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Residuation of tropical series: rationality issues

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***Abstract*— Decidability of existence, rationality of delay controllers and robust delay controllers are investigated for systems with time weights in the tropical and interval semirings. Depending on the (max,+) or (min,+)-rationality of the series specifying the controlled system and the control objective, cases are identified where the controller series defined by residuation is rational, and when it is positive (i.e., when delay control is feasible). When the control objective is specified by a tolerance, i.e. by two bounding rational series, a nice case is identified in which the controller series is of the same rational type as the system specification series.**

I. INTRODUCTION

Timed discrete-event systems are discrete-event systems whose behavior depends on timing constraints and not only on logical constraints such as the ordering of events. Such systems are often modeled by weighted automata [6], also called automata with multiplicities, where weights (multiplicities) may range over an arbitrary semiring. E.g., the (max,+)-automata proposed in [7] are weighted in $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$ (the tropical semiring), while the (min,+)-automata are weighted in $\mathbb{R}_{\min} = (\mathbb{R} \cup \{\infty\}, \min, +)$. The latter are also called price automata, because the multiplicity of a transition often represents a cost. These automata and their behaviors (series) have also been studied in [9]. D. Krob has shown in [13] that equality of two (max,+) or (min,+)-rational series is undecidable in general. However, some of the inequalities and equalities for sequential (deterministic) series are decidable [14]. Recent results in [11] show that sequentiality is decidable for polynomially ambiguous (min,+) automata.

In order to increase the expressive power, one may consider automata with weights taken from semirings of intervals, therefore called interval automata. Such automata were first introduced in [5] where they were defined as Büchi automata over an alphabet of pairs made of an event and a real time interval.

Our scope is not limited to deterministic (max,+) or interval automata, because their expressive power is too limited. However, nondeterministic weighted automata suffer from several drawbacks: there is no (finite state) determinization procedure, no general (state) minimization algorithm, and their behaviors (rational formal power series) can in general not be checked for equality by effective procedures.

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In this paper, we aim at extending to abstract systems and specifications both given by pairs of lower and upper bound series the supervisory control approach proposed in [12] for (max,+) automata. In [12], the behavior of the closed-loop system is represented by the Hadamard product of the system and controller series, and the controller series is formally computed using residuation theory [4]. Namely, when the controller can delay the controllable transitions but it cannot prevent the firing of transitions, the residuation S_1/S_2 of the (specification) series S_1 by the (system) series S_2 amounts to the Hadamard product of S_1 with the series $-S_2$ with all coefficients multiplied by -1. Residuated series may have both positive and negative coefficients, hence they do not always define feasible delay controllers.

A major problem with the above recalled approach using Hadamard inversion is that the residuated series needs not be rational. Changing all coefficients to their opposite sends a (max,+)-rational series to a (min,+)-rational series and vice versa, but the multiplication of coefficients by -1 is neither a (max,+)-rational nor a (min,+)-rational operation. It was indeed shown by Lombardy and Mairesse [16] that the opposite of a (max,+)-rational series is (max,+)-rational iff it is *unambiguous*, *i.e.* there is at most one successful path in the (max,+) automaton labeled by w for every word w. It seems reasonable to assume that specification series are unambiguous, but it would be very restrictive to require also non-ambiguity from the system series. In this paper we show that if the specification series is (min,+)-rational and the system series is (max,+)-rational, then the controller series defined by residuation is (min,+)-rational (and similarly for the opposite polarities), hence in particular one can decide whether this series is non-negative.

We shall try to extend residuation further to interval valued formal power series and to intervals of formal power series. Interval valued series can serve to model behaviors of systems whose transitions have uncertain costs or durations. Intervals of series may serve to the same effect, but with some added flexibility since the two bounding series of an interval are structurally independent. For series of intervals, we show that the controller series has generally not the same type or polarity as the system series, unless assuming that both the system and the controller series are sequential, an assumption which is even stronger than unambiguity. The situation turns out to be more favorable with intervals of series, *i.e.* when the expected behavior of the closed-loop system is specified by a tolerance made of a lower bound series and an upper bound series, and the behavior of the uncontrolled system is described similarly. We are interested in *robust* control, *i.e.* in finding bounds on controller (cost

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or) delay series such that the specified tolerance is met by the closed loop system for all possible behaviors of the uncontrolled system within its defining bounds. We identify a situation in which the controller series interval is guaranteed to be rational and of the same type as the specification series interval.

Deciding about non-emptiness of the residuated series interval is crucial for applications. Fortunately, this can be done since the inequality $S \leq S'$ can be decided for S (max,+)-rational and S' (min,+)-rational (unlike the opposite inequality).

II. (MAX,+) AND (MIN,+) ALGEBRAS

In this section, we recall elements of the theory of idempotent semirings, also called dioids (see [1]), a basic structure used throughout the paper.

A. Definition

A dioid is a set $\mathcal{D}$ equipped with two internal operations, denoted by $\oplus$ and $\otimes$, such that the addition $\oplus$ is commutative, associative, idempotent, and has a zero element ϵ, while the multiplication $\otimes$ is associative, has a unit element e, has the absorbing element ϵ, and distributes over $\oplus$. The addition $\oplus$ induces a natural order $\preceq$, namely $a \preceq b \Leftrightarrow a \oplus b = b$. Dioid operations may be extended to dioids of matrices as follows. Let $A, B \in \mathcal{D}^{m,n}$ and $C \in \mathcal{D}^{n,\ell}$. Then:

- $\forall i, j \in \{1, \dots, n\}$, $(A \oplus B)_{i,j} = A_{i,j} \oplus B_{i,j}$;
- $\forall i, k \in \{1, \dots, \ell\}$, $(A \otimes C)_{i,k} = \bigoplus_{j=1}^{n} A_{i,j} \otimes C_{j,k}$.

