Probabilistic cellular automata and perfect simulation

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Cellular automata

Cellular automata (CA), introduced in the 50’s by S. Ulam and J. von Neumann, are dynamical systems in which space and time are discrete.

- $E$: a set of cells (ex. $Z^d$ or $Z/nZ$).
- Each cell contains a letter from a finite alphabet $A$.
- The states of all the cells evolve synchronously: the content of each cell evolving as a function of the contents of the cells in its finite neighborhood and according to a local rule.

Example 1. $E = Z, A = \{0, 1\}$, and $f : A^E \to A^E$ defined by

$$f(x) = 2x \mod 2.$$


Probabilistic cellular automata (PCA)

Motivations:

- Fault-tolerant computational models [8, 5].
- PCA appear in combinatorial problems related to the enumeration of directed animals [3, 1].
- In classification of deterministic CA (Wolfram’s program): robustness to random errors [4].

Assumption: $E = Z^d$ or $Z/nZ$.

Definitions and notations:

- $X = A^E$ equipped with the product topology (generated by cylinders).
- A cylinder is a subset of $X$ having the form $\{x \in X : x_i = y_i \}$ for a given finite subset $K$ of $E$ and a given sequence $(y_i)_{i \in K}$.
- $M(A)$ and $M(A)$ are resp. the state probabilities on $A$ and $X$.

Def. Given a finite neighborhood $V \subseteq E$, a local transition function of neighborhood $V$ is a function $f : A^V \to M(A)$.

Def. The probabilistic cellular automaton (PCA) $P$ of transition function $f$ is the application $M(X) \to M(X)$, $\mu \mapsto \mu^P$ defined on cylinders by:

$$\mu^P(V) = \sum_{x \in \pi(V)} \mu(x) f\left(\left(x_{\pi(V)}\right)\right),$$

where $V(X) = \{x \in A^E : x \in V\}$.

Interpretation: A PCA $P$ is a Markov chain on the state space $A^E$. If $E$ is finite, the transition probabilities are given by:

$$P(x,y) = \sum_{x \in \pi(V)} \mu(x) f\left(\left(x_{\pi(V)}\right)\right).$$

Example 2. $A = \{0, 1\}$, $V = \{0, 1\}$, and $f(x,y) = \min(p_x, 1-p_y) \cdot \min(p_x, 1-p_y)$.


Ergodic PCA

Def. $\pi \in M(X)$ is a stationary measure of the PCA $P$ if $\pi^P = \pi$.

A PCA has at least one stationary measure.

Example 3. APC from Example 2 has $\delta_{0,0}$ and $\delta_{1,1}$ as stationary measures.

Def. The PCA $P$ is ergodic if it has exactly one stationary measure $\pi$, and if for any measure $\mu \in M(A)$, the sequence $\mu^P$ converges weakly to $\pi$ (i.e. $\mu^P(C)$ cons. to $\pi(C)$ for any cylinder $C$).

Ergodicity of a PCA is undecidable [8].

Sufficient conditions [8, Chap. 3]. There exists a constant $\eta_0$ depending only on the size $n$ of the neighborhood, such that

$$\sum_{x \in \pi(V)} \min_{y \in \pi(V)} f((x_{\pi(V)})\mid y) > \eta_0 \Rightarrow P \text{ ergodic}.$$

The value of $\eta_0$ is not known exactly, but satisfies $\eta_0 < 1 - 1/n$.

How to sample the stationary measure of an ergodic $ACP$?

Perfect simulation of PCA

Assumptions: $E = Z^d/nZ$ and $A = \{0, 1\}$.

Let $P$ be an ergodic PCA and $\pi$ its stationary measure on $X = A^E$.

Perfect sampling: a random algorithm which returns a state $x \in X$ with probability $\pi(x)$.

Coupling from the past (Propp and Wilson, 1996):

- $\phi : X \times [0,1]^E \to X$ an update function. Example:

  $$\phi(x,r) = \begin{cases} 0 & \text{if } 0 \leq r < f((x_{\pi(V)})\mid 0) \\ 1 & \text{otherwise} \end{cases}$$

- $(\vec{r},\vec{x})_{i\in E}$ is a sequence of i.i.d. random variables, each with $[0,1]$ uniform on $[0,1]^E$.

- Compute the sets $\{x \mid \phi(x,\vec{r}) = 1\}$ and $\{x \mid \phi(x,\vec{r}) = 0\}$, $x \in X$.

- Stop when the computed set is a singleton and return its value.

Prop. [7] If the procedure stops a.s., then it returns state $x$ with probability $\pi(x)$.

Example. Toy example of a PCA on the alphabet $\{0,1\}$ and the set of cells $Z/Z$. The state space is $X = \{x_0 = 0, x_1 = 0, x_2 = 10, x_3 = 11\}$. On this sample, the algorithm returns $x_0$.


Envelope PCA

New alphabet $B = \{0, 1, ?\}$ (unknown letters replaced by ‘?’). Can be seen as a subset of the power set of $A$.

- 0 is in the singleton $\{0\}$.
- 1 is in the singleton $\{1\}$.
- $\pi \in A \cup \{0, 1\}$.

The envelope PCA of $P$, is the PCA $env(f)$ of alphabet $B$, defined on the set of cells $E$, with the same neighborhood $V$ as for $P$, and a local function $env(f) : B^E \to M(B)$, defined for each $y \in B^E$ by

$$\begin{cases} env(f)(y)(0) = \min_{x \in \pi(V)} f((x_{\pi(V)})\mid 0) \\ env(f)(y)(1) = \min_{x \in \pi(V)} f((x_{\pi(V)})\mid 1) \end{cases}$$

Construction of an update function for the envelope PCA: $\phi : B^E \times [0,1]^E \to B^E$.

$$\phi'(x') = \begin{cases} y & \text{if } 0 \leq r < f((((x')_{\pi(V)})\mid 0) \\ 1 & \text{otherwise} \end{cases}$$

Properties.

1. For any $x \in A^E$ and $y \in B^E$ such that $x \in y$:
   $$\forall y \in [0,1]^E, \phi(x,r) \neq \phi(y,r)$$.

2. If the algorithm stops almost surely, then the PCA $P$ is ergodic and the output of the algorithm is distributed according to the stationary measure of $P$.

3. The algorithm stops almost surely if and only if
   $$\min_{x \in B^E} f((x')_{\pi(V)}) > \alpha > 0$$.

Extensions

- Alphabet $A$ with more than two elements.
- Non-homogeneous finite PCA.

Infinite case

Assumptions: $E = Z, A = \{0, 1\}$, ergodic PCA $P$ with stationary distribution $\pi$.

A perfect sampling procedure is a random algorithm taking as input a finite subset $K$ of $E$ and returning a cylinder $x \in C(K)$ with probability $\pi(x)$.

Theorem. There exists a critical value $0 < \alpha \leq 1$, depending only on $[V]$, such that $env(P)$ is ergodic if

$$\max_{x \in B^E} f((x')_{\pi(V)}) > \alpha > 0$$

and non-ergodic if

$$\min_{x \in B^E} f((x')_{\pi(V)}) < \alpha > 0$$.

Alternative approach: restriction to finite windows & boundary conditions.

Open questions and further directions

- Coupling times PCA vs. envelope PCA?
- Open problem:
  For a PCA on $E = Z^d, d \geq 1$, does the uniqueness of the stationary measure imply ergodicity?

References