpha_u)_{u\in A\cup B} \in \{\pm 1\}^n$ to the popularity of M, where the α_u's are the n new variables that were added for the extended formulation.

For any $(\vec{x}, \vec{\alpha}) \in \mathcal{P}'_G$, where $\vec{x}$ is a popular fractional

[1] We remark that the same LP was introduced in [26]. However, its self-duality is observed and explicitly used for the first time in the present work.

matching in G and $\vec{\alpha}$ its witness, in order to find popular matchings in G whose convex hull contains $\vec{x}$, we use the following new idea: for any vertex u, use the value of α_u to divide into two sub-arrays the ordered array of partners that u gets assigned in $\vec{x}$ and swap these two sub-arrays to get a reordered array. We obtain from these reordered arrays the popular matchings $M_1,\ldots,M_k$ such that $\vec{x} = \sum_{i=1}^{k} p_i M_i$ where $\sum_{i=1}^{k} p_i = 1$ and $p_i \geq 0$ for all i. The popularity of these matchings will be proved by assigning an appropriate witness $\vec{\alpha}^i \in \{\pm 1\}^n$ to each M_i and showing that each M_i along with its witness $\vec{\alpha}^i$ belongs to $\mathcal{P}'_G$.

The half-integrality of $\mathcal{P}_G$ for the general case (when G has no perfect stable matching) and for non-bipartite graphs follows from the integrality of $\mathcal{P}_G$ in this special case. For non-bipartite graphs, we show a simple reduction from the vertex cover problem to show that it is NP-hard to compute a max-utility popular matching in a roommates instance.

1.2 Background and related work. As mentioned earlier, popular matchings were introduced by Gärdenfors [18] in bipartite graphs with two-sided preferences, i.e., vertices on both sides of the graph have preferences over their neighbors. For one-sided preferences (i.e., vertices of only one side have preferences over their neighbors), similar notions have been suggested by Kreweras [28] and Fishburn [14]. Popular matchings need not always exist in this setting and an efficient algorithm was given in [2] to determine if the given instance admits a popular matching or not. It was shown in [26] that popular mixed matchings always exist here and such a mixed matching can be efficiently computed. Popular mixed matchings as an allocation mechanism have been studied in [3, 4, 5].

In the domain of two-sided preferences, when preference lists involve ties, the problem of determining if $G = (A \cup B, E)$ admits a popular matching or not is NP-hard [6, 9]. When preference lists are strict, popular matchings always exist in G and efficient algorithms for computing a max-size popular matching were given in [20, 24]. A subclass of max-size popular matchings called dominant matchings were studied in [10]. The convex hull of the $\{0, \frac{1}{2}, 1\}$-edge incidence vectors of popular half-integral matchings in G was described in [25] – this was done via the stable matching polytope of a larger graph G^* that was constructed using G.

Stable matchings have been extensively studied and there are several monographs [19, 27, 29, 34] on this subject. Roth [31, 32] discusses how stable matchings compare in practice with other types of matchings in the two-sided matching markets. The first description of the polytope $\mathcal{S}_G$ for $G = (A \cup B, E)$ was given by Vande Vate [37] in 1989 and several descriptions of $\mathcal{S}_G$ are now known [11, 15, 30, 33, 36].

Stable matchings need not always exist in the roommates problem and efficient algorithms to determine if a roommates instance $G = (V, E)$ admits a stable matching or not were given in [21, 36]. Stable half-integral matchings always exist in G [35] and it was shown in [1] that the polytope of all stable fractional matchings in G is half-integral.

Organization of the paper. Section 2 describes the extended popular fractional matching polytope $\mathcal{P}'_G$ and shows the self-duality of the LP that gives rise to this description. Section 3 shows the integrality of the popular fractional matching polytope $\mathcal{P}_G$ when G is bipartite and admits a perfect stable matching. Section 4 shows the half-integrality of $\mathcal{P}_G$ in any instance $G = (A \cup B, E)$. Section 5 shows the half-integrality of $\mathcal{P}_G$ in a roommates instance $G = (V, E)$ with strict preference lists and Section 6 shows the NP-hardness of the max-utility popular matching problem in $G = (V, E)$.

2 The extended popular fractional matching polytope

Our input instance is $G = (A \cup B, E)$ on n vertices and m edges. Let $\vec{x}$ be any fractional matching in G, that is, $x \in \mathbb{R}^m_{\geq 0}$ and $\sum_{e \in E(u)} x_e \leq 1$ for every vertex u. It will be convenient for us to assume that each vertex is *fully matched* in any fractional matching. So for each vertex u, a new vertex $\ell(u)$ called u's *last resort neighbor* will be added at the bottom of u's preference list.

In the fractional matching $\vec{x}$, we will set $x_{(u,\ell(u))} = 1 - \sum_{e \in E(u)} x_e$ for each vertex u. Let $\tilde{E}$ denote the edge set $E \cup \{(u, \ell(u)) : u \in A \cup B\}$ and let $\tilde{E}(u)$ be the set of edges in $\tilde{E}$ that are incident on u. Let $\mathcal{M}_G$ be the matching polytope of G: so $\mathcal{M}_G = \{x \in \mathbb{R}^{m+n}_{\geq 0} : \sum_{e \in \tilde{E}(u)} x_e = 1 \ \forall u \in A \cup B.\}$

The popular fractional matching polytope of G is:

$$\mathcal{P}_G = \{\vec{x} \in \mathcal{M}_G : \Delta(\vec{x}, M) \geq 0 \ \ \forall \text{matchings } M \text{ in } G\}, \tag{2.1}$$

where $\Delta(\vec{x}, M) = \Delta(\Pi, M)$ and Π is a mixed matching corresponding to the fractional matching $\vec{x}$ (recall that in a bipartite graph, a fractional matching is equivalent to a mixed matching). Alternately, $\Delta(\vec{x}, M) = \sum_{u \in A \cup B} \mathsf{vote}_u(\vec{x}, M(u))$, where $M(u)$ is u's partner in M and the function $\mathsf{vote}_u(v, v')$ is defined as follows:

$$\mathsf{vote}_u(v, v') = \begin{cases} 1 & \text{if } u \text{ prefers } v \text{ to } v', \\ -1 & \text{if } u \text{ prefers } v' \text{ to } v, \\ 0 & \text{otherwise (i.e., } v = v'). \end{cases}$$

This extends by linearity to $\mathsf{vote}_u(\vec{x}, v')$ as follows:

$$\begin{aligned} \mathsf{vote}_u(\vec{x}, v') &= \sum_{v} x_{(u,v)} \mathsf{vote}_u(v, v') \\ &= \sum_{v : v \succ_u v'} x_{(u,v)} - \sum_{v : v \prec_u v'} x_{(u,v)}, \end{aligned}$$

where $\{v : v \succ_u v'\}$ consists all neighbors of u that are ranked better than v' in u's preference list and the set $\{v : v \prec_u v'\}$ consists of all those who are ranked worse; the vertex $\ell(u)$ is a member of the latter set.

The description of $\mathcal{P}_G$ in (2.1) above involves possibly exponentially many constraints – one for each matching in G. Hence we will use the *extended* popular fractional matching polytope $\mathcal{P}'_G$ of G (from [26]) that uses n new variables α_u for $u \in A \cup B$ along with the $m+n$ variables x_e for $e \in \tilde{E}$. Note that the edge utilities given in the input instance G play no part in the description of either $\mathcal{P}_G$ or $\mathcal{P}'_G$.

The graph $\tilde{G}_x$. The graph $\tilde{G}_x$ is the graph G augmented with last resort vertices and with edge set $\tilde{E}$ where the weight of edge (a,b) (denoted by $\mathsf{wt}_x(a,b)$) is equal to $\mathsf{vote}_a(b,\vec{x}) + \mathsf{vote}_b(a,\vec{x})$, where $\mathsf{vote}_u(v,\vec{x}) = -\mathsf{vote}_u(\vec{x},v')$ for any $(u,v) \in \tilde{E}$. Thus

$$\begin{aligned} \mathsf{wt}_x(a,b) &= \sum_{b' : b' \prec_a b} x_{(a,b')} - \sum_{b' : b' \succ_a b} x_{(a,b')} \\ &\quad + \sum_{a' : a' \prec_b a} x_{(a',b)} - \sum_{a' : a' \succ_b a} x_{(a',b)}. \end{aligned}$$

For any $u \in A \cup B$, the weight of the edge $(u, \ell(u))$ is

$$\mathsf{wt}_x(u,\ell(u)) = \mathsf{vote}_u(\ell(u),\vec{x}) = -\sum_{v \in \mathsf{Nbr}(u)} x_{(u,v)},$$

where $\mathsf{Nbr}(u)$ is the set of neighbors of u in the original G, so $\ell(u) \notin \mathsf{Nbr}(u)$. For any matching M in $\tilde{G}_x$ that matches all vertices in $A \cup B$, observe that the weight of M in $\tilde{G}_x$ is exactly the same as $\Delta(M,\vec{x})$.

So $\vec{x} \in \mathcal{M}_G$ is popular if and only if the maximum weight of a matching in the graph $\tilde{G}_x$ that matches all vertices in $A \cup B$ is at most 0. Note that this value cannot be less than 0 since the weight of the fractional matching $\vec{x}$ in $\tilde{G}_x$ is 0. Thus $\vec{x} \in \mathcal{M}_G$ is popular if and only the maximum weight of a matching in the graph $\tilde{G}_x$ that matches all vertices in $A \cup B$ is exactly 0.

The maximum weight matching problem in the graph $\tilde{G}_x$ that matches all vertices in $A \cup B$ is the following linear program in variables y_e, for $e \in \tilde{E}$.

$$\text{maximize} \sum_{e \in \tilde{E}} \mathsf{wt}_x(e)\, y_e$$

$$\begin{aligned} \textstyle\sum_{e \in \tilde{E}(u)} y_e &= 1 \ \ \forall u \in A \cup B \\ y_e &\geq 0 \ \ \forall e \in \tilde{E}. \end{aligned}$$

Consider the dual program (in variables α_u for $u \in A \cup B$) to the above primal program. This is LP1 described below.

$$\textbf{(LP1)} \quad \text{minimize} \sum_{u \in A \cup B} \alpha_u$$

$$\begin{aligned} \alpha_a + \alpha_b &\geq \mathsf{wt}_x(a,b) \ \ \forall (a,b) \in E \\ \alpha_u &\geq \mathsf{wt}_x(u,\ell(u)) \ \ \forall u \in A \cup B. \end{aligned}$$

For any point $\vec{x} \in \mathcal{P}_G$, the optimal primal value is 0 (since $\vec{x}$ is a popular fractional matching) and hence by LP-duality, the optimal dual value is also 0. So there is an optimal solution $\vec{\alpha}^x = (\alpha^x_u)_{u \in A \cup B}$ to LP1 such that $\sum_{u \in A \cup B} \alpha^x_u = 0$. We will regard $\vec{\alpha}^x$ as the *witness* to the popularity of $\vec{x}$.