In the sequel, we use the extended notations $\mathcal{D}^{m,Q}$, $\mathcal{D}^{P,n}$ and $\mathcal{D}^{P,Q}$ for finite sets P and Q. Let $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ and $\mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\}$. The basic dioids used in this paper are $(\mathbb{R}_{\max}, \max, +)$ and $(\mathbb{R}_{\min}, \min, +)$. The unit element is 0 for both dioids. The zero element is $-\infty$ for $\mathbb{R}_{\max}$ and $+\infty$ for $\mathbb{R}_{\min}$. The order induced by $\mathbb{R}_{\max}$ is the usual order, whereas the order induced by $\mathbb{R}_{\min}$ is the reverse of the usual order. The completions of $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$ w.r.t. the induced order relations are noted $\overline{\mathbb{R}_{\max}}$ and $\overline{\mathbb{R}_{\min}}$, respectively. Thus, $\overline{\mathbb{R}_{\max}}$ and $\overline{\mathbb{R}_{\min}}$ have the same carrier set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, but the supremum of $\overline{\mathbb{R}_{\max}}$ is $+\infty$ whereas the supremum of $\overline{\mathbb{R}_{\min}}$ is $-\infty$. In the sequel, $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$ are called the (max,+)-dioid and the (min,+)-dioid, respectively. $\overline{\mathbb{R}_{\max}}$ is also a dioid, equipped with the product operation $\otimes_{max}$ specified in Table I (see Section III-A), and similarly for $\overline{\mathbb{R}_{\min}}$.

In order to represent intervals, we use products of semirings. The two product semirings that we consider are the following.

Definition 2.1 (adapted from [15]): Let $\mathcal{I}_{max}^{max}$ denote the idempotent semiring with carrier set $\{[\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \overline{\mathbb{R}_{\max}}\} \wedge \underline{x} \leq \overline{x}\}$ defined with:

$$\begin{aligned} [\underline{x_1}, \overline{x_1}] \oplus [\underline{x_2}, \overline{x_2}] &= [\max(\underline{x_1}, \underline{x_2}), \max(\overline{x_1}, \overline{x_2})], \\ [\underline{x_1}, \overline{x_1}] \otimes [\underline{x_2}, \overline{x_2}] &= [\underline{x_1} + \underline{x_2}, \overline{x_1} + \overline{x_2}], \end{aligned}$$

$\varepsilon = [-\infty, -\infty]$ (zero interval) and $e = [0, 0]$ (unit interval).

This dioid is usually employed for time event graphs when the timings of each transition lays in a interval. The $\oplus$ operator is a synchronization operation.

Definition 2.2: Let $\mathcal{I}_{max}^{min}$ denote the idempotent semiring with carrier set $\{[\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \overline{\mathbb{R}}\}$ defined with:

$$\begin{aligned} [\underline{x_1}, \overline{x_1}] \oplus [\underline{x_2}, \overline{x_2}] &= [\max(\underline{x_1}, \underline{x_2}), \min(\overline{x_1}, \overline{x_2})], \\ [\underline{x_1}, \overline{x_1}] \otimes [\underline{x_2}, \overline{x_2}] &= [\underline{x_1} + \underline{x_2}, \overline{x_1} + \overline{x_2}], \end{aligned}$$

zero and unit are resp. $\varepsilon = [-\infty, +\infty]$ and $e = [0, 0]$.

The $\oplus$ operator can be seen as an intersection operation.

In Definition 2.2, as opposed to Definition 2.1, we do not exclude imaginary intervals $[\underline{x}, \overline{x}]$ where the lower bound $\underline{x}$ is greater than the upper bound $\overline{x}$. For instance, $[1, 3] \oplus [2, 4] = [2, 3]$ is a well formed interval but $[1, 3] \oplus [4, 5] = [4, 3]$ is an imaginary interval. Indeed, well formed intervals are naturally preserved by addition in $\mathcal{I}_{max}^{max}$ but they are not preserved by addition in $\mathcal{I}_{max}^{min}$. Algebraically, $\mathcal{I}_{max}^{min}$ is just the direct product of the (idempotent) semirings $\overline{\mathbb{R}_{\max}}$ and $\overline{\mathbb{R}_{\min}}$ but $\mathcal{I}_{max}^{max}$ is not the direct product of $\overline{\mathbb{R}_{\max}}$ with itself.

Now we recall the notions of $\mathcal{D}$-series and $\mathcal{D}$-automaton over an arbitrary dioid $\mathcal{D}$ and the equivalence between the properties of recognizability by $\mathcal{D}$-automata and rationality for $\mathcal{D}$-series. We also recall the construction of $\mathcal{D}(\Sigma)$, the dioid of all $\mathcal{D}$-series over an alphabet Σ.

Given a dioid $\mathcal{D}$ and a finite alphabet Σ, a $\mathcal{D}$-series over Σ is a function $S : \Sigma^* \to \mathcal{D}$ where Σ^* is the set of all finite words on Σ. We denote by $\mathcal{D}(\Sigma)$ the set of all $\mathcal{D}$-series over Σ. The support of the series S is the set $supp(S)$ of all words w such that $S(w) \neq \epsilon$. By convention, we write series as formal sums $S = \bigoplus_{w \in \Sigma^*} S(w)w$ or $S = \bigoplus_{w \in supp(S)} S(w)w$. Let Q be a finite set of states. A finite $\mathcal{D}$-automaton over Σ and Q is a triple $\mathcal{A} = (\alpha, \mu, \beta)$ where $\alpha \in \mathcal{D}^{1,Q}$, $\beta \in \mathcal{D}^{Q,1}$ and μ is a morphism of monoids from Σ^* to $\mathcal{D}^{Q,Q}$. The series recognized by $\mathcal{A}$ is defined as $\bigoplus_{w \in \Sigma^*} (\alpha \otimes \mu(w) \otimes \beta)w$. A famous Schützenberger's theorem [18] states that the series which are recognized by finite $\mathcal{D}$-automata coincide with the rational $\mathcal{D}$-series, *i.e.* $\mathcal{D}$-series generated from finite $\mathcal{D}$-series using the rational operations of sum, Cauchy product and iterated Cauchy product. Recall that the Cauchy product of two series $S, T \in \mathcal{D}(\Sigma)$ is defined as $S \otimes T = \bigoplus_{w \in \Sigma^*} (\oplus_{uv=w} S(u) \otimes T(v))w$. We denote by $\mathcal{D}Rat(\Sigma)$ the set of the rational $\mathcal{D}$-series over Σ. A rational $\mathcal{D}$-series S is *unambiguous* if it is recognized by a finite $\mathcal{D}$-automaton (α, μ, β) with set of states Q such that, for any word $w = \sigma_1 \dots \sigma_n \in supp(S)$, there exists a unique sequence of states $q_0, q_1, \dots, q_n q_{n+1}$ such that $\alpha(q_0)$, $\mu(\sigma_i)(q_i, q_{i+1})$ and $\beta(q_n)$ differ from ϵ for all $0 \leq i \leq n-1$. A rational $\mathcal{D}$-series is *sequential* if it is recognized by a $\mathcal{D}$-automaton (α, μ, β) such that the underlying automaton on Σ^* has a single initial state q_0 ($\alpha(q) = \epsilon$ for all $q \neq q_0$) and it has a deterministic transition relation (for all σ and q, $\mu(\sigma)(q, q') \neq \epsilon$ for at most one state q'). The following result due to Lombardy and Mairesse shows the interest of unambiguous series in the context of tropical semirings.

