On a coverage model in communications and its relations to a Poisson-Dirichlet process

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Simons Conference on Networks and Stochastic Geometry
UT Austin Austin, 17–21 May 2015

OUTLINE

Yesterday:

- “Germ-grain” coverage models in stochastic geometry,
- SINR (or shot-noise) coverage model,
- Palm and stationary coverage characteristics.

Today:

- Poisson-Dirichlet processes,
- Relations to SINR coverage.

Poisson-Dirichlet processes

Size-biased permutations

Consider a sequence of numbers $(P_n) = (P_n)_{n=1}^{\infty}$, with $\sum_n P_n = 1$, $0 \leq P_n \leq 1$. In fact (P_n) is a distribution on $\mathbb{N} = \{1, 2, \ldots\}$.

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A size-biased permutation (SBP) $(\tilde{P}_n)$ of (P_n), is a random permutation of the sequence (P_n) with distribution

$$\mathsf{P}\{\tilde{P}_1 = P_k\} = P_k$$

$$\vdots$$

$$\mathsf{P}\{\tilde{P}_n = P_j | \tilde{P}_{i,\, i \le n-1}\} = \frac{P_j}{1 - \sum_{i=1}^{n-1} \tilde{P}_i} \qquad P_j \ne \tilde{P}_1 \ldots, \tilde{P}_{n-1} \quad n \ge 1\,.$$

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$$\begin{aligned} \mathsf{P}\{\tilde{P}_1 &= P_k\} = P_k \\ &\vdots \\ \mathsf{P}\{\tilde{P}_n &= P_j | \tilde{P}_i, {}_{i \le n-1}\} = \frac{P_j}{1 - \sum_{i=1}^{n-1} \tilde{P}_i} \qquad \begin{array}{c} P_j \neq \tilde{P}_1 \ldots, \tilde{P}_{n-1} \\ n \ge 1\,. \end{array} \end{aligned}$$

We say (P_n) is invariant with respect to SBP (ISBP) if $(P_n) =_{\text{distr.}} (\tilde{P}_n)$. Clearly (P_n) needs to be a random.

Also, $(\tilde{\tilde{P}}_n)$ is ISBP for any (P_n).

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ISBP is a notion of stochastic equilibrium. Appears naturally in models of genetic populations that evolve under the influence of mutation and random sampling.

Stick-braking (SB) model

Consider the following "stick braking" (SB) model, also called residual allocation model:

$$P_1 = U_1, \quad P_n = (1 - U_1) \dots (1 - U_{n-1}) U_n, \quad n \geq 2\,,$$

for some independent $U_1, U_2, \dots \in (0, 1)$. Note $\{P_n\}$ is a distribution.

Again, such constructions appear naturally in population models.

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When (for what distribution of $\boldsymbol{(U_n)}$) $\boldsymbol{(P_n)}$ is ISBP?

Kingman's Poisson-Dirichlet process

THM Consider SB model (P_n) with independent, identically distributed (U_n). Then (P_n) is ISBP iff $U_n \sim \text{Beta}(1, \theta)$ for some $\theta > 0$.

Mc Closky (1965)

Recall, $\text{Beta}(\alpha, \beta) \sim \Gamma(\alpha + \beta)/\Gamma(\alpha)\Gamma(\beta)t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t$, $t \in (0, 1)$.

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$\{V_i\}$ (considered as a point process on $(0, \infty)$) is called Poisson-Dirichlet PD$(0, \theta)$ point process.

Two-parameter Poisson-Dirichlet process

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$\{V_i\}$ (considered as a point process on $(0, \infty)$) is called Poisson-Dirichlet PD$(\alpha, 0)$ point process.
Similar construction of Poisson-Dirichlet PD(α, θ) point process? Slightly more involved.

$\mathbf{PD}(\alpha, 0)$ vs $\mathbf{PD}(0, \theta)$

FACT

- For PD$(0, \theta)$, $\sum_j Y_j$ has Gamma(θ) distribution and is independent of $\{V_i = \frac{Y_i}{\sum_j Y_j}\}$.