For example, consider the instance in Fig. 1 with the half-integral matching $\vec{q} = (I_S + I_M)/2$ where $S = \{(a_1,b_1),(a_2,b_2)\}$ and $M = \{(a_1,b_2),(a_2,b_1)\}$, where I_X is the 0-1 edge incidence vector of matching X for $X \in \{S,M\}$. The fractional matching $\vec{q}$ is popular as witnessed by the following α-values: $\alpha_{a_0} = \alpha_{a_1} = \alpha_{b_2} = 0$, $\alpha_{a_2} = -1$, and $\alpha_{b_1} = 1$. We have $\alpha_{a_0} + \alpha_{a_1} + \alpha_{a_2} + \alpha_{b_1} + \alpha_{b_2} = 0$.

Instead of fixing a particular fractional matching $\vec{x}$ and regarding LP1 as a linear program in the n variables α_u, for $u \in A \cup B$, we could regard LP1 as a linear program in the $m+2n$ variables x_e, for $e \in \tilde{E}$, and α_u, for $u \in A \cup B$. This yields the following LP (where $\mathsf{wt}_x(a,b)$ and $\mathsf{wt}_x(u,\ell(u))$ have been explicitly written in terms of x_e's).

$$\textbf{(LP2)} \quad \text{minimize} \sum_{u \in A \cup B} \alpha_u$$

$$\begin{aligned} \alpha_a + \alpha_b &\geq \sum_{b' : b' \prec_a b} x_{(a,b')} - \sum_{b' : b' \succ_a b} x_{(a,b')} \\ &\quad + \sum_{a' : a' \prec_b a} x_{(a',b)} - \sum_{a' : a' \succ_b a} x_{(a',b)} \ \ \forall (a,b) \in E \\ \alpha_u &\geq -\sum_{v \in \mathsf{Nbr}(u)} x_{(u,v)} \quad \forall u \in A \cup B \\ \sum_{e \in \tilde{E}(u)} x_e &= 1 \quad \forall u \in A \cup B \quad \text{and} \quad x_e \geq 0 \quad \forall e \in \tilde{E}. \end{aligned}$$

The polytope $\mathcal{P}'_G$ is the set of optimal solutions to LP2. Hence $\sum_{u \in A \cup B} \alpha_u = 0$ for all the points $(\vec{x},\vec{\alpha})$ in $\mathcal{P}'_G$ and thus the description of $\mathcal{P}'_G$ consists of the following constraints:

$$\begin{aligned} \sum_{u \in A \cup B} \alpha_u &= 0 \\ \alpha_a + \alpha_b &\geq \mathsf{vote}_a(b,\vec{x}) + \mathsf{vote}_b(a,\vec{x}) \ \ \forall (a,b) \in E \\ \alpha_u &\geq \mathsf{vote}_u(\ell(u),\vec{x}) \ \ \forall u \in A \cup B \\ \sum_{e \in \tilde{E}(u)} x_e &= 1 \ \ \forall u \in A \cup B \quad \text{and} \quad x_e \geq 0 \quad \forall e \in \tilde{E}. \end{aligned}$$

Observe that the description of $\mathcal{P}'_G$ involves just $O(m+n)$ constraints, far fewer than the exponentially many constraints in the description of $\mathcal{P}_G$. The description of $\mathcal{P}'_G$ and the above formulation of LP2 were given in [26]. In this paper we show a very interesting and special property of LP2.

LEMMA 2.1. *LP2 is its own dual program, i.e., LP2 is* self-dual.

Proof. Consider the dual program corresponding to LP2. The dual variables are non-negative y_e for each $e \in \tilde{E}$ and β_u for each $u \in A \cup B$. This linear program is:

(LP3) maximize $\sum_{u\in A\cup B}\beta_u$

$$\begin{aligned}\beta_a + \beta_b + \sum_{b':b'\prec_a b} y_{(a,b')} - \sum_{b':b'\succ_a b} y_{(a,b')}&\\ + \sum_{a':a'\prec_b a} y_{(a',b)} - \sum_{a':a'\succ_b a} y_{(a',b)} &\leq 0 \quad \forall (a,b)\in E\\ \beta_u - \sum_{v\in\mathsf{Nbr}(u)} y_{(u,v)} &\leq 0 \quad \forall u\in A\cup B\end{aligned}$$

$$\sum_{e\in\tilde{E}(u)} y_e = 1 \quad \forall u\in A\cup B \qquad \text{and} \qquad y_e\geq 0 \quad \forall e\in\tilde{E}.$$

Let us substitute $\gamma_u = -\beta_u$ for each $u\in A\cup B$. This makes LP3 exactly the same as LP2: the objective function is minimize $\sum_{u\in A\cup B}\gamma_u$ and in the constraints, the variable γ_u has replaced α_u for each $u\in A\cup B$ and the variable y_e has replaced x_e for each $e\in\tilde{E}$. □

Thus LP2 is self-dual and this property will be crucial to us (see Lemma 3.2). We now show the relationship between $\mathcal{P}'_G$ and the stable matching polytope $\mathcal{S}_G$.

The typical formulation of $\mathcal{S}_G$ contains the constraints that $\sum_{e\in\tilde{E}(u)} x_e = 1$ for all $u\in A\cup B$ and $x_e\geq 0$ for all $e\in\tilde{E}$ along with the *stability* constraint for each edge $(a,b)\in\tilde{E}$. The stability constraint for the edge (a,b) in the description of $\mathcal{S}_G$ from [30] is given by constraint (2.2) below and the stability constraint for the edge (a,b) in the description of $\mathcal{S}_G$ from [36] is given by constraint (2.3) below.

$$(2.2)\qquad \sum_{b':b'\succ_a b} x_{(a,b')} + x_{(a,b)} + \sum_{a':a'\succ_b a} x_{(a',b)} \geq 1.$$

$$(2.3)\qquad \sum_{b':b'\prec_a b} x_{(a,b')} + x_{(a,b)} + \sum_{a':a'\prec_b a} x_{(a',b)} \leq 1.$$

In fact, the above two stability constraints are not independent and one can be derived from the other. This is because we can obtain (2.3) from (2.2) by replacing $\sum_{b':b'\succ_a b} x_{(a,b')}$ with $1 - x_{(a,b)} - \sum_{b':b'\prec_a b} x_{(a,b')}$ and $\sum_{a':a'\succ_b a} x_{(a',b)}$ with $1 - x_{(a,b)} - \sum_{a':a'\prec_b a} x_{(a',b)}$ (since $\sum_{e\in\tilde{E}(u)} x_e = 1$ for all $u\in A\cup B$).

By subtracting (2.2) from (2.3), we get the following constraint which is equivalent to either of these two (by an appropriate substitution):

$$(2.4)\qquad \left(\sum_{b':b'\prec_a b} x_{(a,b')} - \sum_{b':b'\succ_a b} x_{(a,b')}\right) + \left(\sum_{a':a'\prec_b a} x_{(a',b)} - \sum_{a':a'\succ_b a} x_{(a',b)}\right) \leq 0.$$

Thus constraint (2.4) for each edge $(a,b)\in\tilde{E}$ along with the constraints that $\sum_{e\in\tilde{E}(u)} x_e = 1$ for all $u\in A\cup B$ and $x_e\geq 0$ for all $e\in\tilde{E}$ is a description of $\mathcal{S}_G$. Observe that the first term in constraint (2.4) is $\mathsf{vote}_a(b,\vec{x})$ and the second term there is $\mathsf{vote}_b(a,\vec{x})$.

Thus the description of $\mathcal{P}'_G$ is a natural generalization of the description of $\mathcal{S}_G$ where we have $\alpha_a+\alpha_b$ on the right side of constraint (2.4) for $(a,b)\in E$ and we have only α_u on the right side in the constraint corresponding to $(u,\ell(u))$ for any $u\in A\cup B$; $\mathcal{P}'_G$ also has the constraint that the sum of all α_u's has to be 0. So $\mathcal{S}_G = \{\vec{x} : (\vec{x},\vec{0})\in\mathcal{P}'_G\}$. While $\mathcal{S}_G$ is integral, we know that $\mathcal{P}_G$ is not integral (as shown by the example in Fig. 1). However we will be able to show in Section 3 that $\mathcal{P}_G$ is integral in an important special case.

3 Integrality of $\mathcal{P}_G$ in a special case

We will prove the following theorem in this section.

THEOREM 3.1. *Let $G=(A\cup B,E)$ be an instance of the stable marriage problem with strict preference lists such that G admits a perfect stable matching. Then $\mathcal{P}_G$ is integral.*

Our assumption that G admits a perfect stable matching implies that every stable matching S in G is *perfect* [17], i.e., every $u\in A\cup B$ is matched in S. In fact, this implies that every popular matching in G has to be perfect – this is due to the fact that a stable matching is a minimum-size popular matching in G (see Corollary 1 in [20]). Also, this extends to all popular fractional matchings as well. That is, if $\vec{x}\in\mathcal{P}_G$, then $\vec{x}$ has to fully match every vertex in $A\cup B$ to genuine neighbors, otherwise we have $\Delta(\vec{x},S)<0$ where S is any stable matching[2], contradicting the popularity of $\vec{x}$. Lemma 3.1 shows an important property satisfied by popular matchings in G that are perfect.

For any edge $(a,b)\in E$, we will refer to the constraint $\alpha_a+\alpha_b\geq\mathsf{vote}_a(b,\vec{x})+\mathsf{vote}_b(a,\vec{x})$ (similarly, $\alpha_u\geq\mathsf{vote}_u(\ell(u),\vec{x})$ for any $u\in A\cup B$) in the description of $\mathcal{P}'_G$ as the *covering constraint* for edge (a,b) (resp., $(u,\ell(u))$).

LEMMA 3.1. *If M is a popular matching in $G=(A\cup B,E)$ such that M is perfect, then M has a witness $\vec{\alpha}^M\in\{\pm1\}^n$ to its popularity.*

Proof. A popular matching that is perfect is a *dominant* matching in G, i.e., a popular matching M with the property that M is strictly more popular than every larger matching in G. It was shown in [10] that every dominant matching allows a partition $A_0\cup A_1$ of A and $B_0\cup B_1$ of B such that the following two properties are satisfied:

(1) every blocking edge with respect to M is present in $A_0\times B_1$

[2] $\vec{x}\in\mathcal{M}_G$, so $\vec{x}$ is a convex combination of some matchings $M'_1,\ldots,M'_t$ in G and if some M'_j is *not* perfect, then it means $\Delta(M'_j,S)<0$ and we also have $\Delta(M'_i,S)\leq 0$ for all matchings M'_i by the popularity of S, so this implies $\Delta(\vec{x},S)<0$.