Theorem 2.3 ([16]): A rational (max,+) series is a rational (min,+) series if and only if it is unambiguous.

The sequential $\mathcal{D}$-series, *i.e.* the series recognized by $\mathcal{D}$-automata with underlying deterministic automata, are of course unambiguous. The set $\mathcal{D}(\Sigma)$ of all $\mathcal{D}$-series over Σ

may be endowed with two operations so as to form a dioid. One way to obtain this is to use point-wise addition and the Cauchy product. The other way is to use point-wise addition and the Hadamard product. Therefore, for $S, T \in \mathcal{D}(\Sigma)$, we let:

- $S \oplus T = \bigoplus_{w \in \Sigma^*} (S(w) \oplus T(w))w$;
- $S \odot T = \bigoplus_{w \in \Sigma^*} (S(w) \otimes T(w))w$ (Hadamard product).

Since the above operations preserve the rationality of series, both $(\mathcal{D}(\Sigma), \oplus, \odot)$ and $(\mathcal{D}Rat(\Sigma), \oplus, \odot)$ are dioids. Note that $(\mathcal{D}(\Sigma), \oplus, \odot)$ is complete if $\mathcal{D}$ is complete, but this is not the case for $(\mathcal{D}Rat(\Sigma), \oplus, \odot)$.

III. Residuation of (max,+) and (min,+) series and rational series

In this section, we recall the definition of residuation in dioids and in particular in dioids of $\mathcal{D}$-series over Σ. Then, we focus on the residuation of (max,+) series. After reviewing the results obtained in [12], we examine to what extent they can be applied in the context of supervisory control and underline some drawbacks. We then turn to consider *hybrid* residuation operations of (max,+) series by (min,+) series and conversely. We observe that such operations preserve rationality, and that an easy decision of the control problem ensues when the specification series and the system series have different polarities. For the sake of simplicity, we assume that all transitions of the system are uncontrollable in the sense of this term defined by Ramadge and Wonham [17], *i.e.* that the transitions of the plant may be delayed but cannot be disabled otherwise.

A. Residuation of (max,+) series

In any dioid $\mathcal{D}$, the (right) residue of an element b by an element a, denoted b/a, is the greatest solution of the inequality $a \otimes x \preceq b$ (where $\preceq$ is the order relation induced by the addition operation), if such a greatest solution exists. The existence of residues is guaranteed for all b and a in any *complete* dioid $\mathcal{D}$, *i.e.* a dioid in which arbitrary subsets have least upper bounds. Table I shows the residuation map b/a for the complete (max,+) dioid $\overline{\mathbb{R}_{\max}}$. Note that $\preceq$ coincides in $\overline{\mathbb{R}_{\max}}$ with the usual order relation $\leq$ whereas it coincides in $\overline{\mathbb{R}_{\min}}$ with the reverse order relation $\geq$.

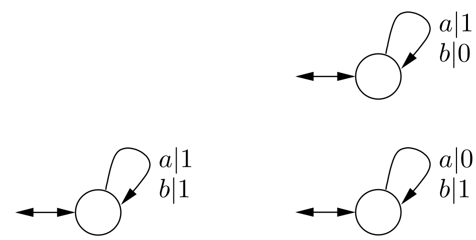
$\otimes_{\max}$	$-\infty$	a	$+\infty$
$-\infty$	$-\infty$	$-\infty$	$-\infty$
b	$-\infty$	$a+b$	$+\infty$
$+\infty$	$-\infty$	$+\infty$	$+\infty$

$/_{\max}$	$-\infty$	a	$+\infty$
$-\infty$	$+\infty$	$-\infty$	$-\infty$
b	$+\infty$	$b-a$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	$+\infty$

TABLE I

(MAX,+) PRODUCT AND THE CORRESPONDING RESIDUATION.

In a complete dioid $\mathcal{D}$, the operation of residuation in $\mathcal{D}$ extends pointwise to $(\mathcal{D}(\Sigma), \oplus, \odot)$, the dioid of $\mathcal{D}$-series with the Hadamard product. Namely, for any series S_1, S_2 and for any word $w \in \Sigma^*$, $(S_1 \circ\!\!\!/ S_2)(w) = S_1(w)/S_2(w)$. Based on this fact, it was proposed in [12] to use (max,+) residuation for computing delay controllers. Given a specification series S_1 and a system series S_2, the residuated (max,+) series gives for each word $w \in \Sigma^*$ in $supp(S_2)$ the maximum delay $(S_1 \circ\!\!\!/ S_2)(w)$ that can be added to the worst-case duration $S_2(w)$ of the sequence of actions w in the plant without exceeding the specified upper bound $S_1(w)$. This proposal has the outcome that the behavior of the closed loop system may be defined as the product of two (max,+) series, namely $S_2 \otimes (S_1 \circ\!\!\!/ S_2)$, but there are some drawbacks. First, it is not clear that one can decide whether the controller series $S_1 \circ\!\!\!/ S_2$ is non-negative. Second, it is not always possible to represent $S_1 \circ\!\!\!/ S_2$ with a (max,+)-automaton, as the following example shows, hence the controller obtained by residuation may have no finite representation.



(a) (b)

Fig. 1. Two automata that recognize (a) the length of a word and (b) the maximum number of occcurences of a letter in a word.