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  Indeed, $\sum_j Y_j = L^{-1/\alpha}$, where L is P$_{\alpha,0}$-almost surely existing limit
  $$L := \lim_{n \to \infty} n V_{(n)}^\alpha$$
  with $V_{(1)} > V_{(2)} > \ldots$ order statistics of $\{V_i\}$.

Proof of the existence of $L := \lim_{n\to\infty} nV_{(n)}^{\alpha}$

Recall $V_i := Y_i / \sum_j Y_j$ where $\{Y_i\} = \Theta_\alpha$ is Poisson process on $(0, \infty)$ with intensity $t^{-1-\alpha}\, dt$, $\alpha \in [0, 1)$.

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Kingman's argument: It is easy to see that $\{Y_j^{-\alpha}\}$ is homogeneous Poisson process on $(0,\infty)$ of intensity $1/\alpha$.
Indeed: $\mathsf{E}[\#\{Y_i^{-\alpha} \le s\}] = \mathsf{E}[\#\{Y_i \ge s^{-1/\alpha}\}] = \int_{s^{-1/\alpha}}^{\infty} t^{-1-\alpha}\,dt = s/\alpha.$

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Consequently, $Y_{(i+1)}^{-\alpha} - Y_{(i)}^{-\alpha}$ are iid exponential variables with mean α and thus by the LLN a.s. $\lim_{n\to\infty} Y_{(n)}^{-\alpha}/n = \lim_n 1/n \sum_{i=1}^{n} (Y_{(i+1)}^{-\alpha} - Y_{(i)}^{-\alpha}) = \alpha$.

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Finally,

$$nV_{(n)}^{\alpha} = n\Big(\frac{\sum_j Y_j}{Y_{(n)}^{-\alpha}}\Big)^{-\alpha} = \frac{n}{Y_{(n)}^{-\alpha}}\Big(\sum_j Y_j\Big)^{-\alpha} \to \frac{(\sum_j Y_j)^{-\alpha}}{\alpha}.$$

Change-of-measure representation

Pitman, Yor (1997).

$$\mathsf{E}_{\alpha,\theta}[f(\{V_i\})] = C_{\alpha,\theta}\mathsf{E}_{\alpha,0}[L^{\theta/\alpha}f(\{V_i\})]\,,$$

where

$C_{\alpha,\theta} = 1/\mathsf{E}_{\alpha,0}[L^{\theta/\alpha}] = \Gamma(1-\alpha)^{\theta/\alpha}\Gamma(\theta+1)/\Gamma(\theta/\alpha+1)$.

SINR and Poisson-Dirichlet processes

STIR process is PD$(\mathbf{0}, \boldsymbol{\theta} = \mathbf{2}/\boldsymbol{\beta})$

- Denote SINR process $\boldsymbol{\Psi} := \{\boldsymbol{Z_i}\}$, with $\boldsymbol{Z_i} := \frac{\boldsymbol{S_i/\ell(|X_i|)}}{\boldsymbol{W+\sum_{j\neq i} S_j/\ell(|X_j|)}} = \frac{\boldsymbol{Y_i}}{\boldsymbol{W+\sum_{j\neq i} Y_j}}$.

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- Recall $\Theta = \{Y_i\}$ is Poisson pp of intensity $2a/\beta t^{-1-2/\beta}\, dt$, on $(0, \infty)$, equal (modulo irrelevant in this context constant $2a/\beta$) to this of Θ_α, with $\alpha = 2/\beta$. Recall, Θ_α gives rise to PD$(\alpha, 0)$ via similar (to SINR) points' normalization $V_i = \frac{Y_i}{\sum_j Y_j}$.

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- Recall SINR process $\Psi := \{Z_i\}$ can be easy related to STINR process $\Psi' := \{Z_i' := \frac{Y_i}{W + \sum_j Y_j}\}$ via $Z_i' = \frac{Z_i}{1 + Z_i}$.