(2) if (a,b) is an edge in $A_1 \times B_0$ then both a and b prefer their respective partners in M to each other.

We will define $\vec{\alpha}^M$ as follows: set $\alpha_u^M = 1$ for each $u \in A_0 \cup B_1$ and set $\alpha_u^M = -1$ for each $u \in A_1 \cup B_0$. Observe that the covering constraints of all edges in E get satisfied by properties (1) and (2) given above. Also $\alpha_u \geq -1 = \mathsf{vote}_u(\ell(u), M(u))$ for each $u \in A \cup B$ since each vertex is matched in M to a genuine neighbor and so u's vote for $\ell(u)$ versus $M(u)$ is -1.

We also have $\sum_{u \in A \cup B} \alpha_u^M = \sum_{(a,b) \in M} (\alpha_a^M + \alpha_b^M)$ since M is a perfect matching and $\alpha_a^M + \alpha_b^M = 0$ for each (a,b) in M by our assignment of α^M-values. Thus $(I_M, \vec{\alpha}^M) \in \mathcal{P}'_G$ where I_M is the 0-1 edge incidence vector of M, in other words, the vector $\vec{\alpha}^M$ witnesses M's popularity. □

Let $\vec{x} \in \mathcal{P}_G$. We seek to express $\vec{x}$ as a convex combination of some popular (integral) matchings $M_1, \ldots, M_k$ in G. We know from LP1 that there exists a witness $\vec{\alpha}^x$ to the popularity of $\vec{x}$. Since $\vec{x}$ has to fully match every vertex in $A \cup B$ to genuine neighbors, the covering constraint for $(u, \ell(u))$ in the description of $\mathcal{P}'_G$ becomes $\alpha_u^x \geq -1$. So $\alpha_u^x \geq -1$ for each $u \in A \cup B$. It can also be shown that $\alpha_u^x \leq 1$ for each $u \in A \cup B$. Before we prove this, we need the following very useful lemma.

LEMMA 3.2. *For every $(a,b) \in E$, if $x_{(a,b)} > 0$ then the covering constraint in $\mathcal{P}'_G$ for (a,b) is tight. That is, we have:*

$$\begin{aligned} \alpha_a^x + \alpha_b^x &= \sum_{b': b' \prec_a b} x_{(a,b')} - \sum_{b': b' \succ_a b} x_{(a,b')} \\ &+ \sum_{a': a' \prec_b a} x_{(a',b)} - \sum_{a': a' \succ_b a} x_{(a',b)}. \end{aligned}$$

Proof. This follows directly from Lemma 2.1 which proved that LP2 is self-dual. So $(\vec{x}, \vec{\alpha}^x)$ which is an optimal solution to LP2 is also an optimal solution to its dual. Thus the following condition is implied by complementary slackness: if $x_{(a,b)} > 0$ then the constraint in LP2 for $x_{(a,b)}$ is tight. That is,

$$\begin{aligned} \alpha_a^x + \alpha_b^x &= \sum_{b': b' \prec_a b} x_{(a,b')} - \sum_{b': b' \succ_a b} x_{(a,b')} \\ &+ \sum_{a': a' \prec_b a} x_{(a',b)} - \sum_{a': a' \succ_b a} x_{(a',b)}. \end{aligned}$$

□

LEMMA 3.3. *For every vertex $u \in A \cup B$, we have $\alpha_u^x \leq 1$.*

Proof. Let $\{v_1, v_2, \ldots, v_d\}$ be the set of all neighbors of u such that $x_{(u,v_i)} > 0$. Let v_d be the least preferred neighbor of u in this set. If $x_{(u,v_d)} = \delta$, then $\mathsf{vote}_u(v_d, \vec{x}) = -(1-\delta)$. We will now show an upper bound for $\mathsf{vote}_{v_d}(u, \vec{x})$. We know that $\sum_{u': u' \prec_{v_d} u} x_{(u',v_d)} + \sum_{u': u' \succ_{v_d} u} x_{(u',v_d)} = 1 - \delta$. So $\mathsf{vote}_{v_d}(u, \vec{x}) = \sum_{u': u' \prec_{v_d} u} x_{(u',v_d)} - \sum_{u': u' \succ_{v_d} u} x_{(u',v_d)}$ is at most $1 - \delta$.

Lemma 3.2 tells us that the covering constraint for edge (u, v_d) is tight. So $\alpha_u^x + \alpha_{v_d}^x = \mathsf{vote}_u(v_d, \vec{x}) + \mathsf{vote}_{v_d}(u, \vec{x})$ and we have shown that the right side is at most $-(1-\delta) + (1-\delta) = 0$. Since $\alpha_{v_d}^x \geq -1$, it follows that $\alpha_u^x \leq 1$. □

Thus there exists a witness $\vec{\alpha}^x = (\alpha_u^x)_{u \in A \cup B} \in [-1,1]^n$ to the popularity of $\vec{x}$. We will use $\vec{\alpha}^x$ as follows:

- for each $a \in A$: determine the value r_a such that $r_a \cdot 1 + (1 - r_a) \cdot (-1) = \alpha_a^x$, i.e., $2r_a - 1 = \alpha_a^x$.
- for each $b \in B$: determine the value r_b such that $r_b \cdot (-1) + (1 - r_b) \cdot 1 = \alpha_b^x$, i.e., $1 - 2r_b = \alpha_b^x$.

The values r_a and r_b. The interpretation of r_a is as follows: we would like to come up with popular matchings $M_1, \ldots, M_k$ whose convex hull contains $\vec{x}$. We know from Lemma 3.1 that every popular matching M in G has a witness vector $\vec{\alpha}^M \in \{\pm 1\}^n$ and so for any $a \in A$, we have $\alpha_a^M \in \{\pm 1\}$. Suppose r_a fraction of these k matchings assign a's α-value to 1 and so $1 - r_a$ of these assign a's α-value to -1. Then $r_a \cdot 1 + (1 - r_a) \cdot (-1) = 2r_a - 1 = \alpha_a^x$.

Let X_a denote the array containing a's assignment in $\vec{x}$: each cell in the array X_a corresponds to a neighbor b of a such that $x_{(a,b)} > 0$. These neighbors of a are arranged in X_a in *increasing* order of a's preferences and the cell containing b has length $x_{(a,b)}$. Thus the total length of X_a is 1. We use the value r_a to partition X_a into a *positive* sub-array and a *negative* sub-array as defined below.

DEFINITION 3. *For any $a \in A$, the initial r_a fraction of X_a (i.e., the least preferred r_a fraction of X_a) will be called the* positive *sub-array of X_a and the remaining part (i.e., the most preferred $1 - r_a$ fraction of X_a) will be called the* negative *sub-array of X_a.*

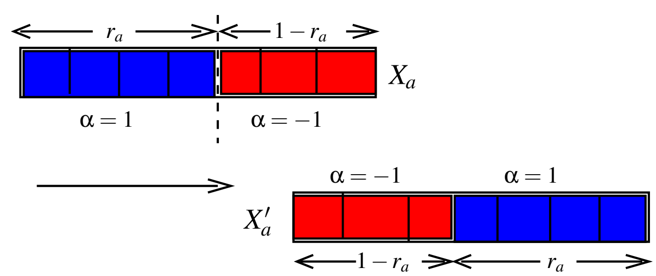

Figure 2: We will reorder X_a (the array on the left) by swapping the positive and negative sub-arrays as shown above. The reordered array will be called X'_a.

In Fig. 2, the array on the left is X_a and the positive sub-array of X_a is colored blue and the negative sub-array of X_a

is colored red. We will assume that the positive sub-array of X_a has a's α-value set to 1 and the negative sub-array of X_a has a's α-value set to -1.

Our main idea here is the following: *reorder X_a* as shown in Fig. 2. That is, we cut X_a at the end of its positive sub-array and move its entire negative sub-array (in the same order) to the left of its positive sub-array. Note that neither the order within the positive sub-array nor within the negative sub-array is changed by this. The reordered array is shown on the right on Fig. 2, call this array X'_a. In case the line cutting X_a into these 2 sub-arrays went through a cell, that cell is now split into 2 cells (one negative and one positive). Thus each cell in X'_a is either positive or negative. Recall that each negative cell corresponds to a's α-value being -1 and each positive cell corresponds to a's α-value being 1.

Similar to the interpretation of r_a, the interpretation of r_b is that if we assume r_b fraction of the matchings $M_1, \ldots, M_k$ assign b's α-value to -1 and so $(1-r_b)$ of them assign b's α-value to 1, then $r_b \cdot (-1) + (1-r_b) \cdot 1 = 1 - 2r_b = \alpha_b$. As done for each $a \in A$, here also we form the array X_b which is b's assignment in $\vec{x}$, however here the neighbors of b are arranged in *decreasing* order of b's preference.

DEFINITION 4. *For any $b \in B$, the initial r_b fraction of X_b (i.e., the most preferred r_b fraction of X_b) will be called the* negative *sub-array of X_b and the remaining part (i.e., the least preferred $1-r_b$ fraction of X_b) will be called the* positive *sub-array of X_b.*

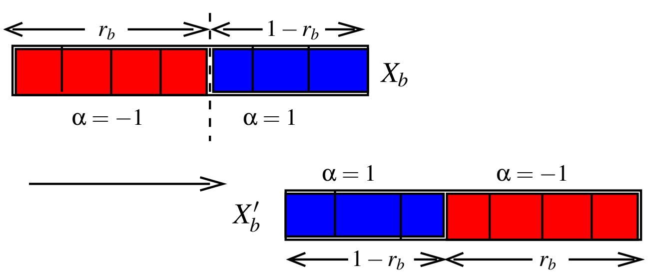

Figure 3: The array on the left is X_b (b's neighbors in $\vec{x}$ in decreasing order of b's preference) and the array on the right is X'_b.

Refer to Fig. 3 – in the array on the left, the red part is the negative sub-array of X_b and the blue part is the positive sub-array of X_b. As before, we will assume that the negative sub-array of X_b has b's α-value set to -1 and the positive sub-array of X_b has b's α-value set to 1. We will cut X_b at the end of its negative sub-array and move its entire positive sub-array to the left of its negative sub-array as shown in Fig. 3. Call this reordered array X'_b.

3.1 Finding the popular matchings whose convex hull contains $\vec{x}$. Form the table T whose rows are the reordered arrays X'_u, for $u \in A \cup B$. The table T has width 1 and the number of cells in u's row is at most $\deg(u)+1$, where $\deg(u)$ is u's degree in the original G. For any $t \in [0,1)$, define the set $M_t \subseteq E$ as follows:

- draw the vertical line L_t at distance t from the left end of T;
- the line L_t intersects or touches the left boundary of some cell in X'_u (call this cell $c_u(t)$) for each $u \in A \cup B$;

$$M_t = \{(u,v) : u \in A \cup B \text{ and } v \text{ is in cell } c_u(t)\}.$$

We will show in Theorem 3.2 that M_t is a matching, i.e., if b is in the cell in X'_a at distance t from the left end of T, then a has to be in the cell in X'_b at distance t from the left. We first show the following simple lemma which will be used in the proof of Theorem 3.2.