Example 1: Take $\Sigma = \{a, b\}$. Consider the two automata shown on the left resp. right in Figure 1. The automaton (a) recognizes the (max,+) series $S_1 = \bigoplus_{w \in \Sigma^*} |w| w$. The automaton (b) recognizes the series $S_2 = \bigoplus_{w \in \Sigma^*} \max(|w|_a, |w|_b) w$. Clearly, for any word w, $S_1(w)/_{\max} S_2(w) = |w| - \max(|w|_a, |w|_b) = \min(|w|_a, |w_b|)$. Therefore, $S_1 \circ\!\!\!/_{\max} S_2 = \bigoplus_{w \in \Sigma^*} \min(|w|_a, |w_b|) w$. This series can be recognized by the automaton (b) seen as a (min,+) automaton. Now, it has been shown in [10] that the (min,+) series recognized by the automaton (b) is ambiguous. Therefore $S_1 \circ\!\!\!/ S_2$ is an ambiguous (min,+) rational series and in view of Theorem 2.3, this series cannot be a (max,+) rational series.

B. Residuation of a (max,+) series by a (min,+) series

Example 1 has shown a case where residuating rational (max,+) series does not produce rational (max,+) series but rational (min,+) series. In that example, S_1 was clearly unambiguous, hence in fact it was also a rational (min,+) series. Generalizing over this example, we are going to study the residuation of (min,+) rational series by (max,+) rational series and symmetrically. We will show that when S_1 and S_2 are rational series with different polarities, then $S_1 \circ\!\!\!/ S_2$ is a rational series with the same polarity as S_1. Now, there is a price to pay: if $S_1 \circ\!\!\!/ S_2$ is interpreted as a controller enforcing the specification S_1 on the system S_2, then it is generally not possible to represent the closed loop system as a rational power series since S_2 (the system) and $S_1 \circ\!\!\!/ S_2$ (the controller) live in different algebras.

In order to achieve the above goal, we need a hybrid product of elements from the (max,+) and (min,+) semirings, and a corresponding residuation. As $\overline{\mathbb{R}_{\max}}$ and $\overline{\mathbb{R}_{\min}}$ have the same carrier set $\mathbb{R} \cup \{-\infty, +\infty\}$, the operations $b \otimes_{\max} a$ and $b/_{\max} a$ already defined for the (max,+) semiring (see Table I) may as well be seen as operations with profile $\overline{\mathbb{R}_{\max}} \times \overline{\mathbb{R}_{\min}} \to \overline{\mathbb{R}_{\max}}$. It is worth noting that the operation $/_{\max} : \overline{\mathbb{R}_{\max}} \times \overline{\mathbb{R}_{\min}} \to \overline{\mathbb{R}_{\max}}$ restricts on the incomplete dioids $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$ and co-restricts on the incomplete dioid $\mathbb{R}_{\max}$ (because it never produces the result $+\infty$). Dual operations $\otimes_{\min}$ and $/_{\min}$ can be defined similarly by exchanging $+\infty$ and $-\infty$ in Table I.

We are now ready to extend the operations $\otimes_{\max}$ and $/_{\max}$ to (max,+) and (min,+) series. As our product of series is the Hadamard product $S \odot_{\max} T = \bigoplus_{w\in\Sigma^*}(S(w) \otimes_{\max} T(w))w$, the corresponding residuation operation $\oslash_{\max}$ on series is given by pointwise extension of the operation $/_{max}$, thus $S \oslash_{\max} T = \bigoplus_{w\in\Sigma^*}(S(w)/_{\max}T(w))w$. A symmetric operation $\oslash_{\min}$ can be defined similarly. The following theorem shows that both residuation operations / preserve rationality, *i.e.* a finite automaton recognizing S/T may be constructed from finite automata recognizing S and T.

Theorem 3.1: Let $S \in \overline{\mathbb{R}_{\min}}Rat(\Sigma)$ and $T \in \overline{\mathbb{R}_{\max}}Rat(\Sigma)$. Then

- $S \oslash_{\max} T \in \overline{\mathbb{R}_{\min}}Rat(\Sigma)$;
- $T \oslash_{\min} S \in \overline{\mathbb{R}_{\max}}Rat(\Sigma)$.

Proof: Due to lack of space, we give here only the intuition of the proof. A complete proof can be found in [2]. To show the (min,+) of (max,+) rationality, we construct the automaton recognizing the residuated series. The first step is to compute the product automaton of S and T, seen as untimed automata. Then, for each transition of this new automaton, consider the weights of the two transition of S and T used to build it. The weight of this transition is the residuation of these weights. ■

It would also be natural to look at $T \oslash_{\max} S$. When dealing with non-complete dioids, the same kind of results holds: if $S \in \mathbb{R}_{\min}Rat(\Sigma)$ and $T \in \mathbb{R}_{\max}Rat(\Sigma)$, $T \oslash_{\max} S \in \mathbb{R}_{\max}Rat(\Sigma)$. Unfortunately, some difficulties arise for non-complete dioids, in particular with the support of the residuation. Indeed, in T, there can be a word w with weight $+\infty$, but not in S, as it is the zero element of the (min,+) algebra. But $+\infty \oslash_{\max} + \infty = +\infty$ and w is in the support of the residuation but not in that of S. Moreover, this residuation of dioids with different sets does not correspond to any natural product.

Suppose that S is (min,+) rational and T is (max,+) rational. As before, the (min,+) series $S \oslash_{\max} T$ defines for each w the maximal delay that can be added to the actual duration $T(w)$ needed by the plant without exceeding the specified upper bound $S(w)$, but now one can decide whether $S \oslash_{\max} T(w) \geq 0$ for all w in $supp(S) \cap supp(T)$ in view of the following proposition, the proof of which is recalled for the sake of completeness.

Proposition 3.2: Let $\mathcal{A} = (\alpha, \mu, \beta)$ be a weighted automaton. Let $S_{\min}$ and $S_{\max}$ be the respective (min,+) and (max,+) series recognized by $\mathcal{A}$. It is decidable in time $O(n^3)$, where n is the dimension of μ, whether $S_{\min}(w) < 0$ (resp. $S_{\max}(w) > 0$) for some w in $supp(S_{\min})$ (resp. in $supp(S_{\max})$).

