Modèles et algorithmes des réseaux

Allocation de ressources et protocole TCP

Ana Busic
Inria Paris - DI ENS

http://www.di.ens.fr/~busic/

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Motivating example: Distributed control of data transport in the Internet

- How to assign bandwidth in networks
  - Understanding TCP, the protocol regulating most Internet traffic
  - Still an active research topic, in the context of datacenter networks (see « DC-TCP »)
Other application scenarios of current interest

- Allocation of \{storage, bandwidth, CPU\} resources in cloud computing

![A Google datacenter](image)

- Allocation of energy to consumers in the smart grid, under demand-response scenarios

![Smart Grid](image)
TCP in one slide

Source dynamics:
- Maintain Nb of (sent&not acked pkts)=cwnd (congestion window)
- Update cwnd
  \[ \text{cwnd} + 1/cwnd \text{ upon receipt of pkt ack} \]
  \[ \text{cwnd}/2 \text{ upon detection of pkt loss} \]
- "Congestion avoidance" alg introduced in 1993
- After Internet congestion collapse
Network model

- Resources, or links, $\ell \in \mathcal{L}$, each with capacity $C_\ell > 0$
- Users, or transmissions, or flows, $s \in S$
- User $s$ uses same rate at all $\ell \in s$ ($s \leftrightarrow$ subset of $\mathcal{L}$)
Allocations

- **max-min fairness**: feasible $x^{mm}$ such that $orall s \in S, \exists \ell \in s$ with $\sum_{t \in \ell} x_{t}^{mm} = C_\ell$ and $x_{s}^{mm} = \max_{t \in \ell} x_{t}^{mm}$ ("no envy": each $s$ can find competing $t$ at least as poor as $s$)

- **Proportional fairness**: feasible $x^{pf}$ such that for all feasible $y$, $\sum_{s} \frac{y_{s}-x_{s}^{pf}}{x_{s}^{pf}} \leq 0$

Alternative characterization:
Unique maximizer of $\sum_{s} \ln(x_{s})$ among feasible $x$

Notion introduced by F. Kelly (Cambridge University) in 1997
Allocations

Alternative characterization: Nash’s bargaining solution

i.e. unique vector \( \phi(C) \) in feasible convex set \( C \subset \mathbb{R}_+^S \)

s.t.

- Pareto efficiency: \( \phi(C) \leq x \in C \Rightarrow x = \phi(C) \)
- independence of irrelevant alternatives:
  \( \phi(C) \in C' \subset C \Rightarrow \phi(C) = \phi(C') \)
- symmetry: \( C \) symmetric \( \Rightarrow \phi(C)_i \equiv \phi(C)_1 \)
- scale invariance: for diagonal \( D \) with \( D_{ii} > 0 \),
  \( \phi(DC) = D\phi(C) \)
Allocations

Network Utility Maximization $x^*$: solution of

$$\begin{align*}
\text{Max} & \quad \sum_s U_s(x_s) \\
\text{Over} & \quad x_s \geq 0 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (P) \\
\text{Such that} & \quad \forall \ell, \sum_{s \in \ell} x_s \leq C_\ell
\end{align*}$$

for concave, increasing utility functions $U_s : \mathbb{R}_+ \rightarrow \mathbb{R}$

$\rightarrow$ A concave optimization program

**Examples**

Proportional fair $x^{pf}$: $U_s = \ln$

For $w, \alpha > 0$, $(w, \alpha)$-fair $x = x(w, \alpha)$: $U_s(x_s) = w_s \frac{x_s^{1-\alpha}}{1-\alpha}$
Allocations

Network Utility Maximization $x^*$: solution of

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$$\text{Over} \quad x_s \geq 0 \quad (P)$$
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**Examples**
Proportional fair $x^{pf}$: $U_s = \ln$
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[Exercise: $\lim_{\alpha \rightarrow 1} x(1, \alpha) = x^{pf}$ and $\lim_{\alpha \rightarrow +\infty} x(1, \alpha) = x^{mm}$]
Relaxed problem and primal algorithm

Relaxed problem: \[
\text{Max } \sum_s U_s(x_s) - \sum_\ell C_\ell(y_\ell) \\
\text{Over } x_s \geq 0 \quad \text{(RP)}
\]
with \[y_\ell = \sum_{s \in \ell} x_s\]

for concave increasing utility functions \(U_s\) and convex increasing cost functions \(C_\ell\).