LEMMA 3.4. *For any $(a,b) \in E$, we have:*

$$x_{(a,b)} + \sum_{b' \prec_a b} x_{(a,b')} + \sum_{a' \prec_b a} x_{(a',b)} \;\le\; r_a + (1 - r_b).$$

Moreover, if $x_{(a,b)} > 0$ then this constraint is tight.

Proof. We know by the covering constraint in $\mathcal{P}'_G$ that for any $(a,b) \in E$:

$$\begin{aligned}\alpha^x_a + \alpha^x_b \quad \ge \quad & \sum_{b' \prec_a b} x_{(a,b')} - \sum_{b' \succ_a b} x_{(a,b')} \\ + \quad & \sum_{a' \prec_b a} x_{(a',b)} - \sum_{a' \succ_b a} x_{(a',b)}.\end{aligned}$$

Rewrite the above constraint in a simpler form by substituting $\sum_{b' \succ_a b} x_{(a,b')} = 1 - x_{(a,b)} - \sum_{b' \prec_a b} x_{(a,b')}$ and similarly substitute $\sum_{a' \succ_b a} x_{(a',b)} = 1 - x_{(a,b)} - \sum_{a' \prec_b a} x_{(a',b)}$. Replace α^x_a with $2r_a - 1$ and α^x_b with $1 - 2r_b$. This results in the following simpler looking constraint:

$$\begin{aligned}(3.5) \qquad & x_{(a,b)} + \sum_{b' \prec_a b} x_{(a,b')} + \sum_{a' \prec_b a} x_{(a',b)} \\ & \le\; 1 + r_a - r_b \;=\; r_a + (1 - r_b).\end{aligned}$$

Note that it follows from Lemma 3.2 that when $x_{(a,b)} > 0$, (3.5) is tight. Thus we have shown the lemma. $\square$

We are now ready to show that M_t is a valid matching in G. For any edge (a,b), if $x_{(a,b)} > 0$, we need to show that the cell containing b in X'_a and the cell containing a in X'_b are perfectly *aligned* in the vertical direction.

THEOREM 3.2. *M_t is a matching in G.*

Proof. Let (a,b) be any edge such that $x_{(a,b)} > 0$. Recall that in X'_a, a's increasing order of preference of partners in $\vec{x}$ begins from the start of its positive sub-array (the blue region) in a left to right orientation and it wraps around. Suppose $\sum_{b' \prec_a b} x_{(a,b')} \geq r_a$. Let $d = \sum_{b' : b' \prec_a b} x_{(a,b')} - r_a$. Then after traversing length d from the start of X'_a (refer to Fig. 4), we reach the cell in X'_a that contains b – this is the darkened red cell in X'_a in Fig. 4 and it has length $x_{(a,b)}$.

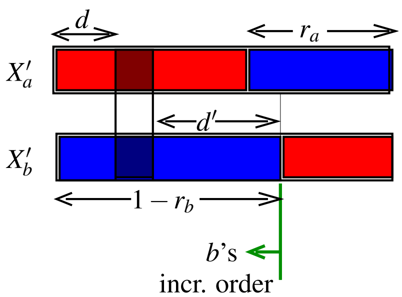

Figure 4: The top array is X'_a and the darkened cell there contains b. The cell exactly below this in the blue sub-array of X'_b contains a.

Similarly in X'_b, b's increasing order of preference of partners in $\vec{x}$ begins from the end of its positive sub-array or the blue region and this order is from right to left (as indicated by the arrow in Fig. 4). Let $d' = \sum_{a' : a' \prec_b a} x_{(a',b)}$. After traversing length d' from this vertical line in X'_b (marking the end of the positive sub-array) from right to left, we reach the cell that contains a. This cell is within the positive sub-array of X'_b since $d' + x_{(a,b)} \leq 1 - r_b$ because $\sum_{b' \prec_a b} x_{(a,b')} + d' + x_{(a,b)} = r_a + (1 - r_b)$ (by Lemma 3.4) and we are in the case where $\sum_{b' \prec_a b} x_{(a,b')} \geq r_a$.

Refer to Fig. 4, where the cell in X'_b that contains a is the darkened blue cell and it has length $x_{(a,b)}$. Since $d + x_{(a,b)} + d' = 1 - r_b$, it follows that the cell containing a in X'_b and the cell containing b in X'_a are exactly aligned with each other in the vertical direction.

The picture is absolutely symmetric when $\sum_{a' \prec_b a} x_{(a',b)} \geq (1 - r_b)$. Then the cell containing b is in the positive sub-array of X'_a (its blue region) and the cell containing a is in the negative sub-array of X'_b (its red region).

The only case left is when $\sum_{b' \prec_a b} x_{(a,b')} < r_a$ *and* $\sum_{a' \prec_b a} x_{(a',b)} < (1 - r_b)$. Using Lemma 3.4, it is easy to see that in this case we also have the following inequalities: $\sum_{b' \succ_a b} x_{(a,b')} < 1 - r_a$ and $\sum_{a' \succ_b a} x_{(a',b)} < r_b$. In other words, the line separating the positive sub-array from the negative sub-array in X_a went through the cell containing b and similarly, the line separating the negative sub-array from the positive sub-array in X_b went through the cell containing a. Let $d_0 = \sum_{b' \prec_a b} x_{(a,b')}$ and $d_1 = \sum_{a' \prec_b a} x_{(a',b)}$ (see Fig. 5).

Let the length of the rightmost cell in X'_a be $x^0_{(a,b)}$ and

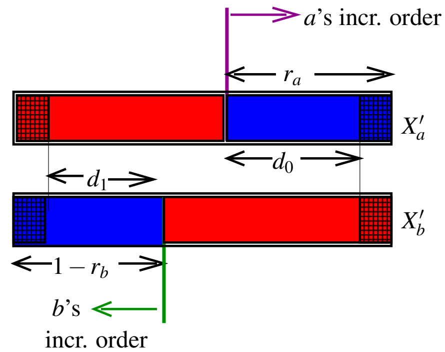

Figure 5: Both the leftmost (the dashed red) cell and the rightmost (the dashed blue) cell in X'_a contain b. Symmetrically both the leftmost (the dashed blue) cell and the rightmost (the dashed red) cell in X'_b contain a.

let the length of the leftmost cell in X'_b be $x^1_{(a,b)}$. So $x^0_{(a,b)} = r_a - d_0$ and $x^1_{(a,b)} = (1 - r_b) - d_1$. Lemma 3.4 tells us that $x_{(a,b)} + d_0 + d_1 = r_a + (1 - r_b)$. Hence $x_{(a,b)} = x^0_{(a,b)} + x^1_{(a,b)}$. We know that the length of the cell containing b in X_a is $x_{(a,b)}$. So is the length of the cell containing a in X_b. Hence the length of the leftmost cell in X'_a is $x_{(a,b)} - x^0_{(a,b)} = x^1_{(a,b)}$ and the length of the rightmost cell in X'_b is $x_{(a,b)} - x^1_{(a,b)} = x^0_{(a,b)}$. Thus in this case as well we have perfect alignment between the two cells in X'_a that contain b and the two cells in X'_b that contain a. So no matter which of these cells is intersected by the line L_t, we have exact alignment between the cell containing a in X'_b and the cell containing b in X'_a.

Thus we can conclude that if b belongs to cell $c_a(t)$, then a has to belong to cell $c_b(t)$. Hence for each vertex $u \in A \cup B$, there is exactly one edge $(u,v) \in M_t$ that is incident on u. So M_t is a matching in G. □

The popularity of matching M_t. We now need to show that M_t is a popular matching in G. We do this by showing a witness vector $\vec{\alpha}^t$ such that $\vec{\alpha} = \vec{\alpha}^t$ and $\vec{x} = I_{M_t}$ satisfy the constraints of $\mathcal{P}'_G$, where I_{M_t} is the 0-1 edge incidence vector of M_t. We define $\vec{\alpha}^t$ as follows. Recall that we defined the matching M_t via the vertical line L_t that intersected table T.

- For each $u \in A \cup B$ do: if the cell intersected by L_t is in the *negative* sub-array of X'_u, then set $\alpha^t_u = -1$; else (the cell is in the *positive* sub-array of X'_u) set $\alpha^t_u = 1$.

Observe that $\alpha^t_a + \alpha^t_b = 0$ for each edge $(a,b) \in M_t$. This is because it follows from the proof of Theorem 3.2 that, for each edge $(a,b) \in M_t$, either b's cell is in the negative sub-array of X'_a and a's cell is in the positive sub-array of X'_b or vice-versa. Since M_t is a perfect matching, Corollary 3.1 follows.

COROLLARY 3.1. $\sum_{u \in A \cup B} \alpha^t_u = 0$.

We now need to show that $\vec{\alpha}^t$ and I_{M_t} also satisfy the "covering constraints" in $\mathcal{P}'_G$. Refer to Fig. 6. The vertical line in X'_a (with the left to right arrow adjacent to it) denotes the start of a's preference order of its partners in $\vec{x}$ in increasing order and this wraps around. Similarly, the vertical line in X'_b (with the right to left arrow adjacent to it) denotes the start of b's preference order of its partners in $\vec{x}$ in increasing order and this also wraps around.

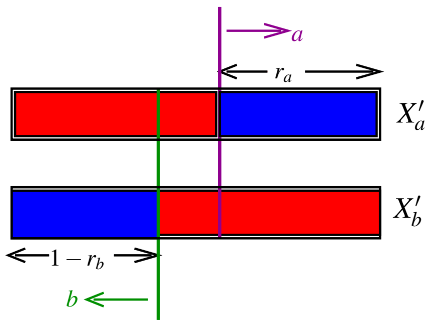

Figure 6: The rightwards arrow in X'_a denotes a's increasing order. The leftwards arrow in X'_b denotes b's increasing order (this is from right to left).

Recall that we used the symbol $c_u(t)$ to denote the cell in X'_u that is intersected by line L_t, for any $u \in A \cup B$. Using Lemma 3.4, it is easy to make the following observations:

(I) suppose $x_{(a,b)} = 0$: so $\sum_{b' \prec_a b} x_{(a,b')} + \sum_{a' \prec_b a} x_{(a',b)} \le r_a + (1 - r_b)$. Note that $r_a + (1 - r_b)$ is the sum of lengths of positive sub-arrays in X'_a and X'_b. Hence for both a and b to be matched in M_t to worse partners than each other, both $c_a(t)$ and $c_b(t)$ must be in their respective *positive* sub-arrays. So if only one of $c_a(t), c_b(t)$ is in its positive sub-array, then at least one of a, b is matched in M_t to a better partner than the other.

