

Two is easy, three is hard: extending convexity to higher dimensions and multi-server queues

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A well-known result

- Single queue, Poisson arrivals, a fast and a slow server
- Control: when to assign customer to slow server
- Result: threshold optimality (Lin & Kumar '84)
- Method: dp (K '95)



Another well-known result

- Two parallel (heterogeneous) queues, Poisson arrivals
- Control: where to assign an arriving customer
- Result: switching curve
- Method: dp (Hajek '84)



Yet another well-known result

- Two queues in tandem, Poisson arrivals
- Control: server speeds
- Result: monotone server speeds
- Method: dp (Weber & Stidham '87)



Method

- Formulate dp value function
- Write down equations needed for anticipated result
- try to propagate value functions
- add equations as needed



Example (Hajek model)

- Value function:

$$V_{n+1}(x) = |x| + \lambda \min_{i=\{1,2\}} \{V_n(x + e_i)\} + \sum_{i=1}^2 \mu_i V_n((x - e_i)^+)$$

- monotonicity: if route to Q2 in x , then also in $x + e_1$
- route to Q2 in $x \Rightarrow V_n(x + e_2) - V_n(x + e_1) \leq 0$
- monotonicity if $V_n(x + e_1 + e_2) - V_n(x + 2e_1) \leq V_n(x + e_2) - V_n(x + e_1)$
- thus propagate $V_n(x + e_1 + e_2) + V_n(x + e_1) \leq V_n(x + 2e_1) + V_n(x + e_2)$



Example (Lin-Kumar model)

- Value function: $V_{n+1}(x) = |x| + \lambda V'_n(x + e_1) + \sum_{i=1}^2 \mu_i V'_n((x - e_i)^+)$
 $V'_n(x) = \min\{V_n(x - e_1 + e_2), V_n(x)\}$ if $x_1 > 0$, $V_n(x)$ otherwise
- monotonicity: if route to server 2 in x , then also in $x + e_1$
- route to server 2 in $x \Rightarrow V_n(x + e_2) - V_n(x + e_1) \leq 0$
- monotonicity if $V_n(x + e_1 + e_2) - V_n(x + 2e_1) \leq V_n(x + e_2) - V_n(x + e_1)$
- thus propagate $V_n(x + e_1 + e_2) + V_n(x + e_1) \leq V_n(x + 2e_1) + V_n(x + e_2)$



Same equations

- "Conclusion": Lin-Kumar model \equiv Hajek model \equiv Weber-Stidham model
- Central role to set of equations
- For each set of equations a set of "operators" that propagate (Operators: things that happen in system such as arrivals, departures, environment changes,...)
- Dp equation = concatenation of operators



Example (Hajek model)

Value function:

$$V_{n+1}(x) = T_{\text{costs}}(T_{\text{cc}}(T_R, T_{D1}, T_{D2}))V_n(x)$$

with

$$T_{\text{costs}}f(x) = C(x) + f(x),$$

$$T_{\text{cc}}(f_1, \dots, f_m)(x) = \sum_i p_i f_i(x),$$

$$T_R f(x) = \min_i f(x + e_i),$$

$$T_{Di} f(x) = f((x - e_i)^+)$$



My contribution

- Identified many interesting operators
- Identified relevant (in)equalities
- Matched them
- Wrote an overview about it
(Foundations and Trends on Stochastic Systems 1:1–73, 2006)



Operators

- (controlled) environment
- arrivals, admission control, routing
- single server, multiple servers, assignable server
- tandem server



Classes of inequalities

- First-order (e.g., $f(x + e_1) \leq f(x + e_2)$)
- Schur convexity ($x \prec y$ if x more balanced than y)
- Convexity (componentwise convex, sub/supermodular, sub/superconvex, multimodular)



Typical results

- First-order: optimality of μc rule (single server) and LEPT (multiple servers)
- Schur convexity: optimality of join the shortest queue
- Convexity in 1 dimension: monotonicity and optimality of threshold policy for concave service rates
- Convexity in multiple dimensions: monotonicity of control tandem model (W & S), convexity of value function of multi-server tandem system
- Convexity in two dimensions: results of Lin & Kumar, Hajek



