Cellular automata

Cellular automata (CA), introduced in the 50's by S. Ulam and J. von Neumann, are dynamical systems in which space and time are discrete.

- **E**: a set of cells (ext. $Z^d$ or $Z/nZ$).
- Each cell contains a letter from a finite alphabet $A$.
- The contents of all the cells evolve synchronously: the content of each cell evolving as a function of the contents of the cells in its neighborhood and according to a local rule.

Example 1. $E = Z$, $A = \{0, 1\}$, and $F: A^E \to A^E$ defined by $\langle F(x) \rangle_{k} = x_k + x_{k+1} \mod 2$.

Probabilistic cellular automata (PCA)

Motivations:

- Fault-tolerant computational models [8, 5].
- PCA appear in combinatorial problems related to the enumeration of directed animals [3, 1].
- In classification of deterministic CA (Wolfram’s program): robustness to random errors [4].

Assumption: $E = Z^d$ or $Z/nZ$.

Definitions and notations:

- $X = A^E$, equipped with the product topology (generated by cylinders).
- A cylinder $V$ is a subset of $X$ having the form $\{x \in X; x_k = y_k\}$ for a given finite subset $K$ of $E$ and a given sequence $(y_k)_{k \in K} \in A^K$. $C(K)$ is the set of all cylinders of base $K$.
- $M(A)$ and $M(A,X)$ are resp. the sets of measurable functions on $A$ and $X$.

Def. Given a finite neighborhood $V \subset E$, a (local) transition function of neighborhood $V$ is a function $f: A^V \to M(A)$.

Def. The probabilistic cellular automaton (PCA) $P$ of transition function $f$ is the application $M(X) \to M(X)$, $\mu \mapsto \mu^P$ defined on cylinders by:

$$
\mu^P(V) = \sum_{x \in \phi^{-1}(V)} \mu(D_{\phi}(x))
$$

where $\phi(V) = \{x \mid x_k \in V\}$.

Interpretation: $A \times \{\}^E$ is a Markov chain on the state space $A^E$. If $E$ is finite, the transition probabilities are given by:

$$
P(x,y) = \sum_{k \in E} \mu^P(D_{\phi}(x)) \delta_{\phi(x,k),y}.
$$

Example 2. $A = \{0, 1\}$, $V = \{0, 1\}$, and $f(x,y) = px + (1-p)x_k$, $p \in [0,1]$.

Ergodic PCA

Def. $\pi \in M(X)$ is a stationary measure of the PCA $P$ if $\pi^P = \pi$.

A PCA has at least one stationary measure.

Example 3. ACP from Example 2 has $\pi_0$ of $\delta_{0,1}$ as stationary measure.

Def. The PCA $P$ is ergodic if it has exactly one stationary measure $\pi$, and if for any measure $\mu \in M(A)$, the sequence $\mu^P(i)$ converges weakly to $\pi$ (i.e. $\mu^P(i)$ converges to $\pi$ for any cylinder $C$).

Ergodicity of a PCA is undecidable [8].

Sufficient conditions [8, Chap. 3]. There exists a constant $\eta_0$ depending only on the size $n$ of the neighborhood, such that

$$
\sum_{k \in E} \min_{x,y \in A^E} f((\pi_{\phi}(x))^k) \geq \eta_0 \quad \Rightarrow \quad P \text{ ergodic}.
$$

The value of $\eta_0$ is not known exactly, but satisfies $\eta_0 \approx 1 - 1/n$. How to sample the stationary measure of an ergodic ACP?

Perfect simulation of PCA

Assumptions: $E = Z^d$ or $Z/nZ$ and $A = \{0, 1\}$.

Let $P$ be an ergodic PCA and $\pi$ its stationary measure on $X = A^E$.

Perfect sampling: a random algorithm which returns a state $x \in X$ with probability $\pi(x)$.

Coupling from the past (Propp and Wilson, 1996):

$$
\phi: X \times [0, 1]^E \to X
$$

is an update function: Example:

$$
\phi(x, r) = \begin{cases} 
0 \leq y < \phi(x, a, r) \leq 1 \\
\phi(x, a, r) \quad \text{otherwise}
\end{cases}
$$

$$(\{y \leq \phi(x, a, r) \})_0 \subseteq \{x \mid x \in X\}, \{\phi(x, a, r) \}, x \in X \} \ldots
$$

Stop when the computed set is a singleton and return its value.

Prop. [7] If the procedure stops a.s., then it returns state $x$ with probability $\pi(x)$.

Example. Toy example of a PCA on the alphabet $\{0, 1\}$ and the set of cells $Z/2Z$. The state space is $X = \{x_0 = 0, x_2 = 0, x_3 = 0, x_4 = 1\}$. On this sample, the algorithm returns $x_0$.

Envelope PCA

New alphabet $\mathcal{E} = \{0, 1, *\}$ (unknown letters replaced by “?”). Can be seen as a subset of the power set of $A$.

- $\emptyset$ in the singleton $\{\}$.
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- $\emptyset \in A = \{0, 1\}$.

The envelope PCA of $P$, is the PCA envelope of $A$, defined on the set of cells $E$, with the same neighborhood $V$ as for $P$, and a local function $f: \mathcal{E}^V \to M(A)$ defined for each $y \in \mathcal{E}^V$ by

$$
\mathcal{E}^V \times \{\}^E
$$

Construction of an update function for the envelope PCA $\mathcal{E} = \{0, 1\} \to \mathcal{E} = \{0, 1, *\}

Envelope automata

• Perfect simulation of PCA
• Open questions and further directions

Open questions and further directions

- Coupling times PCA vs. envelope PCA?
- Open problem:
  - For a PCA on $E = Z^d$, $d \geq 1$, does the uniqueness of the stationary measure imply ergodicity?

References