(II) suppose both $c_a(t)$ and $c_b(t)$ are in their respective negative sub-arrays: then we claim that *both* a and b get matched in M_t to better partners than each other. This is because $x_{(a,b)} + \sum_{b' \prec_a b} x_{(a,b')} + \sum_{a' \prec_b a} x_{(a',b)} \le r_a + (1 - r_b)$ and $r_a + (1 - r_b)$ is the sum of the lengths of the positive sub-arrays in X'_a and in X'_b.

LEMMA 3.5. *For each* $(a,b) \in E$, *we have* $\alpha^t_a + \alpha^t_b \ge \mathsf{vote}_a(b, M_t(a)) + \mathsf{vote}_b(a, M_t(b))$.

Proof. Let $(a,b) \in E$ and let the line L_t intersect cell $c_a(t)$ in X'_a and cell $c_b(t)$ in X'_b. There are three possible cases regarding the "signs" of the cells $c_a(t)$ and $c_b(t)$:

(1) Both $c_a(t)$ and $c_b(t)$ are positive: so $\alpha^t_a = \alpha^t_b = 1$ and thus $\alpha^t_a + \alpha^t_b = 2$; hence the covering constraint for edge (a,b) holds because the right side of this constraint is always at most 2.

(2) One of $c_a(t), c_b(t)$ is positive and the other is negative: so (α^t_a, α^t_b) is either $(-1, 1)$ or $(1, -1)$. We know from Observation (I) and the proof of Theorem 3.2 that when exactly one of the cells is positive, either (i) at least one of a, b is matched in M_t to a better partner or (ii) a and b are matched to each other. Thus the edge (a,b) is covered in both these sub-cases.

(3) Both cells $c_a(t)$ and $c_b(t)$ are negative: so $\alpha^t_a = \alpha^t_b = -1$. We know from Observation (II) that when both $c_a(t)$ and $c_b(t)$ are negative, then both a and b are matched in M_t to better partners than each other. Thus here also the edge (a,b) is covered.

Thus $\vec{\alpha}^t$ and I_{M_t} together satisfy the covering constraints in $\mathcal{P}'_G$ for all edges (a,b) in E. For each $u \in A \cup B$, we have $\alpha^t_u \ge -1$ and so the covering constraint for the edge $(u, \ell(u))$ is also satisfied because $\mathsf{vote}_u(\ell(u), M_t(u))$ is -1. We have also shown that $\sum_u \alpha^t_u = 0$. Thus we can conclude that $(I_{M_t}, \vec{\alpha}) \in \mathcal{P}'_G$, i.e., M_t is a popular matching in G. □

We are now ready to express $\vec{x}$ as a convex combination of popular matchings: these matchings are obtained by sweeping a vertical line from the left end to the right end of table T. Whenever a new cell begins in some row in T (say, at distance t from the left end of T), we define a new matching M_t as described above. The leftmost cell in T begins at distance 0 from the left end of T, let the second leftmost cell in T begin at distance t_1 from the left side of T, and so on, i.e., let the i-th leftmost cell in T begin at distance t_{i-1} from the left side of T. Thus we construct matchings $M_0, M_{t_1}, \ldots, M_{t_{k-1}}$. So we have:

$$\vec{x} = t_1 \cdot I_0 + (t_2 - t_1) \cdot I_1 + \cdots + (1 - t_{k-1}) \cdot I_{k-1},$$

where $I_0, I_1, \ldots, I_{k-1}$ are the 0-1 edge incidence vectors of the matchings $M_0, M_{t_1}, \ldots, M_{t_{k-1}}$, respectively.

The total number of matchings k that we construct here is at most $m + |A|$ since $m + |A| = \sum_{a \in A} (\deg(a) + 1)$ is an upper bound on the total number of distinct cells in T. Thus every popular fractional matching in G can be expressed as a convex combination of at most $m + n/2$ popular matchings in G (since $|A| = |B| = n/2$ in this section). This finishes the proof of Theorem 3.1, i.e., if $G = (A \cup B, E)$ admits a perfect stable matching then the polytope $\mathcal{P}_G$ is integral.

4 Half-integrality of $\mathcal{P}_G$ in any bipartite instance G

In this section we are in the general case: we have an instance $G = (A \cup B, E)$ with strict preference lists and G need not admit a stable matching that matches all vertices. We know that $\mathcal{P}_G$ need not be integral in such an instance. We will show the following theorem.

THEOREM 4.1. *The popular fractional matching polytope* $\mathcal{P}_G$ *in* $G = (A \cup B, E)$ *is half-integral.*

We will show the above theorem with the help of Theorem 3.1. Using the given instance G, we will construct a new instance $H = (V \cup V', E')$ as follows: let $V = A_0 \cup B_1$ and let $V' = B_0 \cup A_1$, where $A_i = \{a_i : a \in A\}$ and $B_i = \{b_i : b \in B\}$, for $i = 0,1$ (see Fig. 7).

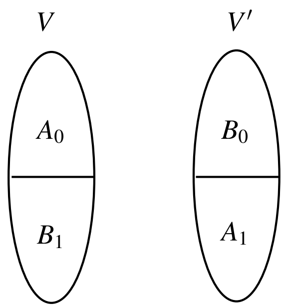

Figure 7: The vertex set of the graph H is two copies of vertex set of the graph G.

The edge set of H is $E' = E_0 \cup E_1 \cup \{(u_0,u_1) : u \in A \cup B\}$, where $E_i = \{(a_i,b_i) : (a,b) \in E\}$, for $i = 0,1$. For $i = 0,1$ and for each vertex u_i in H, u_i's preference list is the same as it was in G, with a subscript i added to each of its neighbors (in the same order of preference) along with u_{1-i} added as u_i's least preferred neighbor in H.

LEMMA 4.1. *H admits a perfect stable matching.*

Proof. Let S be a stable matching in H. Let $S_0 = S \cap E_0$ and $S_1 = S \cap E_1$. By ignoring the subscripts of their vertices, both S_0 and S_1 become stable matchings in G. Since all stable matchings in G match exactly the same vertices [17], it follows that u_0 is left unmatched in S_0 if and only if u_1 is left unmatched in S_1. Thus $S = S_0 \cup S_1 \cup \{(u_0,u_1) : u$ is an unstable vertex in $G\}$. So S is a perfect matching. $\square$

We can now use Theorem 3.1 to conclude that $\mathcal{P}_H$ (the popular fractional matching polytope of H) is integral. The rest of this section will use the integrality of $\mathcal{P}_H$ to prove that $\mathcal{P}_G$ is half-integral. In order to do this, we define a mapping f from $\mathcal{P}_G$ to the set of fractional matchings in H. Let $\vec{x} \in \mathcal{P}_G$, where $\vec{x} = (x_e)_{e \in \tilde{E}}$; we know that there exists a witness $\vec{\alpha}^x = (\alpha^x_v)_{v \in A \cup B}$ such that $\vec{x}$ and $\vec{\alpha}^x$ satisfy the constraints of $\mathcal{P}'_G$.

Define the vector $f(\vec{x}) = \vec{z} = (z_e)_{e \in E'}$ as follows: for every edge $(a,b) \in E$, let $z_{(a_0,b_0)} = z_{(a_1,b_1)} = x_{(a,b)}$ and for every $u \in A \cup B$, let $z_{(u_0,u_1)} = x_{(u,\ell(u))}$. It is easy to see that $\vec{z}$ is a fractional matching in H.

LEMMA 4.2. *For any popular fractional matching $\vec{x}$ in G, the vector $f(\vec{x})$ is a popular fractional matching in H.*

Proof. We need to show a witness vector $\vec{\beta} = (\beta_v)_{v \in V \cup V'}$ such that $\vec{\beta}$ and $f(\vec{x})$ satisfy the constraints in $\mathcal{P}'_H$. We define $\vec{\beta}$ as follows: for each $u \in A \cup B$, let $\beta_{u_0} = \beta_{u_1} = \alpha^x_u$. Let $(a,b) \in E$. Given the fact that $\vec{x}$ and $\vec{\alpha}^x$ satisfy the covering constraints in $\mathcal{P}'_G$ for (a,b), it immediately follows that $f(\vec{x})$ along with the vector $\vec{\beta}$ satisfies covering constraints in $\mathcal{P}'_H$ for the edges (a_0,b_0) and (b_1,a_1).

Consider the edge $(u,\ell(u))$ in G for any $u \in A \cup B$: we have $\alpha^x_u \geq x_{(u,\ell(u))} - 1$. In the graph H, the right side of the edge covering constraint for the edge (u_0,u_1) is $\mathsf{vote}_{u_0}(u_1, f(\vec{x})) + \mathsf{vote}_{u_1}(u_0, f(\vec{x})) = 2(x_{(u,\ell(u))} - 1)$. As the left side of this constraint is $\beta_{u_0} + \beta_{u_1} = \alpha^x_u + \alpha^x_u = 2\alpha^x_u$, it follows that the covering constraint for the edge (u_0,u_1) is also satisfied by $f(\vec{x})$ and $\vec{\beta}$. Thus the covering constraints in the description of $\mathcal{P}'_H$ for all edges in E' are satisfied by $f(\vec{x})$ and $\vec{\beta}$. Since $\beta_{u_i} = \alpha^x_u \geq -1$, the covering constraint in $\mathcal{P}'_H$ for the edge $(u_i,\ell(u_i))$ is trivially satisfied for all $u \in A \cup B$ and $i = 0,1$. $\square$

Our goal now is to define a mapping h from $\mathcal{P}_H$ to $\mathcal{P}_G$ so that $h \circ f(\vec{x}) = \vec{x}$ for any popular fractional matching $\vec{x}$ in G. Thus h is the inverse of f, when restricted to fractional matchings that are in the image of f. Let $\vec{z} = (z_e)_{e \in E'}$ be any popular fractional matching in H. We define $h(\vec{z}) = \vec{y} = (y_e)_{e \in \tilde{E}}$ as given below and note that $h \circ f(\vec{x}) = \vec{x}$ for any $x \in \mathcal{P}_G$.

$$
\begin{aligned}
y_{(a,b)} &= (z_{(a_0,b_0)} + z_{(b_1,a_1)})/2 \ \text{ for every } (a,b) \in E \\
y_{(u,\ell(u))} &= z_{(u_0,u_1)} \ \text{ for every } u \in A \cup B.
\end{aligned}
$$

LEMMA 4.3. *For any popular fractional matching $\vec{z}$ in H, the vector $h(\vec{z})$ is a popular fractional matching in G.*

Proof. Let $\vec{y} = h(\vec{z})$. We will now show that $\vec{y} \in \mathcal{P}_G$. For this to be true, it is necessary that $\vec{y}$ satisfies $\sum_{e \in \tilde{E}(u)} y_e = 1$ for each $u \in A \cup B$. Since $\sum_{e \in E'(v)} z_e = 1$ for each vertex v in H, it is simple to see that the above constraint is satisfied. We will now show that $\vec{y}$ and an appropriate witness vector $\vec{\alpha}$ satisfy the other constraints defining $\mathcal{P}'_G$.