Proof: $S_{\min}(w) < 0$ for some w in the support of S if and only if, the trim automaton, a cycle of negative weight (which can be checked usung Bellman-Ford algorithm) or the automaton contains some elementary path $q_0\, a_1\, q_1 \ldots a_n\, q_n$ such that $\alpha_{q_0} + \sum_{i=1}^{n} \mu(a_i)_{(q_{i-1},q_i)} + \beta_{q_n} < 0$ (which can be checked using Floyd-Warshall algorithm). ■

When S is (min,+) rational and T is (max,+) rational, Theorem 3.1 and Proposition 3.2 provide an effective procedure for deciding whether there exists a non-negative delay-controller series, namely the (min,+) rational series $S \oslash_{\max} T$, and then constructing it. Note that, by a result of Krob presented in [14], any (min,+) automaton recognizing $S \oslash_{\max} T$ can be transformed to an equivalent (min,+) automaton (α, μ, β) in which all entries of α, μ, and β are either non-negative or equal to $-\infty$ [14]. A (min,+) rational series of this type is less unlikely than an arbitrary series to represent a useful delay controller. However, a (min,+) automaton which fails to be sequential cannot easily be used for on-line control, whence the problem to construct a sequential (max,+) series K as large as possible such that $K(w) \leq (S \oslash_{\max} T)(w)$ for all w. For such (non-optimal) controller series K, the closed-loop system $T \odot_{max} K$ could in fact be represented by a (max,+) rational series, but this problem is open.

IV. Interval Weighted Automata

A. Background

Residuation of intervals in a dioid $\mathcal{D}$ has been studied in [15]. In that work, the set $\mathcal{I}(\mathcal{D})$ of all intervals $[\underline{x}, \overline{x}] = \{t \in \mathcal{D} \,|\, \underline{x} \preceq t \preceq \overline{x}\}$ is shown to be a dioid, with (left and right) residuation operations. The residuation operations serve to compute robust compensating controllers for Timed Event Graphs. Timed behaviors are defined by associating to each transition a formal power series in one variable γ over $\overline{\mathbb{Z}}_{max}$, such that the coefficient of γ^k in this series is the date of the k-th firing of the transition. For specifications, intervals express tolerances on the desired behavior. For systems, intervals reflect an imprecise knowledge of the exact timed behavior, whence the need for *robust* controllers. Technically speaking, controllers are computed by residuation in $\mathcal{I}(\mathcal{D})$, where $\mathcal{D} = \overline{\mathbb{Z}}^+_{max}[\![\gamma]\!]$ is the set of so-called *causal elements*. Residuation results in intervals $[\underline{x}, \overline{x}]$ whose bounds $\underline{x}$ and $\overline{x}$ are *realizable* series in $\overline{\mathbb{Z}}^+_{max}[\![\gamma]\!]$, which is much stronger than rational series. Such bounds define indeed controllers that can be realized by Timed Event Graphs.

While intervals of (max,+) rational series are the basic setting used in [15], we will investigate here the alternative setting of rational series of intervals, first in $\mathcal{I}^{max}_{max}$ and then in $\mathcal{I}^{min}_{max}$. In both cases, we examine what residuation can afford, and end up with the conclusion that the results are not worth the effort. Note that, differently from [15], we consider formal power series on alphabets Σ with more than one symbol, and we do not care for the realizability of series by Timed Event Graphs but only for their rationality, and hopefully for non-ambiguity or sequentiality.

B. Residuation of $\mathcal{I}_{max}^{max}$ -series

In order to make the definition of $\mathcal{I}_{max}^{max}$ and of residuation in this dioid precise, let us recall definitions and results adapted from [15].

Definition 4.1: A (closed) interval in dioid $\mathcal{D}$ is a non-empty set of the form $\mathbf{x} = [\underline{x}, \overline{x}] = \{t \in \mathcal{D} \,|\, \underline{x} \preceq t \preceq \overline{x}\}$.

Proposition 4.2: The set of intervals, denoted $I(\mathcal{D})$, endowed with the coordinate-wise operations $[\underline{x}, \overline{x}] \oplus [\underline{y}, \overline{y}] = [max(\underline{x}, \underline{y}), max(\overline{x}, \overline{y})]$ and $[\underline{x}, \overline{x}] \otimes [\underline{y}, \overline{y}] = [\underline{x}+\underline{y}, \overline{x}+\overline{y}]$, is a dioid, where the intervals $\epsilon = [\epsilon, \epsilon]$ resp. $\mathbf{e} = [e, e]$ are the zero resp. the neutral element. If $\mathcal{D}$ is complete, then $I(\mathcal{D})$ is complete.

Proposition 4.3: For any interval $\mathbf{a} \in I(\mathcal{D})$, the right product by $\mathbf{a}$ *i.e.* the operation $\mathbf{x} \to \mathbf{x} \otimes \mathbf{a}$, has a (right) adjoint residual operation $\mathbf{y} \to \mathbf{y} \phi \mathbf{a}$, given by $\mathbf{b} \phi \mathbf{a} = [\underline{b} \phi \underline{a} \wedge \overline{b} \phi \overline{a}, \overline{b} \phi \overline{a}]$ for any intervals $\mathbf{a} = [\underline{a}, \overline{a}]$ and $\mathbf{b} = [\underline{b}, \overline{b}]$.

Take $\mathcal{D} = \overline{\mathbb{R}_{max}}$ and let $\mathcal{D}' = I(\mathcal{D})$ (thus $\mathcal{D}' = \mathcal{I}_{max}^{max}$). In view of the above, $\mathcal{I}_{max}^{max}$ is a complete dioid with residuation. As residuation in $\mathcal{D}'$ extends to series in the dioid $\mathcal{D}'(\Sigma)$ equipped with Hadamard product, we get for free a residuation operation on $\mathcal{I}_{max}^{max}(\Sigma)$. Now, if we call *degenerated* those intervals $\mathbf{x} = [\underline{x}, \overline{x}]$ for which $\underline{x} = \overline{x}$, then the subset of the degenerated intervals induces a restriction of $\mathcal{I}_{max}^{max}$ which is isomorphic to $\overline{\mathbb{R}_{max}}$. Thus, $\overline{\mathbb{R}_{max}}(\Sigma)$ embeds isomorphically into a complete subdioid of $\mathcal{I}_{max}^{max}(\Sigma)$. It then follows from Example 1, after replacing numbers 0 and 1 with corresponding intervals $[0, 0]$ and $[1, 1]$, that residuation of series in $\mathcal{I}_{max}^{max}(\Sigma)$ does not preserve rationality, hence it does not enable an effective computation of compensating controllers.