**Primal algorithm:** for \(U_s\) and \(C_\ell\) differentiable, and positive gain function \(\kappa_s: \mathbb{R}_+ \to \mathbb{R}_+\), let

\[
\frac{d}{dt} x_s = \kappa_s(x_s) \left( U'_s(x_s) - \sum_{\ell \in s} C'_\ell(y_\ell) \right) \quad \text{“gradient ascent”}
\]
Stability via Lyapunov functions

Criterion for convergence of ODE $\dot{x} = F(x)$ with trajectories in $O \subset \mathbb{R}^n$

**Theorem**

Assume $F$ continuous on $O$, and $\exists V : O \to \mathbb{R}$ such that:

(i) $V$ continuously differentiable

(ii) $\forall a \leq A$, $\{x \in O : V(x) \leq A\}$ and $\{x \in O : V(x) \in [a, A]\}$

either compact or empty

(iii) $\forall x \in O \setminus B$, $\nabla V(x) \cdot F(x) < 0$, where $B = \text{argmin}_{x \in O} \{V(x)\}$

Then $\lim_{t \to \infty} V(x(t)) = \inf_{x \in O} V(x)$, $\lim_{t \to \infty} d(x(t), B) = 0$.

If $B = \{x^*\}$ then $\lim_{t \to \infty} x(t) = x^*$. 
Application to gradient ascent/descent dynamics

\[
\frac{d}{dt} x_s = \kappa_s(x_s) \left( U'_s(x_s) - \sum_{\ell \in s} C'_\ell(y_\ell) \right)
\]

Let \( W(x) = \sum_s U_s(x_s) - \sum_\ell C_\ell(y_\ell) \) (system welfare)
and \( V(x) = -W(x) \)

Then: \( \nabla V(x) \cdot F(x) = -\sum_s \kappa_s(x_s) \left[ \frac{\partial}{\partial x_s} W(x) \right]^2 \)

**Theorem**

For \( U_s \) strictly concave, differentiable with \( U'_s(0^+) = +\infty \),
\( C_\ell \) convex, continuously differentiable,
[ \( \Rightarrow \) strict concavity and continuous differentiability of \( W \)]
\( \kappa_s > 0 \), continuous [ \( \Rightarrow \) continuity of \( F \)]
\( \exists x_s > 0 \) s.t. \( U'_s(x_s) < \sum_{\ell \in s} C'_\ell(x_s) \)
[ \( \Rightarrow \) Max of \( W \) achieved at single point \( x^* \in O := (0, \infty)^S \)]

Then “primal” dynamics converge to unique maximizer \( x^* \) of \( W \)
TCP allocation

Approx. \( x_s \approx \frac{cwnd_s}{T_s} \) where \( T_s \): packet round-trip time

Approx. \( \frac{d}{dt} cwnd_s \approx x_s \left( \frac{1}{cwnd_s} \right) - x_s p(s) \left[ \frac{cwnd_s}{2} \right] \)
where \( p(s) \): packet loss probability along path of \( s \)

Approx. \( p(s) \approx \sum_{\ell \in s} p_\ell(y_\ell) \) for link packet loss prob. \( p_\ell(y) \)
[e.g. \( p_\ell(y) = \max(0, 1 - C_\ell/y) \)]

\[ \Rightarrow \dot{x}_s = \left( \frac{x_s^2}{2} \right) \left[ \frac{2}{(x_s T_s)^2} - \sum_{\ell \in s} p_\ell(y_\ell) \right] \]

TCP implicitly runs primal alg. with utility function:
\( U_s(x) = w_s x^{1-\alpha} / (1 - \alpha) \) with \( \alpha = 2, w_s = 2 / T_s^2 \)
→ Leads to \((w, \alpha)\)-fairness with suitable parameters
Convex optimization: Lagrangian, duality, multipliers

Generic convex optimization program
For convex set $C^0$, convex functions $J, f_\ell : C^0 \to \mathbb{R}$,

\[
\begin{align*}
\text{Min} & \quad J(x) \\
\text{Over} & \quad x \in C^0 \quad \quad (P) \\
\text{Such that} & \quad \forall \ell \in \mathcal{L}, f_\ell(x) \leq 0
\end{align*}
\]

Associated Lagrangian
\[
L(x, \lambda) := J(x) + \sum_\ell \lambda_\ell f_\ell(x), \\
\text{where } x \in C^0, \lambda \geq 0
\]

$\lambda$: Lagrange multipliers of $(P)$'s constraints

Dual problem (D): Max $D(\lambda)$ Over $\lambda \geq 0$
where $D(\lambda) := \inf_{x \in C^0} L(x, \lambda)$
Kuhn-Tucker theorem and strong duality

**Def:** \( \lambda^* \geq 0 \) a Kuhn-Tucker vector iff \( \forall x \in C^0, L(x, \lambda^*) \geq J^* \)

where \( J^* \): optimal value of \((P)\).