We know that there exists a witness vector $(\beta_v)_{v \in V \cup V'}$ that is a witness to $\vec{z}$'s popularity in H. For each $u \in A \cup B$, let $\alpha_u = (\beta_{u_0} + \beta_{u_1})/2$. Since $\sum_{u \in A \cup B}(\beta_{u_0} + \beta_{u_1}) = 0$, we have $\sum_{u \in A \cup B} \alpha_u = 0$. Using the fact that $\vec{\beta}$ and $\vec{z}$ satisfy covering constraints for all edges in E', it is straightforward to show that $\vec{\alpha}$ and $\vec{y}$ satisfy covering constraints for all edges in $\tilde{E}$. Thus $(\vec{y}, \vec{\alpha}) \in \mathcal{P}'_G$, i.e., $\vec{y} \in \mathcal{P}_G$. $\square$

Theorem 4.1 follows from Lemma 4.4. Thus the polytope $\mathcal{P}_G$ is half-integral.

LEMMA 4.4. *Let $\vec{x} \in \mathcal{P}_G$. Then $\vec{x} = \sum_{i=1}^{r} \lambda_i \vec{q}_i$, where $\vec{q}_1, \ldots, \vec{q}_r$ are popular half-integral matchings in G and $\lambda_i \geq 0$ for $1 \leq i \leq r$ along with $\sum_i \lambda_i = 1$.*

Proof. Let $\vec{x} \in \mathcal{P}_G$ and let $\vec{z} = f(\vec{x})$. We know from Lemma 4.2 that $\vec{z} \in \mathcal{P}_H$. Since $\mathcal{P}_H$ is integral, it follows that

$\vec{z} = \sum_{i=1}^{r} \lambda_i I_{M_i}$, where $M_1, \ldots, M_r$ are popular matchings in H and $I_{M_1}, \ldots, I_{M_r}$ their respective edge incidence vectors, and for each i, we have $\lambda_i \geq 0$ along with $\sum_i \lambda_i = 1$.

We know that $h(\vec{z}) = \vec{x}$. So $\vec{x} = h(\vec{z}) = \sum_{i=1}^{r} \lambda_i \cdot h(I_{M_i})$. Since $I_{M_1}, \ldots, I_{M_r}$ belong to $\mathcal{P}_H$, it follows from Lemma 4.3 that $h(I_{M_1}), \ldots, h(I_{M_r})$ belong to $\mathcal{P}_G$. Since each coordinate in the vectors $I_{M_1}, \ldots, I_{M_r}$ is either 0 or 1, it follows from the definition of h that $h(I_{M_1}), \ldots, h(I_{M_r})$ are half-integral. Thus $\vec{x}$ is a convex combination of popular half-integral matchings in G. □

Our results imply that we can compute a max-utility popular half-integral matching in $G = (A \cup B, E)$ with a utility function $w : E \to \mathbb{Q}$ by solving a linear program to maximize $\sum_{e \in E} w_e x_e$ and with the description of $\mathcal{P}'_G$ as the set of constraints. The previous polynomial time algorithm for computing a max-utility popular half-integral matching [25] in G involved solving a linear program in $4m$ variables with $8m + 4n$ constraints, where $|E| = m$ and $|A| + |B| = n$. In contrast, the description of $\mathcal{P}'_G$ uses $m + n - 1$ (independent) variables and has $2m + 2n$ constraints.

5 Half-integrality of $\mathcal{P}_G$ in a roommates instance

The input instance here is a graph $G = (V, E)$ (not necessarily bipartite) with strict preferences. Note that popular matchings need not always exist in such an instance. Consider the instance G on 3 vertices a, b, c with cyclic preferences, i.e., a prefers b to c while b prefers c to a, and c prefers a to b. It is easy to see that G has no popular matching.

Let $G = (V, E)$ be any roommates instance with strict preferences on m edges and n vertices. We define the fractional matching polytope FM_G of G below.

$$\mathsf{FM}_G = \{\vec{x} \in \mathbb{R}^m_{\geq 0} : \sum_{e \in E(u)} x_e \leq 1 \ \ \forall u \in V\}.$$

It is known that FM_G is the convex hull of *half-integral* matchings in G. As done in Section 2, it will be convenient to assume that each vertex is fully matched in any fractional matching. So we augment the edge set E with the edges $(v, \ell(v))$ for all vertices v (where $\ell(v)$ is v's last resort neighbor) and let $\tilde{E} = E \cup \{(v, \ell(v)) : v \in V\}$. So we have $\sum_{e \in \tilde{E}(v)} x_e = 1$ for all $v \in V$ and we will continue to use $\vec{x}$ to denote the revised $\vec{x}$ in $[0,1]^{m+n}$.

In order to compare two fractional matchings with respect to popularity, we define $\Delta(\vec{x}, \vec{y})$ for fractional matchings $\vec{x}$ and $\vec{y}$ below.

$$\Delta(\vec{x}, \vec{y}) = \sum_{u \in V} \sum_{\substack{(u,v) \in \tilde{E}(u) \\ (u,v') \in \tilde{E}(u)}} x_{(u,v)} \, y_{(u,v')} \, \mathsf{vote}_u(v, v'). \tag{5.6}$$

Note that when G is bipartite, the above definition coincides with the definition of $\Delta(\Pi, \Lambda)$ given in Section 1 where Π and Λ are the mixed matchings that correspond to $\vec{x}$ and $\vec{y}$, respectively.

A fractional matching $\vec{x}$ is popular if $\Delta(\vec{x}, \vec{y}) \geq 0$ for all $\vec{y} \in \mathsf{FM}_G$. The polytope $\mathcal{P}_G$ is the set of all popular fractional matchings in G, that is

$$\mathcal{P}_G = \{\vec{x} \in \mathsf{FM}_G : \Delta(\vec{x}, \vec{y}) \geq 0 \ \ \forall \vec{y} \in \mathsf{FM}_G\}.$$

Since the polytope FM_G is half-integral, $\mathcal{P}_G$ can also be defined as the set of points $\vec{x} \in \mathsf{FM}_G$ such that $\Delta(\vec{x}, \vec{q}) \geq 0$ for all half-integral matchings $\vec{q}$ in G. Our goal now is to show that the polytope $\mathcal{P}_G$ is half-integral.

We will prove this via the extended formulation $\mathcal{P}'_G$ of $\mathcal{P}_G$ in $\mathbb{R}^{m+2n}$ whose description involves $O(m+n)$ constraints. We give a brief sketch of how we arrive at this description of $\mathcal{P}'_G$ below – this is totally analogous to the discussion in Section 2 for bipartite instances.

Let $\vec{x} \in \mathcal{P}_G$. We consider the max-weight half-integral matching problem that matches all vertices in V in the augmented graph $\tilde{G}_x$: for any edge $(u,v) \in E$, its weight in $\tilde{G}_x$ is $\mathsf{vote}_u(v, \vec{x}) + \mathsf{vote}_v(u, \vec{x})$ and the weight of edge $(u, \ell(u))$ for any $u \in V$ is $\mathsf{vote}_u(\ell(u), \vec{x})$.

The dual LP to the max-weight half-integral matching linear program in $\tilde{G}_x$ that matches all vertices in V is LP1 in Section 2. By regarding LP1 as a linear program in $m + 2n$ variables, we get LP2. The polytope $\mathcal{P}'_G$ is the set of optimal solutions to LP2. Thus we get the same description of $\mathcal{P}'_G$ as we obtained in Section 2: *covering constraints* for all edges in $\tilde{E} = E \cup \{(u, \ell(u)) : u \in V\}$ along with the constraints that $\sum_{u \in V} \alpha_u = 0$, $\vec{x} \geq 0$, and $\sum_{e \in \tilde{E}(u)} x_e = 1$ for all $u \in V$.

We will use the above set of constraints to prove the half-integrality of $\mathcal{P}_G$ as done in Section 4. Corresponding to $G = (V, E)$, we define the bipartite graph $H = (V \cup V', E')$ where V is the vertex set of G and $V' = \{v' : v \in V\}$. So V' is another copy of V. The edge set of H is $E' = \{(u, v'), (v, u') : (u,v) \in E\} \cup \{(v, v') : v \in V\}$.

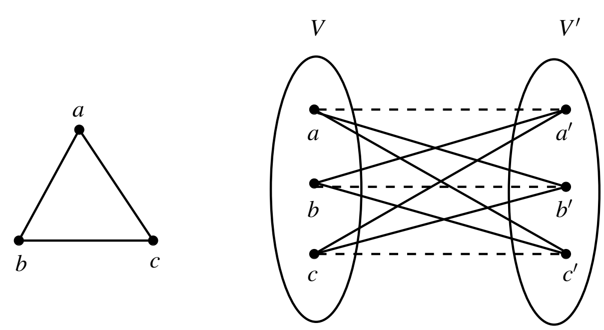

Figure 8: To the left is the graph G which is the triangle on a, b, c. To the right is the graph H which is the 6-cycle a-b'-c-a'-b-c'-a along with the edges $(a, a'), (b, b'), (c, c')$.

Thus H has two edges (u, v') and (v, u') for every edge (u,v) in G, along with the edges (v, v') for all $v \in V$. Fig. 8

has an example where the graph H on the right is the bipartite graph corresponding to the graph G on the left.

The preference lists of vertices in the graph H are as follows: for any vertex $v \in V$, if the preference list of v in G was $u_0 \succ u_1 \succ \cdots \succ u_k$ then the preference list of v in H is $u'_0 \succ u'_1 \succ \cdots \succ u'_k \succ v'$ and symmetrically, the preference list of v' in H is $u_0 \succ u_1 \succ \cdots \succ u_k \succ v$.

Let us define $\mathsf{twin}(v) = v'$ and $\mathsf{twin}(v') = v$ for each $v \in V$; thus v and $\mathsf{twin}(v)$ are each other's last choice neighbors in H.

LEMMA 5.1. *H admits a perfect stable matching.*

Proof. For any edge $e = (u,v')$ in H, let $e' = (v,u')$: it follows from the definition of the edge set E' of H that e' is also an edge in H. For any matching M in H, we define a "mirror image" matching M' in H as follows: $M' = \{e' : e \in M\}$. It is easy to see that if M is stable then M' is also stable. This is because an edge e is a blocking edge to M' only if e' is a blocking edge to M.

