Probabilistic cellular automata and perfect simulation

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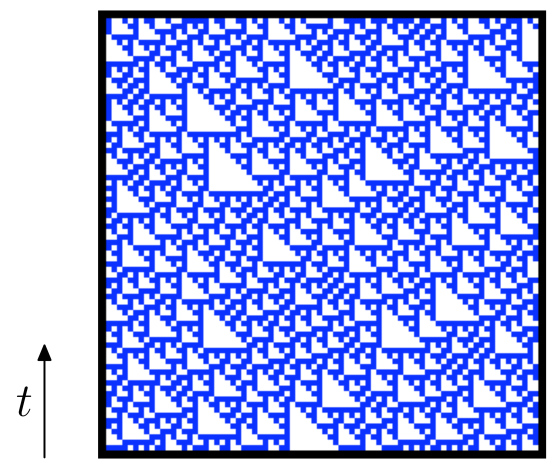
Cellular automata

Cellular automata (CA), introduced in the 50's by S. Ulam and J. von Neumann, are dynamical systems in which space and time are discrete.

- E : a set of cells (ex. $\mathbb{Z}^d$ or $\mathbb{Z}\backslash n\mathbb{Z}$).
- Each cell contains a letter from a finite alphabet $\mathcal{A}$.
- The contents of all the cells evolve *synchronously*: the content of each cell evolving as a function of the contents of the cells in its *finite neighborhood* and according to a *local rule*.

Example 1. $E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, and $F : \mathcal{A}^E \to \mathcal{A}^E$ defined by $(F(x))_k = x_k + x_{k+1} \bmod 2$.



Space-time diagram (the initial configuration is at the bottom).

Probabilistic cellular automata (PCA)

Motivations:

- Fault-tolerant computational models [8, 5].
- PCA appear in combinatorial problems related to the enumeration of directed animals [3, 1].
- In classification of deterministic CA (Wolfram's program): robustness to random errors [4].

Assumption: $E = \mathbb{Z}^d$ or $\mathbb{Z}/n\mathbb{Z}$.

Definitions and notations:

- $X = \mathcal{A}^E$, equipped with the product topology (generated by cylinders).
  A *cylinder* is a subset of X having the form $y_K = \{x \in X; \forall k \in K, x_k = y_k\}$ for a given finite subset K of E and a given sequence $(y_k)_{k\in K} \in \mathcal{A}^K$. $\mathcal{C}(K)$ is the set of all cylinders of base K.
- $\mathcal{M}(\mathcal{A})$ and $\mathcal{M}(X)$ are resp. the sets of probability measures on $\mathcal{A}$ and X.

Def. Given a finite neighborhood $V \subset E$, a (local) transition function of neighborhood V is a function $f : \mathcal{A}^V \to \mathcal{M}(\mathcal{A})$.

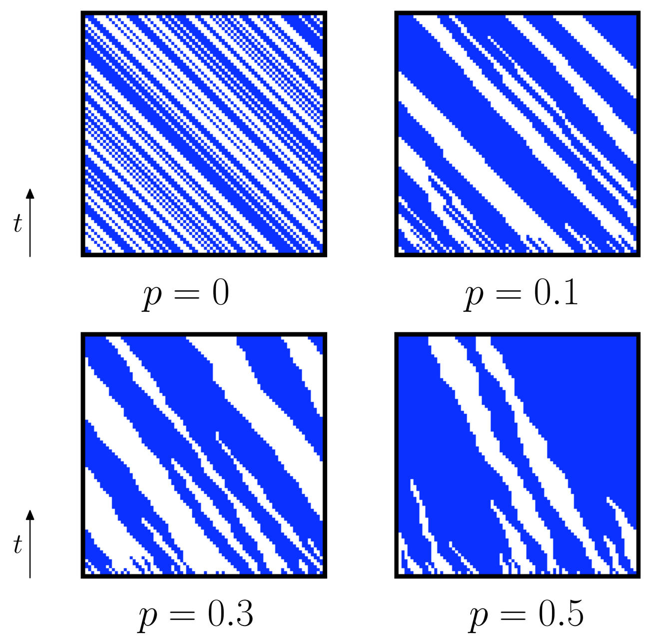
Def. The probabilistic cellular automaton (PCA) P of transition function f is the application $\mathcal{M}(X) \to \mathcal{M}(X)$, $\mu \mapsto \mu P$ defined on cylinders by:

$$\mu P(y_K) = \sum_{x_{V(K)} \in \mathcal{C}(V(K))} \mu(x_{V(K)}) \prod_{k\in K} f((x_i)_{i\in k+V})(y_k),$$

where $V(K) = \cup_{k\in K} k + V$.

Interpretation: A PCA P is a Markov chain on the state space $\mathcal{A}^E$. If E is finite, the transition probabilities are given by

$$P(x, y) = \prod_{k\in E} f((x_i)_{i\in k+V})(y_k), \ x, y \in \mathcal{A}^E.$$



Example 2. $\mathcal{A} = \{0, 1\}$, $V = (0, 1)$, and $f(x, y) = p\,\delta_x + (1 - p)\,\delta_y$, $p \in [0, 1]$.

Ergodic PCA

Def. $\pi \in \mathcal{M}(X)$ is a stationary measure of the PCA P if $\pi P = \pi$.

A PCA has at least one stationary measure.

Example 3. ACP from Example 2 has $\delta_{0^\mathbb{Z}}$ et $\delta_{1^\mathbb{Z}}$ as stationary measures.

Def. The PCA P is ergodic if it has exactly one stationary measure π, and if for any measure $\mu \in \mathcal{M}(X)$, the sequence μP^n converges weakly to π (*i.e.* $\mu P^n(C)$ conv. to $\pi(C)$ for any cylinder C).

Ergodicity of a PCA is undecidable [8].

Sufficient conditions [8, Chap. 3]. There exists a constant η_n depending only on the size n of the neighborhood, such that

$$\left[\sum_{b\in\mathcal{A}} \min_{(a_i)_{i\in V}\in\mathcal{A}^V} f((a_i)_{i\in V})(b) \ > \ \eta_n\right] \implies P \text{ ergodic}.$$

The value of η_n is not known exactly, but satisfies $\eta_n < 1 - 1/n$.

How to sample the stationary measure of an ergodic ACP?

Perfect simulation of PCA

Assumptions: $E = \mathbb{Z}/n\mathbb{Z}$ and $\mathcal{A} = \{0, 1\}$.

Let P be an ergodic PCA and π its stationary measure on $X = \mathcal{A}^E$.

Perfect sampling: a random algorithm which returns a state $x \in X$ with probability $\pi(x)$.

Coupling from the past (Propp and Wilson, 1996) :

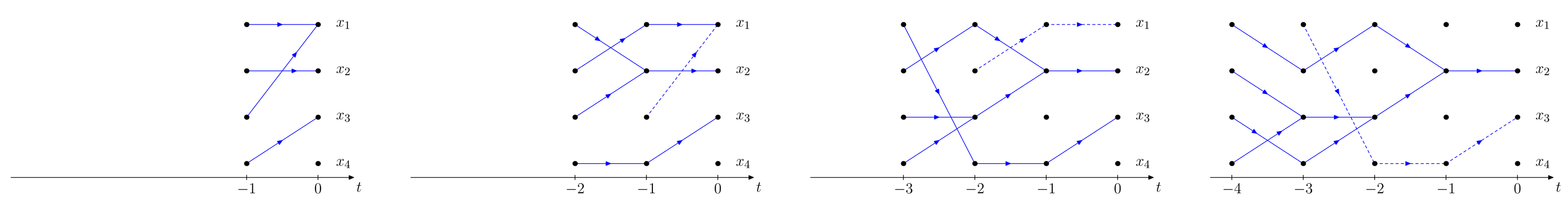
- $\phi : X \times [0, 1]^E \to X$ an update function. Example:

$$\phi(x, r)_k = \begin{cases} 0 \text{ if } 0 \le r_k < f((x_i)_{i\in k+V})(0) \\ 1 \text{ otherwise} \end{cases}$$

