Probabilistic cellular automata
density classification problem

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SAA
Jan 15, 2018
Cellular automata

Cell space: any finite or countable group \((G, +)\).
Alphabet: a finite set \(A\).

Definition by S. Ulam and J. von Neumann (50’s)

A **cellular automaton** is a function \(F : A^G \to A^G\) characterized by
- a finite neighborhood \(V \subset G\),
- a local function \(f : A^V \to A\) such that \(F(x)_k = f((x_{k+v})_{v \in V})\).

Motivation
- very simple description generating complex behaviors, models for physical and biological processes
- distributed computing: exchanging only local information; limited memory
- links with problems in probability and combinatorics

\[G = (\mathbb{Z}, +), A = \{0, 1\}, V = (0, 1), F(x)_n = x_n + x_{n+1} \mod 2.\]
Game of Life

Invented by John Conway in 1970

- Infinite checker-board whose squares (cells) are colored black (alive) or white (dead).
- A living cell stays alive iff there are exactly two or three living neighbors (among eight). Fewer than two causes death by isolation, more than three by overcrowding.
- A dead cell becomes alive if it has precisely three living neighbors

Moore neighborhood
Very complex behaviour:

Still life

Oscillator

Glider gun (Bill Gosper 1970)
Very complex behaviour:

- **Still life**
- **Oscillator**
- **Glider**

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Wolfram’s classification

Elementary CA:
cell space: \((\mathbb{Z}, +)\); alphabet: \(A = \{0, 1\}\); neighborhood: \(V = \{-1, 0, 1\}\).

256 elementary CA (88 essentially different rules).

Empirically classified by S. Wolfram in the 1980’s:

1. Almost all initial configurations lead to the same uniform fixed point configuration.
2. Almost all initial configurations lead to a periodically repeating configuration.
3. Almost all initial configurations lead to essentially random looking behavior.
4. Localized structures with complex interactions emerge.
Wolfram’s classification

class 1: rule 160

class 3: rule 126

class 3: rule 108

class 4: rule 110
Rule 110

Local update rule (8 bit binary expansion of 110 is 01101110):
\[ f(111) = 0; f(110) = 1; f(101) = 1; f(100) = 0; \]
\[ f(011) = 1; f(010) = 1; f(001) = 1; f(000) = 0; \]

This CA has become known since it was proved to be **computationally universal** (or Turing complete) ([Cook 2004](#)); Conj. Wolfram 1985.

Another example of computationally universal CA: Game of Life. ([P. Rendell 2002](#))

Proof outlines of universality [Berlekamp et al. 1982](#).
\( \mathcal{M}(\mathcal{A}) \) the set of probability measures on \( \mathcal{A} \).

A PCA is given by

- a finite neighbourhood \( V \subset G \),
- a local function \( \varphi : \mathcal{A}^V \to \mathcal{M}(\mathcal{A}) \).

The cells are updated synchronously and independently, according to \( \varphi \) (depending on a finite neighbourhood).

This defines an application \( F : \mathcal{M}(\mathcal{A}^G) \to \mathcal{M}(\mathcal{A}^G), \mu \mapsto \mu F \). 
\( (\mathcal{M}(\mathcal{A}^G) \) prob. measures on \( \mathcal{A}^G \)\)
Ergodicity

**Def.** A PCA is **ergodic** if it has a unique invariant measure $\pi$, and if for each measure $\mu \in \mathcal{M}(A^\mathbb{Z})$, the sequence $\mu F^n$ converges weakly to $\pi$ (i.e. $\mu F^n(C)$ conv. to $\pi(C)$ for any finite cylinder $C$).
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A PCA has at least one invariant measure, and the set of invariant measures is convex and compact.
(The set $M(X)$ of measures on $X$ is compact for the weak topology.)

Classification:

- several invariant measures;
- a unique invariant measure without convergence;
  (Example: Chassaing and Mairesse 2011.)
- a unique invariant measure with convergence (ergodic case).
Let $\mathcal{A} = \{0, 1\}$, $G = \mathbb{Z}$, and $|V| \geq 2$. Consider $0 < \alpha < 1$ and the local function:

$$f((x_v)_{v \in V}) = \alpha \delta_{\max(x_v, \ v \in V)} + (1 - \alpha) \delta_0.$$ 

Dirac measure $\delta_0^G$ is an invariant measure.

Using a coupling with a site percolation model, one can prove that there exists $\alpha^* \in (0, 1)$ such that:

\begin{align*}
\alpha < \alpha^* & \implies (iii): \text{ergodicity} \\
\alpha > \alpha^* & \implies (i): \text{several invariant measures.}
\end{align*}

The exact value of $\alpha^*$ is not known but it satisfies $1/|V| \leq \alpha^* \leq 53/54$. 
Ergodicity of PCA on $\mathbb{Z}$

Set of cells: $\mathbb{Z}$. Even the ergodicity of DCA is undecidable. B., Mairesse, Marcovici 2011.