Let S be a stable matching in H, then S' is also a stable matching in H. Suppose S leaves a vertex $u \in V \cup V'$ unmatched. Then S' leaves $\mathsf{twin}(u)$ unmatched, where the function twin has been defined above. Since all stable matchings in H leave the same vertices unmatched, it follows that S leaves *both* u and $\mathsf{twin}(u)$ unmatched. Then the edge $(u, \mathsf{twin}(u))$ is a blocking edge to S, contradicting its stability. Thus S is a perfect matching. $\square$

Hence we can conclude that the popular fractional matching polytope $\mathcal{P}_H$ of H is integral (by Theorem 3.1). As done in Section 4, we define the function f on $\mathcal{P}_G$ as follows: $f(\vec{x}) = \vec{z} = (z_e)_{e \in E'}$ where $z_{(u,v')} = z_{(v,u')} = x_{(u,v)}$ for every edge $(u,v) \in E$ and $z_{(v,v')} = x_{(v,\ell(v))}$ for any $v \in V$. It is easy to see that $\vec{z}$ is a fractional matching in H.

LEMMA 5.2. *For any popular fractional matching x in G, the vector $f(\vec{x})$ is a popular fractional matching in H.*

Lemma 5.2 states that f is a function from $\mathcal{P}_G$ to $\mathcal{P}_H$. Its proof is analogous to the proof of Lemma 4.2 given in Section 4.

We now define a mapping h from $\mathcal{P}_H$ to $\mathcal{P}_G$ such that $h \circ f(\vec{x}) = \vec{x}$ for any popular fractional matching $\vec{x}$ in G. Let $\vec{z} = (z_e)_{e \in E'}$ be any popular fractional matching in H. We define $h(\vec{z}) = \vec{y} = (y_e)_{e \in \tilde{E}}$ as follows: $y_{(v,\ell(v))} = z_{(v,v')}$ for every $v \in V$ and $y_{(u,v)} = (z_{(u,v')} + z_{(v,u')})/2$ for every $(u,v) \in E$. It is easy to see that $h(\vec{z})$ is a fractional matching in G and Lemma 5.3 below states that $h(\vec{z})$ is a popular fractional matching in G.

LEMMA 5.3. *For any popular fractional matching $\vec{z}$ in H, the vector $h(\vec{z})$ is a popular fractional matching in G.*

Lemma 5.3 can be proved in the same way that we proved Lemma 4.3 in Section 4: use the witness vector $\vec{\beta}$ for $\vec{z}$'s membership in $\mathcal{P}'_H$ to construct a witness vector $\vec{\alpha}$ for $h(\vec{z})$'s membership in $\mathcal{P}'_G$. That is, let $\alpha_v = (\beta_v + \beta_{v'})/2$ for each $v \in V$. It is straightforward to show that $h(\vec{z})$ and $\vec{\alpha}$ satisfy all the constraints in $\mathcal{P}'_G$.

Lemma 5.4 proves the half-integrality of $\mathcal{P}_G$. Its proof is the same as the proof of Lemma 4.4 given in Section 4.

LEMMA 5.4. *Let $\vec{x} \in \mathcal{P}_G$. Then $\vec{x} = \sum_{i=1}^{r} \lambda_i \vec{q}_i$, where $\vec{q}_1, \ldots, \vec{q}_r$ are popular half-integral matchings in G and $\lambda_i \geq 0$ for $1 \leq i \leq r$ along with $\sum_i \lambda_i = 1$.*

For any stable fractional matching $\vec{x}$ in G, we have $\vec{x} \in \mathcal{P}_G$. This is because $\vec{x} \in \mathsf{FM}_G$ and $\vec{x}$ satisfies constraint (2.4) for all edges in G, so $(\vec{x}, \vec{0}) \in \mathcal{P}'_G$. It is known that stable half-integral matchings always exist in G [35] and so $\mathcal{P}_G$ is always non-empty. In order to find a max-utility popular fractional matching in G, we solve a linear program to maximize $\sum_{e \in E} w_e x_e$ along with the description of $\mathcal{P}'_G$ as the set of constraints. Since $\mathcal{P}_G$ is half-integral, we can conclude Theorem 5.1.

THEOREM 5.1. *Given an instance $G = (V,E)$ with strict preference lists and a utility function $w : E \to \mathbb{Q}$, there is always a max-utility popular fractional matching $\vec{q} \in \{0, \frac{1}{2}, 1\}^m$ in G, and $\vec{q}$ can be computed in polynomial time.*

6 Hardness of max-utility popular matching in roommates instances

In this section we show the NP-hardness of the max-utility popular matching problem in a roommates instance. Let $H = (V_H, E_H)$ be an instance of VERTEX COVER on n vertices. Based on H, we will construct a roommates instance $G = (V,E)$ with a utility function $w : E \to \{1,2\}$. We will show that H has a vertex cover of size at most k if and only if G has a popular matching M such that $w(M) \geq 4n - 2k$. This reduction will show that the max-utility popular matching problem in a roommates instance is NP-hard.

Let $V_H = \{1, \ldots, n\}$. We now describe the roommates instance G. The vertex set V of G consists of $4n$ vertices: 4 vertices i_0, i_1, i_2, i_3 corresponding to each vertex i where $i \in [n]$. For each $i \in [n]$, we describe the preference lists of the vertices i_0, i_1, i_2, i_3 below (note that $\mathsf{Nbr}_H(i)$ denotes the set of neighbors of vertex i in H):

$$\begin{aligned}
i_0 &: \quad i_1 \succ \pi(j_0 : j \in \mathsf{Nbr}_H(i)) \succ i_2 \succ i_3 \succ \pi(\cdots) \\
i_1 &: \quad i_0 \succ i_2 \succ i_3 \succ \pi(\cdots) \\
i_2 &: \quad i_1 \succ i_0 \succ i_3 \succ \pi(\cdots) \\
i_3 &: \quad i_1 \succ i_2 \succ i_0 \succ \pi(\cdots)
\end{aligned}$$

In order to describe the preference lists of vertices in G in a compact manner, we used the following symbols: $\pi(j_0 : j \in \mathsf{Nbr}_H(i))$ denotes an arbitrary permutation of the vertices $j_0, k_0, \ldots$ where $j, k, \ldots$ are i's neighbors in H and $\pi(\cdots)$ denotes an arbitrary permutation of all neighbors of i_t

in G not explicitly listed so far in the preference list of i_t, for $0 \leq t \leq 3$.

We define edge utilities in G as follows: $w(i_0, i_2) = w(i_1, i_3) = 2$ for every $1 \leq i \leq n$; for every other edge e we set $w(e) = 1$. The instance G has a stable matching $S = \{(i_0, i_1), (i_2, i_3) : i \in [n]\}$ and $w(S) = 2n$.

Since G is a complete graph on $4n$ vertices, every popular matching in G has to be perfect. Given a perfect matching M in G, we need a method to prove the popularity of M. For this, we will use the characterization of popular matchings from [20] that uses *edge labels* on edges outside M: more precisely, (u, v) in $E \setminus M$ gets the label $(\mathsf{vote}_u(v, M(u)), \mathsf{vote}_v(u, M(v)))$.

All edges with the label (1,1) are blocking edges to M. Let G_M be the graph obtained by deleting all edges labeled $(-1, -1)$ from G. It was shown in [20] that a perfect matching M is popular in G if and only if the following two conditions hold in G_M.

(i) There is no alternating cycle with respect to M that contains a blocking edge.

(ii) There is no alternating path with respect to M that contains two or more blocking edges.

We are now ready to show one side of the reduction.

LEMMA 6.1. *Let $C \subseteq [n]$ be a vertex cover of size k in H. Then there exists a popular matching M in G such that $w(M) = 4n - 2k$.*

Proof. We construct a matching M in G according to the vertex cover C in H as follows: start with $M = \emptyset$; for $i = 1$ to n do:

- if $i \in C$, then add (i_0, i_1) and (i_2, i_3) to M;
- if $i \notin C$, then add (i_0, i_2) and (i_1, i_3) to M.

The utility of M is $(1+1)k + (2+2)(n-k) = 4n - 2k$. So what we need to show is that M is a popular matching in G. We will prove this by showing that conditions (i) and (ii) stated above (from [20]) hold in the graph G_M. We will understand the structure of the graph G_M now.

In the graph G_M, for any $i \in [n]$, only the vertex i_0 (among the 4 vertices i_0, i_1, i_2, i_3 corresponding to i) has edges connecting it to the "outside world". Such an edge is of the form (i_0, j_0) where $(i, j) \in E_H$. Since C is a vertex cover of H, one of i, j has to be in C. When both i and j are in C, the edge (i_0, j_0) gets the label $(-1, -1)$ and so it is not present in G_M.

Observe that the graph G_M consists of the following edges (refer to Fig. 9 and Fig. 10):

1. edges in M.
2. the edges $(i_0, i_2), (i_1, i_2), (i_1, i_3)$ for every $i \in C$: each of these edges has the label $(-1, 1)$.

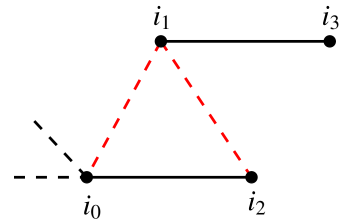

Figure 9: The vertices and edges in G_M corresponding to a vertex $i \notin C$: the bold edges are in M and the red dashed edges are blocking edges.

3. the edges (i_0, i_1) and (i_1, i_2) for every $i \notin C$: each of these is a blocking edge to M.
4. the edge (i_0, j_0) where $(i, j) \in E_H$, and exactly one of i, j is in C: such an edge is labeled $(1, -1)$ or $(-1, 1)$.

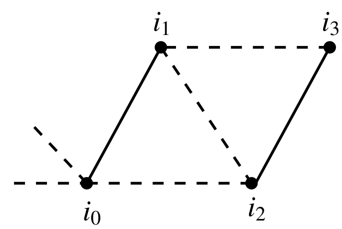

Figure 10: The vertices and edges in G_M corresponding to a vertex $i \in C$. The bold edges are in M. There is no blocking edge here.

We are now ready to show that M is a popular matching in G. The only blocking edges to M in G_M are (i_0, i_1) and (i_1, i_2) for $i \notin C$. It is easy to see that there is no alternating cycle with respect to M in G_M with a blocking edge. This is because any alternating cycle ρ in G_M that contains a blocking edge has to contain either (i_0, i_1) or (i_1, i_2) for some $i \notin C$. This means the vertex i_3 (being the matched partner of i_1) has to belong to ρ. However i_3 has degree 1 in G_M (see Fig. 9) and thus cannot belong to any cycle. Thus there is no such alternating cycle ρ.