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- Recall SINR process $\Psi := \{Z_i\}$ can be easy related to STINR process $\Psi' := \{Z'_i := \frac{Y_i}{W+\sum_j Y_j}\}$ via $Z'_i = \frac{Z_i}{1+Z_i}$.
- Consequently, in case of no noise ($W = 0$), STIR Ψ' is PD$(0, \alpha = 2/\beta)$. Many distributional characteristics of PD$(0,\alpha)$ are developed in Pitman, Yor (1997)!

A few consequences

Denote by $Z'_{(1)} > Z'_{(2)} > \dots$ the ordered points of the STINR process Ψ'.

FACT For the STINR process Ψ' ($W \geq 0$), the random variables

$$R_i := \frac{Z'_{(i+1)}}{Z'_{(i)}} = \frac{Y_{(i+1)}}{Y_{(i)}}, \quad i \geq 1$$

have, respectively, Beta$(2i/\beta, 1)$ distributions. Moreover, $\{R_i\}$ are mutually independent.

BB, Keeler (2014) using Pitman, Yor (1997)

A few consequences, cont'd

Denote for $i = 1, 2, \ldots$

$$A_i := \frac{Z'_{(1)} + \cdots + Z'_{(i)}}{Z'_{(i+1)}} = \frac{Y_{(1)} + \cdots + Y_{(i)}}{Y_{(i+1)}}. \tag{1}$$

$$\Sigma_i := \frac{Z'_{(i+1)} + Z'_{(i+2)} + \cdots}{Z'_i} = \frac{Y_{(i+1)} + Y_{(i+2)} + \cdots}{Y_{(i)}}. \tag{2}$$

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Observe that Σ_i^{-1}corresponds to SIR with successive-interference cancellation in case $W = 0$.;

cf. Zhang, Haenggi (2013).

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Similarly,

$(1 + A_{i-1})/\Sigma_i = (Y_{(1)} + \cdots + Y_{(i)})/(Y_{(i+1)} + Y_{(i+2)} + \cdots)$

corresponds to SIR with signal combination in case $W = 0$.;

cf. BB, Keeler (2015).

A few consequences, cont'd

For $\gamma \geq 0$ let

$$\phi_\beta(\gamma) := \frac{2}{\beta} \int_1^\infty e^{-\gamma x} x^{-2/\beta - 1} dx, \tag{3}$$

$$\psi_\beta(\gamma) := \Gamma(1 - 2/\beta)\gamma^{2/\beta} + \phi_\beta(\gamma). \tag{4}$$

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FACT Consider the STINR process Ψ' $(W \geq 0)$. Then A_{i-1} is distributed as the sum of $i-1$ independent copies of A_1, with the characteristic function $\mathsf{E}[e^{-\gamma A_{i-1}}] = (\phi_\beta(\gamma))^{i-1}$; Σ_i is distributed as the sum of i independent copies of Σ_1, with the characteristic function $\mathsf{E}[e^{-\gamma \Sigma_i}] = (\psi_\beta(\gamma))^{-i}$; and A_{i-1} and Σ_i are independent.

using Pitman, Yor (1997)

A few consequences, cont'd

FACT The inverse of the k th strongest STIR ($W = 0$) value, $1/Z'_{(k)}$, has the Laplace transform

$$\mathsf{E}[e^{-\gamma/Z'_{(k)}}] = e^{-\gamma}(\phi_\beta(\gamma))^{k-1}(\psi_\beta(\gamma))^{-k}.$$

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Observe, $1/Z'_{(k)} \le 1/t$ ($t \le 1$) is equivalent to $Z'_{(k)} \ge t$, and further equivalent to $Z_{(k)} \ge t/(1-t)$ (relation between STINR and SINR). Consequently, the above result gives alternative approach to calculate the stationary SIR ($W = 0$) k-coverage probabilities p_k with $\tau = 1/1 - t$.