First-order example: server assignment

- Two parallel queues, 1 server
- service rates μ_1, μ_2 , holding costs c_1, c_2
- Inequalities ($\bar{\mu}_i = \mu - \mu_i$ with $\mu = \max_i \{\mu_i\}$):

$$\begin{aligned}\mu_1 f(x - e_1) + \bar{\mu}_1 f(x) &\leq \mu_2 f(x - e_2) + \bar{\mu}_2 f(x) \\ f(x - e_1) &\leq f(x), \quad f(x - e_2) \leq f(x)\end{aligned}$$



Allowable cost functions

- Fill in $C(x) = c_1x_1 + c_2x_2$
- Conclusion: $\mu_1c_1 \geq \mu_2c_2, c_1, c_2 \geq 0 \implies \mu c$ rule



Environment

- New feature: environment
 - Add dimension 0, and operator T_{env}
 - $T_{env}(f_1, \dots, f_l)(x) = \sum_y \lambda(x_0, y) \sum_j q^j(x_0, y) f_j(x^*)$, $x_0^* = y$, $x_i^* = x_i$, $i > 0$
 - q^j is used to model arrivals and server speed
 - Arrival process is closed in set all arrival processes (Asmussen & K 1993)
- Extension: controlled environment
 - $T_{Cenv}(f_1, \dots, f_l)(x) = \min_a \{ \sum_y \lambda(x_0, a, y) \sum_j q^j(x_0, a, y) f_j(x^*) \}$



Results

- Environment without control: μc rule
- Environment with control: μc rule and $\mu_1 \leq \mu_2$
- Counterexamples to $\mu_1 > \mu_2$
- Application: 2 " μc " nodes in tandem



Further use

- Comparison of systems (on-off vs. Poisson)
- Monotonicity in parameters (convexity in arrival rate)



Example: Comparison of arrival processes

- $T_A f(x) = f(x + e_1)$, $T'_A f(x) = 0.5f(x) + 0.5f(x + 2e_1)$
- $V_{n+1}(x) = T_{\text{costs}}(T_{\text{cc}}(T_A, T_{D1})V_n(x)$,
 $V'_{n+1}(x) = T_{\text{costs}}(T_{\text{cc}}(T'_A, T_{D1})V'_n(x)$
- All our operators: $f \leq f' \implies Tf \leq Tf'$



Example (continued)

- Result: $f \leq f', f \in Cx \implies T_A f \leq T'_A f'$
- Proof: $T_A f(x) = f(x + e_1) \leq 0.5f(x) + 0.5f(x + 2e_1) = T'_A f(x) \leq T'_A f'(x)$
- Conclusion: $V_n \leq V'_n$ if $V_0 \leq V'_0$
- Thus: costs (such as average queue length) are higher with batch arrivals



"Big" open problems

- Hysteresis
 - dim 1: queue length, dim 2: server speed
 - server costs, holding costs, switching costs
 - optimal switching rule = hysteresis?
 - Lu & Serfozo '84 (again '84!!): hysteresis optimal
 - Hipp & Holzbauer '88: counterex. to a condition in L&S
 - Kitaev & Serfozo '99: "repairs" error without going into detail
 - My opinion: "clean" proof needed
- > 2 servers in Lin-Kumar model \equiv Hajek model



Switching curve with > 2 servers

- Hajek model (2 dim) with multi-server queues
- Switching curve: $f(x + e_1 + e_2) + f(x + e_1) \leq f(x + 2e_1) + f(x + e_2)$
= Superconvexity required
- In recursion needed:
 - Componentwise convexity (apply T_{D2} to SuperC for $x_2 = 0$)
 - Supermodularity = $f(x + e_1) + f(x + e_2) \leq f(x) + f(x + e_1 + e_2)$
(apply T_R to CC)



Switching curve with > 2 servers (continued)

- Towards a contraction: apply multi-server operator to SuperC ($x_2 = 0$)
- $f(x + e_1 + e_2) + f(x + e_1) \leq f(x + 2e_1) + f(x + e_2) \Rightarrow$
- $f(x + e_1 + e_2) + f(x + e_1) + f(x + e_2) + f(x) \leq 2f(x + e_1) + 2f(x + e_2)$
- $\Leftrightarrow f(x + e_1 + e_2) + f(x) \leq f(x + e_1) + f(x + e_2) = \text{Submodularity}$
- Contradiction with supermodularity



Switching curve with > 2 servers (continued)

- failure method confirmed in literature (de Véricourt & Zhou '06)
- Other approaches
 - Weaker inequalities? Recursion for CC and SuperM and certain operators including multi server, no switching curve
 - > 2 single-server queues: no results
 - Other type of methodology?