C. Residuation of $\mathcal{I}_{max}^{min}$ -series

We consider now series of intervals in $\mathcal{I}_{max}^{min}(\Sigma)$. As $\mathcal{I}_{max}^{min}$ is isomorphic to the direct product of $\overline{\mathbb{R}_{\max}}$ and $\overline{\mathbb{R}_{\min}}$, residuation in $\mathcal{I}_{max}^{min}$ operates componentwise, *i.e.* $[\underline{x}, \overline{x}] \phi [\underline{y}, \overline{y}] = [\underline{x} /_{max} \underline{y}, \overline{x} /_{min} \overline{y}]$. The induced restriction of $\mathcal{I}_{max}^{min}(\Sigma)$ on intervals of the form $[\underline{x}, -\infty]$ is a complete subdioid isomorphic to $\overline{\mathbb{R}_{max}}$. It therefore follows from Example 1, where the numbers 0 and 1 are replaced with the intervals $[0, -\infty]$ and $[1, -\infty]$, that residuation in $\mathcal{I}_{max}^{min}(\Sigma)$ does not preserve rationality, hence it does not provide an effective computation of controllers.

V. Robust control of partially known systems against tolerance specifications

In this section, we consider intervals of formal power series over $\overline{\mathbb{R}}$, whose lower and upper bounds $\underline{S} \in \overline{\mathbb{R}_{\max}}\, Rat(\Sigma)$ and $\overline{S} \in \overline{\mathbb{R}_{\min}}\, Rat(\Sigma)$ have the same support and specify a tolerance $[\underline{S}, \overline{S}]$ on the desired behavior of a plant. We assume that the behavior of the uncontrolled plant is described abstractly, hence imprecisely, by an interval $[\underline{T}, \overline{T}]$ of formal power series over $\overline{\mathbb{R}}$, whose lower and upper bounds $\underline{T} \in \overline{\mathbb{R}_{\min}}\, Rat(\Sigma)$ and $\overline{T} \in \overline{\mathbb{R}_{\max}}\, Rat(\Sigma)$ have the same support as the tolerance $[\underline{S}, \overline{S}]$. This is coherent with the general assumption that compensating controllers can delay the plant's actions but cannot otherwise prevent them from firing. Although no procedure is known for deciding whether $\underline{T}(w) \leq \overline{T}(w)$ for all $w \in \Sigma^*$, we do not consider this as a problem since the interval $[\underline{T}, \overline{T}]$ is supposed to describe albeit imprecisely the behavior of a real system which naturally belongs to this interval. Our goal is to compute the largest interval $[\underline{K}, \overline{K}]$ of compensating (or delay) controller series K over $\overline{\mathbb{R}}$ such that $(\overline{T} \odot_{\max} K)(w) \leq \overline{S}(w)$ and $(\underline{T} \odot_{\min} K)(w) \geq \underline{S}(w)$ for all $w \in supp(S)$ and for all formal power series $T \in [\underline{T}, \overline{T}]$. Any such formal power series $K \in [\underline{K}, \overline{K}]$ thus provides a robust control enforcing the specified tolerance $[\underline{S}, \overline{S}]$ on the plant.

For $T \in [\underline{T}, \overline{T}]$, $w \in supp(\overline{S})$, $(T \odot_{max} K)(w) \leq \overline{S}(w)$ iff $K(w) \leq \overline{S}(w)/_{\max} T(w)$, and therefore $(T \odot_{max} K) \leq \overline{S}$ for all $T \in [\underline{T}, \overline{T}]$ iff $K \leq \overline{S} \phi_{\max} \overline{T}$.

For $T \in [\underline{T}, \overline{T}]$, $w \in supp(\underline{S})$, $(T \odot_{min} K)(w) \geq \underline{S}(w)$ iff $K(w) \geq \underline{S}(w)/_{\min} T(w)$, and therefore $(T \odot_{min} K) \geq \underline{S}$ for all $T \in [\underline{T}, \overline{T}]$ iff $K \geq \underline{S} \phi_{\min} \underline{T}$.

For any $w \in \Sigma^*$, $\underline{S}(w) = -\infty$ iff $\underline{T}(w) = +\infty$, and $\overline{S}(w) = +\infty$ if $\overline{T}(w) = -\infty$, as we have assumed that all series $\underline{S}$, $\overline{S}$, $\underline{T}$ and $\overline{T}$ have the same support.

As a result, the interval of robust delay controller series $[\underline{K}, \overline{K}]$ is given by $\underline{K} = \underline{S} \phi_{\min} \underline{T}$ and $\overline{K} = \overline{S} \phi_{\max} \overline{T}$. Now $\underline{S}$ is (max,+) rational and $\underline{T}$ is (min,+) rational, hence $\underline{K}$ is (max,+) rational. Similarly, $\overline{S}$ is (min,+) rational and $\overline{T}$ is (max,+) rational, hence $\overline{K}$ is (min,+) rational. Altogether, the controller $[\underline{K}, \overline{K}]$ is therefore in the same format as the original specification $[\underline{S}, \overline{S}]$.

If $\underline{K}(w) > \overline{K}(w)$ for some w, then the control problem has no solution, *i.e.* the interval of possible controller series K is empty. Seeing that $\underline{K} \in \overline{\mathbb{R}_{\max}}\, Rat(\Sigma)$ and $\overline{K} \in \overline{\mathbb{R}_{\min}}\, Rat(\Sigma)$, the series $\overline{K} \phi_{\max} \underline{K}$ is (min,+) rational, hence by Proposition 3.2, one can decide upon this property (the same technique may be used right at the beginning to check that $[\underline{S}, \overline{S}]$ is a well-formed interval). Note also that the controller series interval $[\underline{K}, \overline{K}]$ may in turn be considered as a specification to be enforced on a new plant component T' that runs concurrently with T.

Before the above results and constructions may be applied to practical control problems, one needs to solve the open problem of finding an unambiguous rational controller series K in $[\underline{K}, \overline{K}]$. Even better, one should search in this interval for a sequential controller series K that is moreover increasing, *i.e.* such that $K(wt) \geq K(w)$ for all $w \in \Sigma^*$ and $t \in \Sigma$. At present, we do not know whether one can decide upon the existence of these two types of controller series. However, the next Proposition 5.1 may help to find an unambiguous rational controller series K in $[\underline{K}, \overline{K}]$ (when both relations $\underline{K} \leq S_{\min}$ and $S_{\max} \leq \overline{K}$ are satisfied for the series $S_{\min}$ and $S_{\max}$ defined in the proposition - note that if $S_{\min} = S_{\max}$ then the considered series is in fact non-ambiguous).