A few consequences, cont'd

FACT For the STINR process ($\boldsymbol{W \geq 0}$),
$\boldsymbol{W/I = \left(\sum_{i=1}^{\infty} Z'_{(i)}\right)^{-1} - 1}$, and $\boldsymbol{W + I = (L/a)^{-\beta/2}}$, with
$\boldsymbol{L := \lim_{i\to\infty} i(Z'_{(i)})^{2/\beta}}$ existing almost surely.
Thus, (theoretically) one can recover the values of the received powers and the noise from the SINR measurements.

using Pitman, Yor (1997)

“Introducing $W > 0$ to Poisson-Dirichlet” from STIR to STINR

Factorial moments of the SINR process

Very much as for $\mathsf{E}[\mathcal{N}^{(n)}] = \mathsf{E}[\sum_{\substack{(Z'_1,\dots,Z'_n)\in(\Psi')^{\times n}\\ \text{distinct}}} \mathbf{1}(0 \in \bigcap_i C_i)]$,

$$M'^{(n)}(t'_1,\dots,t'_n) \quad := \mathsf{E}\Big[\sum_{\substack{(Z'_1,\dots,Z'_n)\in(\Psi')^{\times n}\\ \text{distinct}}} \prod_{j=1}^{n} \mathbb{1}(Z'_j > t'_j)\Big]$$

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We have

$$M'^{(n)}(t'_1,\dots t'_n)$$
$$= n!\left(\prod_{i=1}^{n} \hat{t}_i^{-2/\beta}\right) \mathcal{I}_{n,\beta}((W)a^{-\beta/2}) \mathcal{J}_{n,\beta}(\hat{t}_1,\dots,\hat{t}_n),$$

when $\sum_{i=1}^{n} t'_n < 1$ and $M'^{(n)}(t'_1,\dots t'_n) = 0$ otherwise,
where $\hat{t}_i = \hat{t}_i(t'_1,\dots,t'_n) := \frac{t'_i}{1-\sum_{j=1}^{n} t'_j}$;

Observe factorization of the noise contribution.

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Indeed, for the fist moment measure $M_{\mathsf{PD}}(\mathrm{d}t)$,

$$\int_0^1 f(t) M_{\mathsf{PD}}(\mathrm{d}t) := \mathsf{E}\Big[\sum_i f(V_i)\Big]$$

$$= \mathsf{E}\Big[\sum_i \frac{f(V_i)}{V_i} V_i\Big]$$

$$= \mathsf{E}\Big[\frac{f(\tilde{V}_1)}{\tilde{V}_1}\Big] \text{ where } \{\tilde{V}_i\} \text{ SBP of } \{V_i\}$$

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$$= \mathsf{E}\Big[\sum_i \frac{f(V_i)}{V_i} V_i\Big]$$

$$= \mathsf{E}\Big[\frac{f(\tilde{V}_1)}{\tilde{V}_1}\Big] \text{ where } \{\tilde{V}_i\} \text{ SBP of } \{V_i\}$$

hence $M_{\text{PD}}(\mathrm{d}t) = 1/t \times F_{\tilde{V}_i}(\mathrm{d}t)$ and and we know that $\tilde{V}_i = U_1 \sim \text{Beta}(1-\alpha, \theta + 1 \times \alpha)$.

Factorial moments of PD processes, cont'd

Similarly, by the induction, using ISBP representation of PD, the density $\mu^{(n)}_{\mathrm{PD}}(t_1, \ldots, t_n)$ of the n th factorial moment measure of the PD$(\alpha, 0)$ process can be easily shown to be

$$\mu^{(n)}_{\mathrm{PD}}(t_1, \ldots, t_n) = c_{n,2/\beta,0} \left(\prod_{i=1}^{n} (t'_i)^{-(2/\beta+1)} \right) \left(1 - \sum_{j=1}^{n} (t'_j) \right)^{2n/\beta - 1},$$

where

$$c_{n,\alpha,\theta} = \prod_{i=1}^{n} \frac{\Gamma(\theta + 1 + (i-1)\alpha)}{\Gamma(1-\alpha)\Gamma(\theta + i\alpha)};$$

(related to the Beta distributions of independent $\{U_n\}$ in SB model of PD).