- $(r^j)_{j\in\mathbb{N}}$ a sequence of i.i.d. r.v's, with each r^j uniform on $[0, 1]^E$.
- Compute the sets $\{\phi(x, r^1), x \in X\}$, $\{\phi(\phi(x, r^2), r^1), x \in X\}$, $\{\phi(\phi(\phi(x, r^3), r^2), r^1), x \in X\}, \ldots$
  Stop when the computed set is a singleton and return its value.

Prop. [7] If the procedure stops a.s., then it returns state x with probability $\pi(x)$.

Example. Toy example of a PCA on the alphabet $\{0, 1\}$ and the set of cells $\mathbb{Z}/2\mathbb{Z}$. The state space is $X = \{x_1 = 00, x_2 = 01, x_3 = 10, x_4 = 11\}$. On this sample, the algorithm returns x_2.



Envelope PCA

New alphabet $\mathcal{B} = \{0, 1, ?\}$ (unknown letters replaced by "?"). Can be seen as a subset of the power set of $\mathcal{A}$:

- 0 is the singleton $\{0\}$,
- 1 is the singleton $\{1\}$,
- ? is $\mathcal{A} = \{0, 1\}$.

The envelope PCA of P, is the PCA env(P) of alphabet $\mathcal{B}$, defined on the set of cells E, with the same neighborhood V as for P, and a local function env$(f) : \mathcal{B}^V \to \mathcal{M}(\mathcal{B})$, defined for each $y \in \mathcal{B}^V$ by

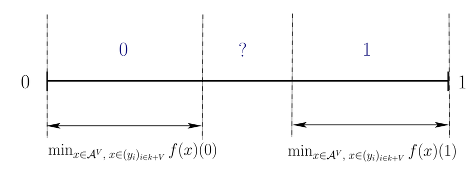
$$\text{env}(f)(y)(0) = \min_{x\in\mathcal{A}^V,\, x\in y} f(x)(0)$$

$$\text{env}(f)(y)(1) = \min_{x\in\mathcal{A}^V,\, x\in y} f(x)(1)$$

$$\text{env}(f)(y)(?) = 1 - \min_{x\in\mathcal{A}^V,\, x\in y} f(x)(0) - \min_{x\in\mathcal{A}^V, x\in y} f(x)(1).$$

Construction of an update function for the envelope PCA: $\tilde{\phi} : \mathcal{B}^E \times [0, 1]^E \to \mathcal{B}^E$

$$\tilde{\phi}(y, r)_k = \begin{cases} 0 \text{ if } 0 \le r_k < \text{env}(f)((y_i)_{i\in k+V})(0) \\ 1 \text{ if } 1 - \text{env}(f)((y_i)_{i\in k+V})(1) \le r_k \le 1 \\ ? \text{ otherwise.} \end{cases}$$



Data: the pre-computed function $\tilde{\phi}$, and a sequence $(r_i^{-j})_{(i,-j)\in E\times\mathbb{N}}$ of i.i.d. uniform in $[0, 1]$.
begin
  $c = ?^E$;
  $t = 1$;
  while $c \notin \{0, 1\}^E$ **do**
    $c = ?^E$;
    for $j = -t$ **to** -1 **do**
      $c = \tilde{\phi}(c, (r_i^{-j})_{i\in E})$
    $t = 2t$
  return c;
end

Properties:

1. For any $x \in \mathcal{A}^E$ and $y \in \mathcal{B}^E$ such that $x \in y$: $\forall r \in [0, 1]^E, \phi(x, r) \in \tilde{\phi}(y, r)$.
2. If the algorithm stops almost surely, then the PCA P is ergodic and the output of the algorithm is distributed according to the stationary measure of P.
3. The algorithm stops almost surely if and only if env$(f)(?^V)(?) < 1$, *i.e.*

$$\min_{x\in\mathcal{A}^V} f(x)(0) + \min_{x\in\mathcal{A}^V} f(x)(1) > 0.$$

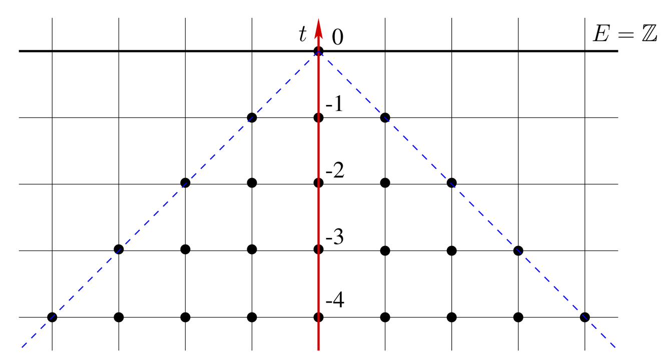
Extensions

- Alphabet $\mathcal{A}$ with more than two elements.
- Non-homogeneous finite PCA.

Infinite case

Assumptions: $E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, ergodic PCA P with stationary distribution π.

A perfect sampling procedure is a random algorithm taking as input a finite subset K of E and returning a cylinder $x_K \in \mathcal{C}(K)$ with probability $\pi(x_K)$.



Dependence cone of a cell

Theorem. There exists a critical value $0 < \alpha^* < 1$, depending only on $|V|$, such that env(P) is ergodic if

$$\max_{x\in\mathcal{B}^V} \text{env}(f)(x)(?) = \text{env}(f)(?^V)(?) < \alpha^*$$

and non-ergodic if

$$\min_{x\in\mathcal{B}^V\backslash\{0,1\}^V} \text{env}(f)(x)(?) > \alpha^*.$$

Alternative approach: restriction to finite windows & boundary conditions.

Open questions and further directions

- Coupling times PCA vs. envelope PCA?
- Open problem:
  For a PCA on $E = \mathbb{Z}^d$, $d \ge 1$, does the uniqueness of the stationary measure imply ergodicity?

References

1. M. Bousquet-Mélou. New enumerative results on two-dimensional directed animals. *Discrete Math.*, 180:73–106, 1998.
2. A. Bušić, B. Gaujal, and J.-M. Vincent. Perfect simulation and non-monotone markovian systems. In *Proceedings 3rd Int. Conf. Valuetools*. ICST, 2008.
3. D. Dhar. Exact solution of a directed-site animals-enumeration problem in three dimensions. *Phys. Rev. Lett.*, 51(10):853–856, 1983.
4. N. Fatès, E. Thierry, M. Morvan, and N. Schabanel. Fully asynchronous behavior of double-quiescent elementary cellular automata. *Theoret. Comput. Sci.*, 362(1-3):1–16, 2006.
5. P. Gács. Reliable cellular automata with self-organization. *J. Statist. Phys.*, 103(1-2):45–267, 2001.
6. M. Huber. Perfect sampling using bounding chains. *Ann. Appl. Probab.*, 14(2):734–753, 2004.
7. J. Propp and D. Wilson. Exact sampling with coupled Markov chains and applications to statistical mechanics. *Random Structures and Algorithms*, 9:223–252, 1996.
8. A. Toom, N. Vasilyev, O. Stavskaya, L. Mityushin, G. Kurdyumov, S. Pirogov. Stochastic cellular systems: ergodicity, memory, morphogenesis (Part I: Discrete local Markov systems). Nonlinear Science: theory and applications. Manchester University Press, 1990.