Consider any alternating path ρ with respect to M in G_M that contains either (i_0, i_1) or (i_1, i_2) for $i \notin C$. Our first observation is that ρ cannot contain *both* (i_0, i_1) and (i_1, i_2) as there is no such alternating path in G_M. So for ρ to contain more than one blocking edge, some edge (i_0, j_0) has to be present in ρ as that is the only way to get out of the set $\{i_0, i_1, i_2, i_3\}$.

Since $i \notin C$ and $(i, j) \in E_H$, the vertex $j \in C$. So j_0's partner in M is j_1 and as j_1 has only two neighbors j_2 and j_3 in G_M, it is easy to see that this alternating path ρ will get "trapped" inside the vertices corresponding to j (see Fig. 10). As there is no blocking edge incident on any of the vertices

j_0, j_1, j_2, j_3 and since this alternating path ρ cannot continue on to other vertices in G_M, it follows that ρ has at most one blocking edge. Thus there is no alternating path in G_M with two or more blocking edges. So conditions (i) and (ii) stated earlier hold for M in G_M and thus M is popular in G. $\square$

Our goal now is to show that if G has a popular matching with utility at least $4n-2k$, then H has a vertex cover of size at most k. The following lemma will be useful to us.

LEMMA 6.2. *Let M be a popular matching in G. Then M has no edge (i_s, j_t) where $i \neq j$ and $0 \le s,t \le 3$.*

Proof. We will prove this lemma by case analysis. Our first observation is that M has to be a perfect matching in G since G is a complete graph on $4n$ vertices. Suppose an edge (i_s, j_t) belongs to M where $i \neq j$.

Case 1. Let $s,t \in \{0,1\}$. Then both (i_0,i_1) and (j_0,j_1) are blocking edges to M and thus we get an alternating path of length 3 with 2 blocking edges (i_0,i_1) and (j_0,j_1) and the matching edge (i_s,j_t) in between them; this is a contradiction to the popularity of M by condition (ii) stated earlier.

Case 2. Let $s \in \{0,1\}$ and $t=2$. If $(j_1,j_3) \notin M$, then there is an alternating path i_{1-s}-i_s-j_2-j_3 in G_M with two blocking edges (i_0,i_1) and (j_2,j_3). If $(j_1,j_3) \in M$, then there is another alternating path i_{1-s}-i_s-j_2-j_0 with two blocking edges (i_0,i_1) and (j_0,j_2). Observe that (j_0,j_2) is a blocking edge to M because we have already established in Case 1 that j_0 cannot be matched in M to any k_0 and so j_0 is matched in M to some neighbor worse than j_2.

Case 3. Let $s \in \{0,1\}$ and $t=3$. If j_2 is matched to neither j_0 nor j_1, then i_{1-s}-i_s-j_3-j_2 has two blocking edges (i_0,i_1) and (j_2,j_3). If j_2 is matched to j_r where $r \in \{0,1\}$, then i_{1-s}-i_s-j_3-j_{1-r} has two blocking edges (i_0,i_1) and (j_3,j_{1-r}).

So we have established that for any $i \in [n]$, neither i_0 nor i_1 can be matched in M to any vertex in $\{j_0,j_1,j_2,j_3\}$ where $j \neq i$.

Case 4. Let $s,t \in \{2,3\}$. So either i_2 or i_3 is matched to an "outside vertex", i.e., a vertex j_t where $i \neq j$. Then it has to be the case that *both* i_2 and i_3 are matched to outside vertices. Otherwise we have 3 out of the 4 vertices i_0,i_1,i_2,i_3 that have to be matched to each other in M, which is not possible.

Thus i_2 is matched to an outside vertex (say, j_t where $t \in \{2,3\}$) and i_3 is also matched to an outside vertex. So (i_2,i_3) is a blocking edge to M. By the same reasoning, (j_2,j_3) is also a blocking edge to M. Thus we have an alternating path of length 3 that consists of two blocking edges (i_2,i_3) and (j_2,j_3) with the matching edge (i_2,j_t) in between them. This contradicts the popularity of M. $\square$

LEMMA 6.3. *Suppose M is a popular matching in G.*

(i) For the vertices i_0,i_1,i_2,i_3 in G: either $\{(i_0,i_1),(i_2,i_3)\} \subseteq M$ or $\{(i_0,i_2),(i_1,i_3)\} \subseteq M$.

(ii) If $(i,j) \in E_H$, then either $\{(i_0,i_1),(i_2,i_3)\} \subseteq M$ or $\{(j_0,j_1),(j_2,j_3)\} \subseteq M$ (or possibly both).

Proof. We know from Lemma 6.2 that for any $i \in [n]$, the vertices i_0,i_1,i_2,i_3 have to be matched to each other in M. So there are 3 possibilities: (1) M contains (i_0,i_1) and (i_2,i_3), (2) M contains (i_0,i_2) and (i_1,i_3), (3) M contains (i_0,i_3) and (i_1,i_2).

However if M contains (i_0,i_3) and (i_1,i_2), then there is an alternating cycle i_0-i_3-i_2-i_1-i_0 in G_M that contains a blocking edge (i_0,i_1) and this contradicts the popularity of M by condition (i) stated earlier. Thus options (1) and (2) above are the only possibilities. This proves the first part of the lemma.

We now show the second part. Suppose $(i,j) \in E_H$ and M contains the 4 edges (i_0,i_2), (i_1,i_3), (j_0,j_2), and (j_1,j_3). Then there is an alternating path i_1-i_2-i_0-j_0 in G_M with two blocking edges (i_1,i_2) and (i_0,j_0): this contradicts the popularity of M by condition (ii) stated earlier. $\square$

We are now ready to show the following lemma that completes our reduction.

LEMMA 6.4. *If M is a popular matching in G with $w(M) \ge 4n-2k$ then there is a vertex cover C in H with $|C| \le k$.*

Proof. We know from Lemma 6.3 (i) that for every $i \in [n]$: either $\{(i_0,i_1),(i_2,i_3)\} \subseteq M$ or $\{(i_0,i_2),(i_1,i_3)\} \subseteq M$. We will construct a vertex cover C in H as follows:

$$C = \{i \in [n] : M \text{ contains the edges } (i_0,i_1) \text{ and } (i_2,i_3)\}.$$

It follows from Lemma 6.3 (ii) that the set C is indeed a vertex cover in H. If $|C| = n_0$ then $w(M) = (1+1)n_0 + (2+2)(n-n_0) = 4n-2n_0$. We are given that $w(M) \ge 4n-2k$, hence $n_0 \le k$. $\square$

The NP-hardness of the max-utility popular matching problem follows from Lemmas 6.1 and 6.4. In fact, our reduction shows the following stronger result.

THEOREM 6.1. *It is NP-hard to compute a max-utility popular matching in a roommates instance $G=(V,E)$, even when all preferences are strict and complete, each edge utility is either 1 or 2, and G admits a stable matching.*

6.1 An inapproximability result. We will now show that there is no polynomial time $O(1)$-approximation algorithm for the max-utility popular matching problem in a roommates instance unless the unique games conjecture fails and this is the case even when all preferences are strict and complete. Let $H=(V_H,E_H)$ be an instance of the VERTEX COVER problem on n vertices and we assume that H admits a perfect matching. It is known that VERTEX COVER is hardest to approximate in such instances [8].

Using H, we will construct a roommates instance $G = (V,E)$ as done at the beginning of Section 6. So Lemmas 6.2 and 6.3 hold. The only change that we make from the description of G given earlier is with respect to the edge utility function w in G. Our new function has $w(i_0,i_2) = w(i_1,i_3) = 1$ for all $i \in [n]$ and $w(e) = 0$ for all other edges.

As before, G has a stable matching $S = \{(i_0,i_1),(i_2,i_3) : i \in [n]\}$ and now $w(S) = 0$. Lemmas 6.1 and 6.4 now yield the following conclusion: *there is a vertex cover in H of size $\leq k$ if and only if there is a popular matching in G of utility $\geq 2n-2k$*. Also, given a popular matching in G of utility at least $2n-2k$, we can easily find a vertex cover of size at most k in H (as done in the proof of Lemma 6.4).

THEOREM 6.2. *There is no polynomial time $O(1)$-approximation algorithm to compute a max-utility popular matching in a roommates instance with non-negative edge utilities unless the unique games conjecture fails.*

Proof. Consider the vertex cover instance H on n vertices. Recall that we assumed H to contain a perfect matching. Let opt be the size of an optimal vertex cover in H, so $\mathsf{opt} \geq n/2$ (due to the perfect matching in H).

Suppose there is a polynomial time ε-approximation algorithm for the max-utility popular matching problem in G, for some constant $\varepsilon > 0$. We will compute a matching M in G by this approximation algorithm and using M, we will obtain a valid vertex cover C in H (as done in the proof of Lemma 6.4). We will now bound the size of C.

If $\mathsf{opt} \leq (1+\varepsilon)n/2$ then there is a popular matching in G with utility $\geq 2n-(1+\varepsilon)n = (1-\varepsilon)n$. So $w(M) \geq \varepsilon(1-\varepsilon)n$ and this implies $|C| \leq (2-\varepsilon+\varepsilon^2)n/2$ (since given a matching with utility $\geq 2n-2k$ in G, we can find a vertex cover in H of size $\leq k$). By assumption $n/2 \leq \mathsf{opt}$, so we have a vertex cover in H of size at most $(2-\varepsilon+\varepsilon^2)\mathsf{opt}$.

The case left is when $\mathsf{opt} \geq (1+\varepsilon)n/2$. However in this case even if $C = V_H$, the approximation factor would be $\frac{n}{(1+\varepsilon)n/2} \leq 2(1-\varepsilon+\varepsilon^2)$. Thus we always have a vertex cover of size at most $(2-\varepsilon+\varepsilon^2)\mathsf{opt}$. Since ε is a constant in $(0,1)$, this will break the unique games conjecture. $\square$

Conclusions and Open problems. Given a bipartite instance $G = (A \cup B, E)$ with strict preference lists, we showed that its popular fractional matching polytope $\mathcal{P}_G$ is half-integral (and in the special case where a stable matching in G is a perfect matching, the polytope $\mathcal{P}_G$ is integral). Thus when there are edge utilities in G, there is always a max-utility popular fractional matching in G that is half-integral. The main open problem here is to settle the complexity of the max-utility popular matching problem in G.

When G is a roommates instance, (i.e., the graph need not be bipartite), we showed that it is NP-hard to find a max-utility popular matching in G. We showed a polynomial time algorithm to compute a max-utility popular half-integral matching in a roommates instance $G = (V,E)$ with a utility function $w : E \to \mathbb{Q}$. A popular matching need not always exist in a roommates instance and the main open problem is to settle the complexity of the *popular matching problem* in a roommates instance, i.e., given an instance $G = (V,E)$ with strict preferences lists, does G admit a popular matching?

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