Handa (2009).

Relating moments of STINR and PD

For $\sum_{i=1}^{n} t'_n < 1$, the density of the n th factorial moment measure of the STINR process is

$$\mu'^{(n)}(t'_1, \ldots t'_n) := (-1)^n \frac{\partial^n M'^{(n)}(t'_1, \ldots t'_n)}{\partial t'_1 \ldots \partial t'_n}$$

$$= \bar{\mathcal{I}}_{n,\beta}((W)a^{-\beta/2})\mu^{(n)}_{\mathrm{PD}}(t'_1, \ldots, t'_n)\,,$$

where $\bar{\mathcal{I}}_{n,\beta}(x) = \frac{\mathcal{I}_{n,\beta}(x)}{\mathcal{I}_{n,\beta}(0)}$.

General factorial moment expansions

Expansions of general characteristics ϕ of the STINR process

$$\mathsf{E}[\phi(\Psi')] = \phi(\emptyset) + \sum_{n=1}^{\infty} \int_{(0,1)^n} \phi_{t'_1,\dots,t'_n}\, \mu'^{(n)}(t'_1,\dots,t'_n)\, dt'_n \dots dt'_1$$

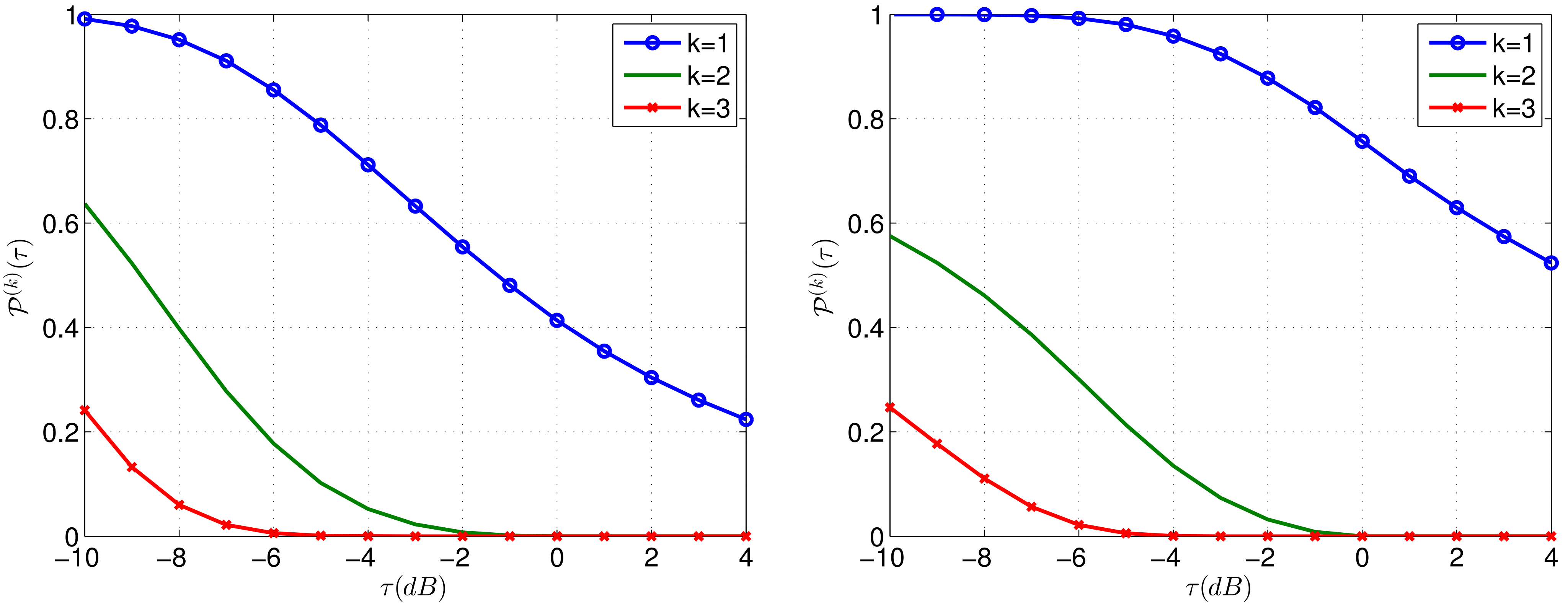
where

$$\begin{aligned}
\phi_{t'_1} &= \phi(\{t'_1\}) - \phi(\emptyset) \\
\phi_{t'_1,t'_2} &= \frac{1}{2}\Big(\phi(\{t'_1,t'_2\}) - \phi(\{t'_1\}) - \phi(\{t'_2\}) + \phi(\emptyset)\Big) \\
\dots \\
\phi_{t'_1,\dots,t'_n} &= \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k} \sum_{\substack{t'_{i_1},\dots,t'_{i_k} \\ \text{distinct}}} \phi(\{t'_{i_1},\dots,t'_{i_k}\})\,.
\end{aligned}$$

BB (1995).