Sufficient conditions: using coupling from the past.

New alphabet $B = \{0, 1, ?\}$ (unknown letters replaced by “?”). $g : B^V \rightarrow \mathcal{M}(B)$, defined for each $y \in B^V$ by

$$g(y)(0) = \min_{x \in A^V, x \in y} f(x)(0), \quad g(y)(1) = \min_{x \in A^V, x \in y} f(x)(1),$$

$$g(y)(?) = 1 - \min_{x \in A^V, x \in y} f(x)(0) - \min_{x \in A^V, x \in y} f(x)(1).$$

Theorem. There exists a critical value $0 < \alpha^* < 1$, depending only on $|V|$, such that $F$ is ergodic if

$$g(\cdot^V)(\cdot) < \alpha^*.$$
Density classification

We fix $\mathcal{A} = \{0, 1\}$. Let $p \in [0, 1]$.

Initial configuration: for each cell, we choose independently to write a 1 with probability $p$ and a 0 with probability $1 - p$ (distribution $\mu_p$ on $\mathcal{A}^\mathbb{Z}$).
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**Challenge**

Find a CA (or PCA) such that:

$$\begin{align*}
\begin{cases}
p < 1/2 & \implies \mu_p F^n \xrightarrow{w} \delta_0 \\
p > 1/2 & \implies \mu_p F^n \xrightarrow{w} \delta_1
\end{cases}
\end{align*}$$

The notation $\xrightarrow{w}$ stands for the weak convergence of measures; Equivalently, for any $k \in \mathbb{Z}$, the probability that $F^n(x)_k = 1$ tends to 0 if $p < 1/2$ and to 1 if $p > 1/2$. 
The majority rule

Let $F$ be the majority CA of neighborhood $V = (-1, 0, 1)$ defined by the local rule $f(x, y, z) = \text{maj}(x, y, z)$. 

If $p \in (0, 1)$, there are two consecutive 0 or two consecutive 1 that stay fixed forever. (B., Fatès, Mairesse, Marcovici 2013) Majority rule with any neighborhood of odd size does not classify the density.
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The GKL (Gács-Kurdyumov-Levin) rule

Let $F$ be the GKL CA of neighborhood $V = (-3, -2, -1, 0, 1, 2, 3)$ defined by

- $F(x)_n = \text{maj}(x_n, x_{n+1}, x_{n+3})$ if $x_n = 1$,
- $F(x)_n = \text{maj}(x_n, x_{n-1}, x_{n-3})$ if $x_n = 0$. 

$0 \in \mathbb{Z}$ and $1 \in \mathbb{Z}$ are fixed points of $F$ but also any word that is a concatenation of patterns $001$ and $011$. It is an open problem to know if GKL classifies the density. Easier on $\mathbb{Z}^2$...
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Easier on $\mathbb{Z}^2$...
Forbidden symmetry

The majority rule over a symmetric neighborhood that contains the cell itself has a finite stable pattern.

Stable patterns for von Neumann and Moore neighborhoods.
Definition of Toom’s rule

The alphabet is still $\mathcal{A} = \{0, 1\}$, the set of cells is now $\mathbb{Z}^2$.

Definition of the CA
We denote by $D$ the CA of neighborhood $V = \{(0, 0), (0, 1), (1, 0)\}$ (north-east-center) defined by the majority rule, that is,

$$(D(x))_{i,j} = \text{maj}(x_{i,j}, x_{i,j+1}, x_{i+1,j}).$$

This CA is known as **Toom’s rule**.
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Toom's rule classifies the density. (B., Fatès, Mairesse, Marcovici 2012)
That is, the sequence $(\mu_p D^n)_{n \geq 0}$ converges weakly to $\delta_{0\mathbb{Z}^2}$ if $p < 1/2$ and to $\delta_{1\mathbb{Z}^2}$ if $p > 1/2$. 
Finite clusters
Finite clusters
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Steps of the proof

Add NW-SE diagonals to the grid, and consider the triangular lattice.

- If $p > 1/2$, there exists a.s. no infinite 0-cluster (classical result of percolation theory)
- Two different 0-clusters cannot merge (P. Gacs, 1990)
- Any finite 0-cluster disappears in finite time and always stays in its enveloping rectangle (eroder property, A. Toom 1990)
- A given point belongs a.s. to the enveloping rectangle of an at most finite number of 0-clusters (by the exponential decay of the size of 0-clusters)
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Density classification on regular trees

Group $T_n = \langle a_1, \ldots, a_n \mid a_i^2 = 1 \rangle$. Cayley graph of $T_n$ is the infinite $n$-regular tree.