Proposition 5.1: Let $\mathcal{A} = (\alpha, \mu, \beta)$ be a weighted automaton, and $S_{\max}$ and $S_{\min}$ be the respective (max,+) and (min,+) series recognized by this automaton. Then there exists a non-ambiguous series S such that $\forall w \in Supp(S_{\max})$, $S_{\min}(w) \leq S(w) \leq S_{\max}(w)$.

Proof: The proof is omitted for the sake of space, see the research report [2] for the details. The basic idea is to construct the Schützenberger covering of $\mathcal{A}$ and then to eliminate competitions from this automaton, thus yielding at the end an unambiguous series recognizer. ∎

VI. EXAMPLE

In a jobshop, manufactured pieces must be transported from point A to B and some others from point B to A. Two robots, X and Y, can perform this task on a unique rail (meaning that the robots cannot perform their tasks simultaneously. The trip of any robot from A to B (resp. B to A) is represented by the letter a (resp. b). In order to comply with the other elements of the manufacture, when a^p and b^q trips are planned, with $|p - q| \leq 1$, the last delivery to A (resp. B) must be done between the dates $p\underline{k}_a$ (resp. $q\underline{k}_b$) and $p\overline{k}_a$ (resp. $q\overline{k}_b$). Robot X (resp Y) has the following characteristics: it starts from A (resp. B) and its trip's duration from A to B and from B to A are in $[\underline{x}_{AB}, \overline{x}_{AB}]$ and $[\underline{y}_{AB}, \overline{y}_{AB}]$, respectively. Given a^p and b^q, the role of the planner is to fix the series of tasks to be performed, by shuffling these two sequences into a single word $w \in \{a, b\}^*$, and to fix a start time δ for the corresponding series of trips of the robots, knowing that the planner cannot distinguish between the robots and the execution of the series of tasks cannot be interrupted after it has been started. Robots at rest in A or B serve to accomplish fast secondary tasks that may always be postponed and do not interfere with the main series of tasks considered here.

As regards time, all possible executions of shuffled sequences of a and b are described by a (min,+) automaton $\underline{T}$ and a (max,+) automaton $\overline{T}$ with an identical structure, shown in Figure 2. The constants x_{AB} and the like are replaced by $\underline{x}_{AB}$ in $\underline{T}$ and by $\overline{x}_{AB}$ in $\overline{T}$.

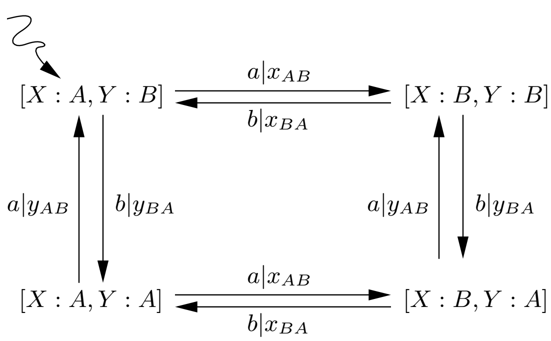


Fig. 2.

The target behaviour of the robots is described by a (max,+) automaton $\underline{S}$ and a (min,+) automaton $\overline{S}$ with an identical structure, shown in Figure 3. The constants k_a and the like are replaced by $\underline{k}_a$ in $\underline{S}$ and with $\overline{k}_a$ in $\overline{S}$.

Let $\underline{K} = \underline{S} \not\phi_{\min} \underline{T}$ and $\overline{K} = \overline{S} \not\phi_{\max} \overline{T}$. These rational series are recognized by two (max,+) and (min,+) automata with a common structure shown of Figure 4 (the constants

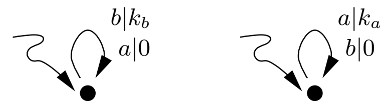


Fig. 3.

k_a, x_{AB} and so on are replaced with $\underline{k}_a, \underline{x}_{AB}$ in the (max,+) automaton, and with $\overline{k}_a, \overline{x}_{AB}$ in the (min,+) automaton).

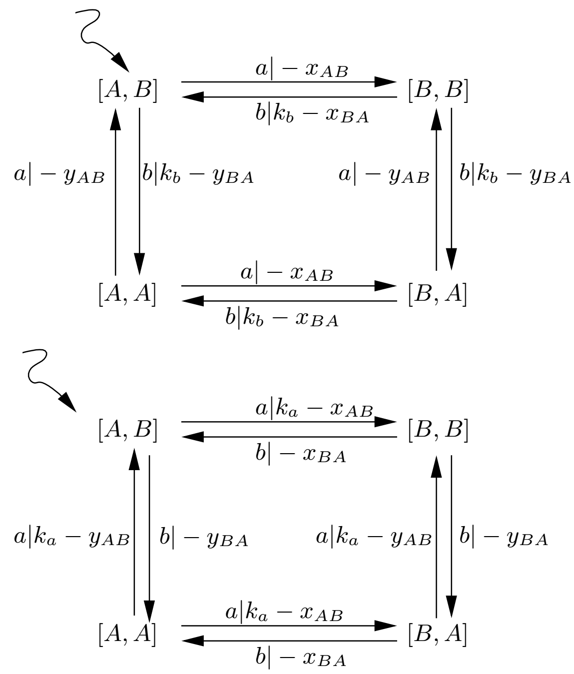


Fig. 4.

In order to enforce the guarantees of timely delivery, the planner must choose $w \in \{a, b\}^*$ such that $[\max\{\underline{K}(w), 0\}, \overline{K}(w)]$ is non-empty. Any non-negative value δ in $[\underline{K}(w), \overline{K}(w)]$ may then be picked as the starting time for executing the plan w.

VII. RESIDUATION W.R.T. CAUCHY PRODUCT

In this section, we propose an adaptation of the constructions developped in Section III to the case when the Hadamard product of formal power series is replaced with the Cauchy product. We propose also an adaptation of the elements suggested in Section V for reasoning on intervals of rational series. We finally discuss possible applications to contracts encountered in theories of software components.

Definition 7.1: Given a dioid $\mathcal{D}$ and a finite alphabet Σ, the Cauchy product of two series $S, T \in \mathcal{D}(\Sigma)$ is defined by: $S \bigotimes T = \bigoplus_{w \in \Sigma^*} (\bigoplus_{uv=w} (S(u) \otimes T(v)))w$.