Numerical examples

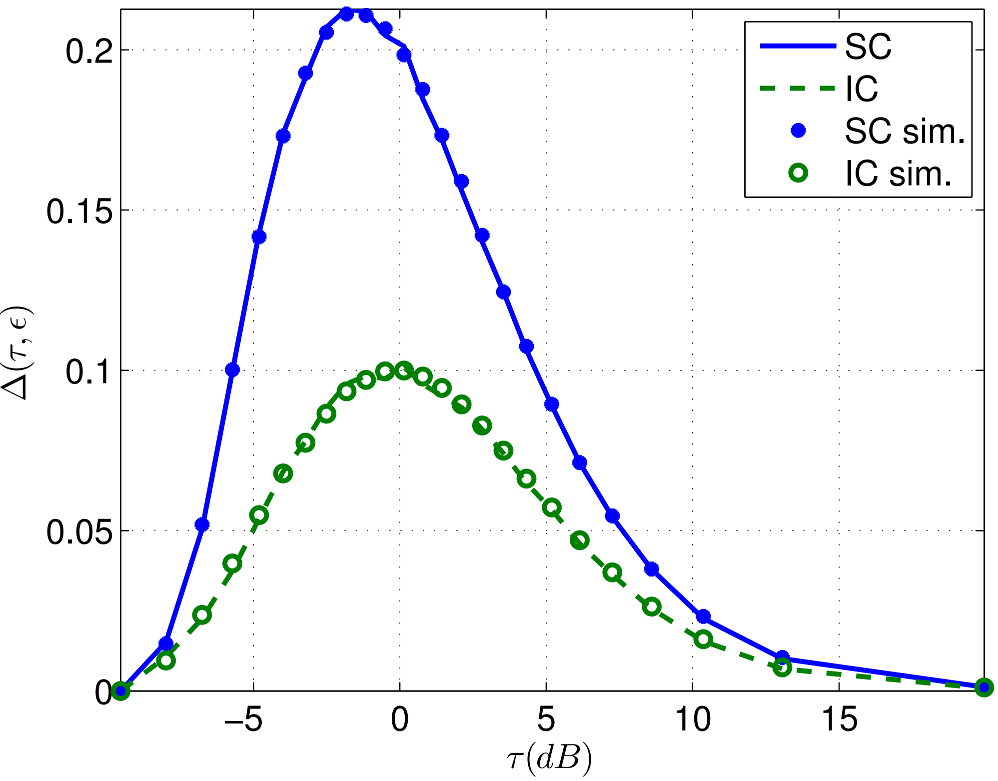
k-coverage probabilities



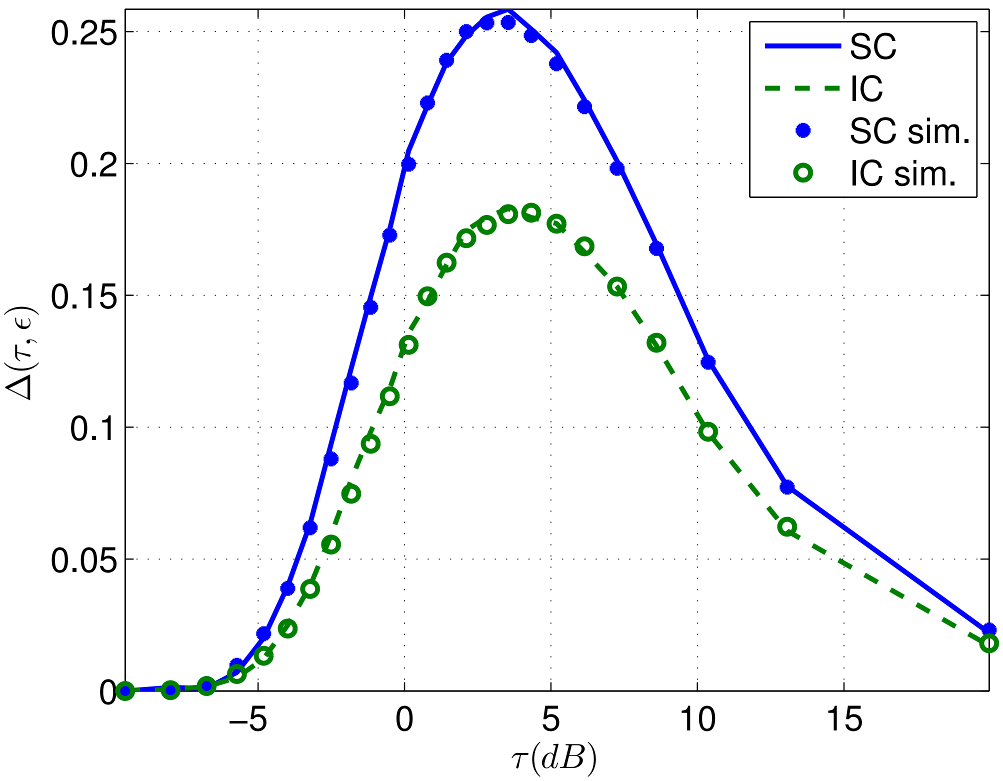
$\boldsymbol{\beta = 3}$ $\boldsymbol{\beta = 5}$

SINR k-coverage probability

Signal combination and interf. cancellation



$\boldsymbol{\beta = 3}$



$\boldsymbol{\beta = 5}$

The increase of the coverage probability when two strongest signals are combined (SC) or the second strongest signal is canceled from the interference (IC).

Conclusions

- We have seen a Poisson-Dirichlet process in some wireless communication model, where it describes "fractions" of the SINR spectrum. But Poisson-Dirichlet processes appear in several apparently different contexts.

Conclusions

- We have seen a Poisson-Dirichlet process in some wireless communication model, where it describes "fractions" of the SINR spectrum. But Poisson-Dirichlet processes appear in several apparently different contexts.
- "Two-parameter" family of Poisson-Dirichlet processes appear naturally in genetic population models in equilibrium ans well as in math/economic models (where it represents e.g. factions of the market owned by different companies).

Conclusions, cont'd

- In math/physics "our" PD$(\alpha, 0)$ process appears as the thermodynamic (large system) limit in the low temperature regime of Derrida's random energy model (REM). It is also a key component of the so-called Ruelle probability cascades, which are used to represent the thermodynamic limit of the Sherrington-Kirkpatrick model for spin glasses (types of disordered magnets).

Relations to the PD processes give some universality to the SINR model, initially motivated by wireless communications. This may hopefully attract some further interest to this model.

thank you