For $n$ even: $T_{2k}' = \langle a, \ldots, a_k \mid \cdot \rangle$ has the same Cayley graph ($T_{2k}'$ is not isomorphic to $T_{2k}$).

Odd values of $n$: natural candidate is majority rule on $n$ neighbours

- CA does not classify the density for $n = 3, 5, \text{ and } 7$. Kanoria, Montanari 2011.
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For $n$ even: $T'_{2k} = \langle a, \ldots, a_k \mid \cdot \rangle$ has the same Cayley graph ($T'_{2k}$ is not isomorphic to $T_{2k}$).

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- IPS does not classify the density. Howard 2000.

Even values of $n$: natural candidate is majority rule on $n$ neighbours and the central cell

- For $n = 4$ and $p \in (1/3, 2/3)$ the trajectories do not converge weakly to a uniform configuration. B. et al 2013.
Free group $T'_4 = \langle a, b | \cdot \rangle$

$F : \mathcal{A}^{T'_4} \to \mathcal{A}^{T'_4}, F(x)_g = \text{maj}(x_{ga}, x_{gab}, x_{gab^{-1}})$. 

\[F : \mathcal{A}^{T'_4} \to \mathcal{A}^{T'_4}, F(x)_g = \text{maj}(x_{ga}, x_{gab}, x_{gab^{-1}}).\]
Theorem

Cellular automaton $F : \mathcal{A}^{T_4} \rightarrow \mathcal{A}^{T_4}$ defined by:

$$F(x)_g = \text{maj}(x_{ga}, x_{gab}, x_{gab}^{-1})$$

for any $x \in \mathcal{A}^{T_4}, g \in T_4$, classifies the density.
Theorem
Cellular automaton $F: \mathcal{A}^{T_4'} \rightarrow \mathcal{A}^{T_4'}$ defined by:

$$F(x)_g = \text{maj}(x_{ga}, x_{gab}, x_{gab^1})$$

for any $x \in \mathcal{A}^{T_4'}, g \in T_4'$, classifies the density.

Steps of the proof:

1. Let $h : [0, 1] \rightarrow [0, 1]$, $h(p)$ the probability that maj($X, Y, Z$) = 1 when $X, Y, Z$ are independent Bernoulli($p$).
   Then $h(p) = 3p^2 - 2p^3$, and the sequence $(h^n(p))_{n \geq 0}$ converges to 0 if $p < 1/2$ and to 1 if $p > 1/2$. 

Family $E_k(n) = \{X^n_{u_1}u_2...u_k | u_1, u_2, ..., u_k \in \{a, ab, ab^{-1}\}\}$ consists of independent Bernoulli($h_n(p)$).

{\{a, ab, ab^{-1}\}} is a code.
Theorem

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for any $x \in \mathcal{A}^{T'_4}$, $g \in T'_4$, classifies the density.

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- Let $h : [0, 1] \rightarrow [0, 1]$, $h(p)$ the probability that $\text{maj}(X, Y, Z) = 1$ when $X, Y, Z$ are independent Bernoulli($p$). Then $h(p) = 3p^2 - 2p^3$, and the sequence $(h^n(p))_{n \geq 0}$ converges to 0 if $p < 1/2$ and to 1 if $p > 1/2$.

- Family $\mathcal{E}_k(n) = \{X_{u_1u_2\ldots u_k}^n \mid u_1, u_2, \ldots, u_k \in \{a, ab, ab^{-1}\}\}$ consists of independent Bernoulli($h^n(p)$).
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- Family $\mathcal{E}_k(n) = \{X_{u_1 u_2 \cdots u_k}^n \mid u_1, u_2, \ldots, u_k \in \{a, ab, ab^{-1}\}\}$ consists of independent Bernoulli($h^n(p)$).
  $\{a, ab, ab^{-1}\}$ is a code.
Group $T_3 = \langle a, b, c \mid a^2 = b^2 = c^2 = 1 \rangle$

$$F : \mathcal{A}^{T_3} \rightarrow \mathcal{A}^{T_3}, \quad F(x)_g = \text{maj}(x_{gab}, x_{gac}, x_{gacb})$$
Back to 1D: 1 bit of memory

If we release some of the constraints, the problem becomes much easier.
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Example of solution using two tapes \( (i.e. \ 4 \ states) \):
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**Example of solution using two tapes (i.e. 4 states):**

- on a first tape compute the traffic CA

```
1 0 1 1 1 0 0 0
111 110 101 100 011 010 001 000
```

- on a second tape write a 0 (resp. 1) if the cells $x$ and $x+1$ are in state 0 (resp. 1) on the first tape, otherwise do not change the state.