It is well known that the Cauchy product preserves rationality of (max,+) or (min,+) series. In the sequel, we let $\mathcal{D} = \overline{\mathbb{R}}_{max}$. The Cauchy product of two series is thus $S \bigotimes_{max} T = \bigoplus_{w \in \Sigma^*} (\bigoplus_{w=uv} (S(u) \otimes_{max} T(v)))w$ where $\otimes_{max}$ is the pruduct in the (max,+) semiring (see Table I). Both $(\mathcal{D}(\Sigma), \oplus, \otimes_{max})$ and $(\mathcal{D}Rat(\Sigma), \oplus, \otimes_{max})$ are dioids. The Cauchy product, as opposed to the Hadamard product, is not commutative, hence one must distinguish left residuals $S \oslash_{max} T$ and right residuals $S \obslash_{max} T$ of formal power series w.r.t. $\bigotimes_{max}$. The two residual operations $(- \oslash_{max} T)$ and $(- \obslash_{max} T)$ are (left) adjoint to the operations $(- \otimes_{max} T)$ and $(T \otimes_{max} -)$ respectively, thus: $S \oslash_{max} T = \bigvee\{X \mid X \otimes_{max} T \leq S\}$, $S \obslash_{max} T = \bigvee\{X \mid T \otimes_{max} X \leq S\}$.

Theorem 7.2: Let $S \in \overline{\mathbb{R}_{\min}}Rat(\Sigma)$ and $T \in \overline{\mathbb{R}_{\max}}Rat(\Sigma)$. Then

- $S \oslash_{\max} T \in \overline{\mathbb{R}_{\min}}Rat(\Sigma)$, and $S \obslash_{\max} T \in \overline{\mathbb{R}_{\min}}Rat(\Sigma)$;
- $T \oslash_{\min} S \in \overline{\mathbb{R}_{\max}}Rat(\Sigma)$, and $T \obslash_{\min} S \in \overline{\mathbb{R}_{\max}}Rat(\Sigma)$.

Proof: The proof is similar to the case of Hadamard product and omitted for the sake of space, see the research report [2] for the details. ■

Consider a tolerance $[\underline{S}, \overline{S}]$, specifying the desired behavior of the sequential composition $K \bigotimes_{max} T$ or $T \bigotimes_{max} K$ of two component systems K and T, where T is given but K is missing. Suppose it is known that the behavior of component T lies between two bounds $\underline{T} \in \overline{\mathbb{R}_{\min}}\, Rat(\Sigma)$ and $\overline{T} \in \overline{\mathbb{R}_{\max}}\, Rat(\Sigma)$. Assuming that all the series $\underline{S}$, $\overline{S}$, $\underline{T}$ and $\overline{T}$ have the same support, we want to compute from $[\underline{S}, \overline{S}]$ and $[\underline{T}, \overline{T}]$ the largest interval $[\underline{K}, \overline{K}]$ such that $K \bigotimes_{max} T$ (resp. $T \bigotimes_{max} K$) lies in $[\underline{S}, \overline{S}]$ for all possible components T.

Consider the sequential composition $T \bigotimes_{max} K$. Thus, we require that $(T \bigotimes_{max} K)(w) \leq \overline{S}(w)$ and $(T \bigotimes_{min} K)(w) \geq \underline{S}(w)$ for all $T \in [\underline{T}, \overline{T}]$ and for all words $w \in \Sigma^*$. The first requirement holds iff $T(u) \otimes_{max} K(v) \leq \overline{S}(uv)$ for all $T \in [\underline{T}, \overline{T}]$ and for all $u, v \in \Sigma^*$, iff $\overline{T}(u) \otimes_{max} K(v) \leq \overline{S}(uv)$ for all u and v, iff $K(v) \leq \overline{S}(uv)/_{max}\overline{T}(u)$ for all u and v, iff $K(v) \leq \bigwedge_{u \in \Sigma^*} \overline{S}(uv)/_{max}\overline{T}(u)$ for all u, iff $K \leq (\overline{S} \obslash_{max} \overline{T})$. The second requirement holds iff $T(u) \otimes_{min} K(v) \geq \underline{S}(uv)$ for all $T \in [\underline{T}, \overline{T}]$ and for all $u, v \in \Sigma^*$, iff $\underline{T}(u) \otimes_{min} K(v) \geq \underline{S}(uv)$ for all u and v, iff $K(v) \geq \underline{S}(uv)/_{min}\underline{T}(u)$ for all u and v, iff $K(v) \geq \bigvee_{u \in \Sigma^*} \underline{S}(uv)/_{min}\underline{T}(u)$ for all u, iff $K \geq (\underline{S} \obslash_{min} \underline{T})$. Finally, $\underline{K} = \underline{S} \obslash_{min} \underline{T}$ and $\overline{K} = \overline{S} \obslash_{max} \overline{T}$. By Theorem 7.2, $\underline{K}$ is a (max,+) series and $\overline{K}$ is a (min,+) series.

Consider now the sequential composition $K \bigotimes_{max} T$. By similar reasoning, one obtains $\underline{K} = \underline{S} \oslash_{min} \underline{T}$ and $\overline{K} = \overline{S} \oslash_{max} \overline{T}$. So, the interval $[\underline{S} \oslash_{min} \underline{T}, \overline{S} \oslash_{max} \overline{T}]$ characterizes exactly the set of all components which fulfil the *contract* $[\underline{S}, \overline{S}] \,/\, [\underline{T}, \overline{T}]$, meaning that whenever they are composed sequentially on the right with a component T satisfying the *assumption* $[\underline{T}, \overline{T}]$, the result of the composition satisfies the *guarantee* $[\underline{S}, \overline{S}]$.

VIII. Conclusion

In this paper, we have shown that residuation preserves rationality of formal power series and intervals thereof whenever the two operands of the residuation have opposite polarities (*e.g.* when one computes the residue S/T of a (max,+) rational series S by a (min,+) rational series T). This approach adds nothing to the theory, but it might help solving some practical control or design problems, if a few central but open questions can be solved, such that deciding whether exists and constructing a sequential series between a (max,+) and a (min,+) rational series.

Although we have only reasoned here with Hadamard product and Cauchy product, remarkably enough, all results and constructions which we have presented extend smoothly to a variety of other products with less obvious practical importance (see report [2] for a thorough presentation of this generalization).

Acknowledgments

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