**Remark:** \( J^* \geq D^* \) where \( D^* \) optimal value of \((D)\)

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**Theorem**

Assume there exists \( \lambda^* \) a Kuhn-Tucker vector. Then

(i) \( \lambda^* \) solves \((D)\), and \( J^* = D^* \) (a.k.a. **strong duality**)

(ii) \( x^* \in C^0 \) if optimal for \((P)\) then achieves \( \min_{x \in C^0} L(x, \lambda^*) \)

(iii) For \( x^* \in \text{int}(C^0) \) an optimum of \((P)\) at which \( \exists \nabla J, \nabla f_\ell \), then

\[
\forall \ell, \lambda^*_\ell f_\ell(x^*) = 0 \quad \text{(complementarity)}
\]

\[
\nabla J(x^*) + \sum_\ell \lambda^*_\ell f_\ell(x^*) = 0 \quad \text{(stationarity)}
\]

Reciprocally assume stationarity + complementarity for some \( \lambda^* \geq 0 \) and some \( x^* \) feasible for \((P)\),
Then \( \lambda^* \): Kuhn-Tucker and \( x^* \) optimal for \((P)\)
Sufficient conditions for KT

**Lemma**

Assume $J^* > -\infty$ and $\exists \hat{x} \in C^0$ such that $\forall \ell, f_\ell(\hat{x}) < 0$. Then a Kuhn-Tucker vector $\lambda^*$ exists.

**In practice:** verify Lemma’s conditions + existence of optimum $x^* \in \text{int}(C^0)$ at which $\exists \nabla J, \nabla f_\ell$. Then characterize $x^*$ that verifies complementarity + stationarity (now guaranteed to exist)
Solving original problem: dual algorithm

Lagrangian: $L(x, \lambda) = \sum_s U_s(x_s) + \sum_{\ell} \lambda_{\ell}[C_{\ell} - \sum_{s \supset \ell} x_s]$

Dual: $D(\lambda) = \sum_s U_s(g_s(\lambda^s)) + \sum_{\ell} \lambda_{\ell}[C_{\ell} - \sum_{s \supset \ell} g_s(\lambda^s)]$

where $\lambda^s := \sum_{\ell \in s} \lambda_{\ell}$ and $g_s := (U'_s)^{-1}$

$\Rightarrow \frac{\partial}{\partial \lambda_{\ell}} D(\lambda) = C_{\ell} - \sum_{s \supset \ell} g_s(\lambda^s)$

Dual algorithm:

$x_s \equiv g_s(\lambda^s)$,
$\dot{\lambda}_{\ell} = \kappa_{\ell} \left[ \sum_{s \supset \ell} x_s - C_{\ell} \right]_{\lambda_{\ell}}^+$

where $[a]_b^+ = a$ if $b > 0$, $\max(a, 0)$ if $b \leq 0$
Solving original problem: dual algorithm

**Theorem**

*Under suitable conditions*

\( U_s \) strictly concave, twice differentiable, \( U'_s(0^+) = +\infty \),
\( U'_s(\infty) = 0 \)

*Trajectories \( x_s \) of dual algorithm converge to unique maximizer \( x^* \) of primal problem.*

[Proof: involved, in particular to show existence and uniqueness of ODE’s solution. "Quasiproof" of convergence: Lyapunov function argument]

Potential implementation: multiplier dynamics \( \equiv \) queue dynamics

⇒ Let \( \lambda^e = \) queueing delay of packets and instantaneously let \( x_s \) to \( g_s(\lambda^s) \)

⇒ Principle underlying TCP-Vegas, an alternative to default TCP (TCP Reno)
Takeaway messages

- For unconstrained convex minimization, gradient descent converges to optimizer [Lyapunov stability]
- Admits distributed implementation in network optimization setting
- TCP implicitly achieves \((w, \alpha)\)-fair allocation by running gradient descent
- Kuhn-Tucker Theorem: Complementarity + Stationarity characterization of (P)'s optima
- Queue dynamics implicitly perform gradient descent for multipliers of constrained program