The second tape of this CA converges to the right answer.
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Finite set of cells

Let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ (set of $n$ cells arranged in a ring).
For $x \in \{0, 1\}^{\mathbb{Z}_n}$, we denote by $\rho(x)$ the proportion of 1 in the configuration $x$. 

Challenge (density classification problem)

Find a CA (or PCA) $F$ such that for each $n$, and for each $x \in \{0, 1\}^{\mathbb{Z}_n}$, 

- if $\rho(x) > \frac{1}{2}$, there exists a time $t_0$ such that $\forall t \geq t_0$, $F_t(x) = 1$
- if $\rho(x) < \frac{1}{2}$, there exists a time $t_0$ such that $\forall t \geq t_0$, $F_t(x) = 0$
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  \forall t \geq t_0, F^t(x) = 1^n
  \]
- if $\rho(x) < 1/2$, there exists a time $t_0$ such that
  \[
  \forall t \geq t_0, F^t(x) = 0^n.
  \]
Finite case: CA

It is not possible to construct such a deterministic CA on the alphabet \( \{0, 1\} \). Land and Belew, 1995

No “perfect” solution of the density classification problem.
Finite case: CA

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No “perfect” solution of the density classification problem.

The performances of different CA have been compared and the GKL rule appears to be one of the best known: for \( n = 149 \), about 80% of well-classified initial conditions.
Finite case: PCA

Still no “perfect” solution of the density classification problem!
(B. et al, 2011)
...but one can answer the question with an arbitrary precision.
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Let us define the Maj-traf PCA by $V = (-1, 0, 1)$ and

$$f(x, y, z) = \alpha \delta_{\text{maj}(x, y, z)} + (1 - \alpha) \delta_{\text{traf}(x, y, z)},$$
Finite case: PCA

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where $\text{traf}(x, y, z)$ is defined by

$$\begin{array}{cccccccc}
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
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```

Ex: ... 0 0 1 0 0 1 0 1 0 0 0 ...
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\[
\begin{array}{cccccccccc}
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\
0 & & & & & & & & \\
\text{Ex: } & & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \ldots \\
\text{T } & \text{M } & \text{T } & \text{M } & \text{M } & \text{T } & \text{M } & \text{T } & \text{M } & \text{T } & \text{T }
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Ex: ...

$$
\begin{array}{cccccccccccc}
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
\end{array}
$$

$$
\begin{array}{cccccccccccc}
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & ... \\
T & M & T & M & M & T & M & T \\
\end{array}
$$
Proposition (Fatès 2011)

For any $r \in (0, 1)$, and any $n \geq 0$, there exists a choice of $\alpha$ such that for any $x \in \{0, 1\}^\mathbb{Z}^n$, the PCA converges to the “good” uniform configuration with a probability greater than $r$.

Examples of diagrams for Traffic and Majority-traffic
Finite case

- On $\mathbb{Z}_n$: two tapes CA as in $\mathbb{Z}$ case.
- On any finite graph: pairwise gossip rule (Bénézit, Thiran, Vetterli 2009)
Fast vs. slow convergence

Let $p^*$ be the threshold between ergodicity and non-ergodicity for the Stavskaya PCA on $\mathbb{Z}$.

\[
\begin{array}{cccc}
0 & 1 (p) & 1 (p) & 1 (p) \\
0 (1-p) & 0 (1-p) & 0 (1-p) & \ \\
0 & 1 & 1 & \\
0 & 0 & 1 & 1 \\
\end{array}
\]

On $\mathbb{Z}_n$: expected time to reach $\delta_0^{\mathbb{Z}_n}$ starting from $1^{\mathbb{Z}_n}$ is

- logarithmic in $n$ if $p < p^*$,
- exponential in $n$ if $p > p^*$.

Taggi (2013)

Positive rates PCA

\[ \varphi : A^V \rightarrow \mathcal{M}(A) \text{ s.t. } \varphi(x)(a) > 0, \forall x \in A^V, \ a \in A. \]

A READER’S GUIDE TO GACS’S “POSITIVE RATES” PAPER

LAWRENCE F. GRAY

Abstract. Peter Gacs’s monograph, which follows this article, provides a counterexample to the important Positive Rates Conjecture. This conjecture, which arose in the late 1960’s, was based on very plausible arguments, some of which come from statistical mechanics. During the long gestation period of the Gacs example, there has been a great deal of skepticism about the validity of his work. The construction and verification of Gacs’s counterexample are unavoidably complex, and as a consequence, his paper is quite lengthy. But because of the novelty of the techniques and the significance of the result, his work deserves to become widely known. This reader’s guide is intended both as a cheap substitute for reading the whole thing, as well as a warm-up for those who want to plumb its depths.

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